UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT4200 — Commutative algebra

Day of examination: Tuesday Dcember 16, 2014

Examination hours: 14.30 – 18.30

This problem set consists of 2 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

EXERCISE 1

a) Let A be a ring, and $\mathbf{m} \subset A$ a maximal ideal. Prove that $\mathbf{m}$ is a prime ideal.

b) Let A be a ring, and $I \subset A$ an ideal. Let $\mathbf{p} \subset A$ be a prime ideal and let S be the difference set $A \setminus \mathbf{p}$. Show that the localization $S^{-1}(A/I) \neq 0$ if and only if $I \subset \mathbf{p}$.

EXERCISE 2 Let A be a ring and $\phi : A \to A$ a ring homomorphism.

a) Show that $B = \{x \in A \mid \phi(x) = x\}$ is a subring of A.

b) Show that if the composition $\phi^2(a) = \phi(\phi(a))$ is the identity map on A, then each element $a \in A$ is the root of a monic polynomial of degree two in $B[x]$.

EXERCISE 3 Let k be a field, and A the graded ring given by

$$A = k[x, y, z]/(xy, xz, yz)$$

a) Compute the Hilbert polynomial of A. What is the dimension of A?

b) Let $\mathbf{m} = (x, y, z)$ be the maximal graded ideal of A. Find an $\mathbf{m}$-primary ideal in A with the least possible number of generators.

EXERCISE 4 Let A be a Noetherian ring and $\phi : A \to A$ a ring homomorphism. Prove that if ϕ is surjective, then it is injective as well.

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EXERCISE 5

a) Let k be a field. Define a ring homomorphism $f : k[x, y] \to k[t]$ by $x \mapsto t^2$, $y \mapsto t^5$. Show that this induces an injective homomorphism $A = k[x, y]/(x^5 - y^2) \to k[t]$.

c) Prove that the integral closure of A is isomorphic to $k[t]$.

b) Let $\mathbf{q} \subset k[t]$ be a non-zero prime ideal. Show that $f^{-1}(\mathbf{q})$ is a non-zero prime ideal. Prove that $dim(A) = 1$.

END