Smoothness, recursion, and derivatives of B-splines

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March 1, 2022

In these notes we define B-splines using divided differences. From this definition we deduce the smoothness of B-splines, the recursion formula, and formulas for derivatives. We also deduce a formula for the value of a B-spline at one of its knots.

1 Divided differences

Let us recall some basic facts about divided differences. The divided difference of a function f at the points $x_0, x_1, \ldots, x_k$ is the leading coefficient of the unique polynomial p of degree at most k that interpolates f at these points. We denote it by $[x_0, x_1, \ldots, x_k]f$ and it is said to have k-th order.

1.1 Distinct points

If the x_i are distinct, p is the Lagrange polynomial interpolant to f. We find $[x_0]f = f(x_0)$. For $k \geq 1$, by expressing p as a weighted average of the interpolants to f over the subsets $x_0, \ldots, x_{k-1}$ and $x_1, \ldots, x_k$, we obtain the recursion

$$[x_0, x_1, \ldots, x_k]f = \frac{[x_1, \ldots, x_k]f - [x_0, \ldots, x_{k-1}]f}{x_k - x_0},$$

The first examples are therefore

$$[x_0]f = f(x_0), \qquad [x_0, x_1]f = \frac{f(x_1) - f(x_0)}{x_1 - x_0},$$

$$[x_0, x_1, x_2]f = \frac{[x_1, x_2]f - [x_0, x_1]f}{x_2 - x_0} = \frac{\frac{f(x_2)-f(x_1)}{x_2-x_1} - \frac{f(x_1)-f(x_0)}{x_1-x_0}}{x_2 - x_0}.$$

From the Lagrange formula for p, we obtain the alternative formula,

$$[x_0, x_1, \ldots, x_k]f = \sum_{i=0}^{k} \frac{f(x_i)}{\prod_{\substack{j=0\\ j\neq i}}^{k}(x_i - x_j)}. \tag{1}$$

So, for example, we can write the second order case as

$$[x_0, x_1, x_2]f = \frac{f(x_0)}{(x_0 - x_1)(x_0 - x_2)} + \frac{f(x_1)}{(x_1 - x_0)(x_1 - x_2)} + \frac{f(x_2)}{(x_2 - x_0)(x_2 - x_1)}.$$

1.2 Arbitrary points

If any of the points $x_0, x_1, \ldots, x_k$ are equal we understand the interpolant p to be the Hermite interpolant to f. By this we mean that if x_i has multiplicity m, i.e., x_i appears m times in the sequence $x_0, x_1, \ldots, x_k$, then p and all its derivatives up to order $m-1$ agree with f at this point x_i. This means that in the special case that all the points are equal, i.e.,

$$x_0 = x_1 = \cdots = x_k,$$

then p is the Taylor approximation to f at x_0, and so

$$[x_0, \ldots, x_k]f = [\underbrace{x_0, \ldots, x_0}_{k+1}]f = \frac{f^{(k)}(x_0)}{k!}.$$

If only some of the points are equal, then in analogy to the case of distinct points, the divided difference can be expressed recursively. If $x_i \neq x_j$, we can use the recursion

$$[x_0, \ldots, x_k]f = \frac{[x_0, \ldots, \hat{x}_i, \ldots, x_k]f - [x_0, \ldots, \hat{x}_j, \ldots, x_k]f}{x_j - x_i},$$

where $x_0, \ldots, \hat{x}_i, \ldots, x_k$ means the sequence $x_0, \ldots, x_k$ with the point x_i removed. Thus, for example, we find

$$[x_0, x_0, x_1]f = \frac{[x_0, x_1]f - [x_0, x_0]f}{x_1 - x_0} = \frac{\frac{f(x_1)-f(x_0)}{x_1-x_0} - f'(x_0)}{x_1 - x_0},$$

which is a linear combination of $f(x_0), f'(x_0), f(x_1)$:

$$[x_0, x_0, x_1]f = c_{00}f(x_0) + c_{01}f'(x_0) + c_{10}f(x_1),$$

where

$$c_{00} = \frac{-1}{(x_1 - x_0)^2}, \quad c_{01} = \frac{-1}{x_1 - x_0}, \quad c_{10} = \frac{1}{(x_1 - x_0)^2}.$$

In general, it follows from the recursion that for a sequence of distinct points $x_0, x_1, \ldots, x_k$ with multiplicities $m_0, m_1, \ldots, m_k$, there are coefficients $c_{i,r}$ such that

$$[\underbrace{x_0, \ldots, x_0}_{m_0}, \ldots, \underbrace{x_k, \ldots, x_k}_{m_k}]f = \sum_{i=0}^{k} \sum_{r=0}^{m_i - 1} c_{ir} f^{(r)}(x_i). \tag{2}$$

In other words, the divided difference is a linear combination of f and its derivatives, where the highest order derivative at x_i is the multiplicity of x_i minus 1.

1.3 Leibniz rule

Later, we will make use of a convenient formula, called the Leibniz rule, for the divided difference of a product of two functions. For the product of functions f and g, the Leibniz rule is

$$[x_0, x_1, \ldots, x_k](fg) = \sum_{i=0}^{k} [x_0, \ldots, x_i]f\,[x_i, \ldots, x_k]g. \tag{3}$$

It is a generalization of the Leibniz rule for derivatives of a product of functions.

2 B-splines

We can define B-splines as follows. For any integers $d \geq 0$ and $n \geq 1$, we call a sequence $\mathbf{t} = (t_1, t_2, \ldots, t_{n+d+1})$, $t_i \in \mathbb{R}$, a *knot vector* if $t_i \leq t_{i+1}$.

For any real number x we write

$$(x)_+ = \begin{cases} x, & \text{if } x > 0, \\ 0 & \text{otherwise.} \end{cases}$$

For $j = 1, 2 \dots, n$, we define the j-th B-spline $B_{j,d}$ by the formula

$$B_{j,d}(x) = (t_{j+d+1} - t_j)[t_j, t_{j+1}, \dots, t_{j+d+1}](\cdot - x)_+^d. \tag{4}$$

Here, x is fixed and the divided difference applies to the function

$$f(y) = (y - x)_+^d.$$

3 Smoothness

Suppose that the knots defining $B_{j,d}$ have the following multiplicities,

$$(t_j, t_{j+1}, \dots, t_{j+d+1}) = (\underbrace{\tau_0, \dots, \tau_0}_{m_0}, \dots, \underbrace{\tau_k, \dots, \tau_k}_{m_k}),$$

where $\tau_0 < \tau_1 < \cdots < \tau_k$. Then we can write $B_{j,d}$ as

$$B_{j,d}(x) = B[\underbrace{\tau, \dots, \tau_0}_{m_0}, \dots, \underbrace{\tau_k, \dots, \tau_k}_{m_k}](x). \tag{5}$$

Theorem 1 *The B-spline $B_{j,d}$ in (5) has smoothness of order C^{d-m_i} at τ_i, $i = 0, 1, \dots, k$.*

Proof. Since

$$\frac{d^r}{dy^r}(y - x)_+^d = \frac{d!}{(d-r)!}(y_i - x)_+^{d-r},$$

it follows from (2) that there are coefficients $c_{i,r}$, independent of x, such that

$$B_{j,d}(x) = (t_{j+d+1} - t_j) \sum_{i=0}^{k} \sum_{r=0}^{m_i - 1} \frac{d!}{(d-r)!} c_{i,r} (\tau_i - x)_+^{d-r}. \tag{6}$$

Since $(\tau_i - x)_+^{d-r}$, as a function of x, has smoothness C^{d-r-1} at τ_i, it follows that $B_{j,d}$ has smoothness of order $C^{d-(m_i-1)-1} = C^{d-m_i}$ at τ_i. □

4 Recursion

From the divided difference definition of B-splines we obtain the recursion formula. For this we will make use of the Leibniz rule (3).

Theorem 2 *For $d \geq 1$,*

$$B_{j,d}(x) = \frac{x - t_j}{t_{j+d} - t_j} B_{j,d-1}(x) + \frac{t_{j+d+1} - x}{t_{j+d+1} - t_{j+1}} B_{j+1,d-1}(x). \tag{7}$$

Proof. Starting from the definition (4), we use the fact that $(\cdot - x)_+^d$ can be written as the product

$$(\cdot - x)_+^d = (\cdot - x)(\cdot - x)_+^{d-1}.$$

We then apply the divided difference $[t_j, t_{j+1}, \ldots, t_{j+d+1}]$ to this product, and use the Leibniz rule. Since

$$[t_j](\cdot - x) = t_j - x, \quad [t_j, t_{j+1}](\cdot - x) = 1,$$

and $[t_j, \ldots, t_k](\cdot - x) = 0$ for any $k \geq j + 2$, we find

$$\begin{aligned} [t_j, \ldots, t_{j+d+1}](\cdot - x)_+^d = (t_j - x)[t_j, \ldots, t_{j+d+1}](\cdot - x)_+^{d-1} \\ + [t_{j+1}, \ldots, t_{j+d+1}](\cdot - x)_+^{d-1}. \end{aligned} \tag{8}$$

Since

$$[t_j, \ldots, t_{j+d+1}] = \frac{[t_{j+1}, \ldots, t_{j+d+1}] - [t_j, \ldots, t_{j+d}]}{t_{j+d+1} - t_j}, \tag{9}$$

multiplying both sides of (13) by $t_{j+d+1} - t_j$ gives

$$\begin{aligned} B_{j,d}(x) &= (t_j - x)([t_{j+1}, \ldots, t_{j+d+1}](\cdot - x)_+^{d-1} - [t_j, \ldots, t_{j+d}](\cdot - x)_+^{d-1}) \\ &\quad + (t_{j+d+1} - t_j)[t_{j+1}, \ldots, t_{j+d+1}](\cdot - x)_+^{d-1} \\ &= (x - t_j)[t_j, \ldots, t_{j+d}](\cdot - x)_+^{d-1} \\ &\quad + (t_{j+d+1} - x)[t_{j+1}, \ldots, t_{j+d+1}](\cdot - x)_+^{d-1}, \end{aligned}$$

which, by the definition of $B_{j,d-1}$ and $B_{j+1,d-1}$, gives the result. □

5 Derivatives

Theorem 3 *For $d \geq 1$,*

$$B'_{j,d}(x) = d\left(\frac{B_{j,d-1}(x)}{t_{j+d} - t_j} - \frac{B_{j+1,d-1}(x)}{t_{j+d+1} - t_{j+1}}\right). \tag{10}$$

Proof. Due to the recursion (9), we can express $B_{j,d}$ in (4) in the form

$$B_{j,d}(x) = [t_{j+1}, \ldots, t_{j+d+1}](\cdot - x)_+^d - [t_j, \ldots, t_{j+d}](\cdot - x)_+^d. \tag{11}$$

Differentiating this with respect to x gives

$$B'_{j,d}(x) = d([t_{j+1}, \ldots, t_{j+d+1}](\cdot - x)_+^{d-1} - [t_j, \ldots, t_{j+d}](\cdot - x)_+^{d-1}),$$

which, again by the definition of $B_{j,d-1}$ and $B_{j+1,d-1}$, yields the result. □

6 Value of a B-spline at a knot

Another useful property of a B-spline is that its value at one of its knots equals the value there of a B-spline of lower degree, more precisely, of the B-spline resulting from removing the knot.

Theorem 4 *For any $i = j, j+1, \ldots, j+d+1$,*

$$B[t_j, \ldots, t_{j+d+1}](t_i) = B[t_j, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j+d+1}](t_i).$$

Proof. Similar to the proof of the recursion formula, we apply the divided difference $[t_j, \ldots, t_{j+d+1}]$ to the product

$$(\cdot - x)_+^d = (\cdot - x)(\cdot - x)_+^{d-1},$$

and use the Leibniz rule. However, using the fact that $[t_j, \ldots, t_{j+d+1}]$ is symmetric with respect to its points, we are at liberty to order these points differently before applying the rule. By ordering them so that t_i goes first, followed by the rest, the rule gives

$$\begin{aligned}
&[t_i, t_j, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j+d+1}](\cdot - x)_+^d \\
&\qquad = (t_i - x)[t_i, t_j, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j+d+1}](\cdot - x)_+^{d-1} \\
&\qquad + [t_j, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j+d+1}](\cdot - x)_+^{d-1}.
\end{aligned} \tag{12}$$

Therefore, letting $x = t_i$,

$$\begin{aligned}[t_i, t_j, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j+d+1}](\cdot - t_i)_+^d \\ = [t_j, \ldots, t_{i-1}, t_{i+1}, \ldots, t_{j+d+1}](\cdot - t_i)_+^{d-1}, \end{aligned} \tag{13}$$

and dividing both sides by $t_{t+d+1} - t_j$ gives the result. □