30th October, 2018

COURSE MAT4200

Mandatory assignment

Submission deadline

Monday 19th November 2018, 14:30 at Devilry (devilry.ifi.uio.no).

Instructions

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. Students who fail the assignment, but have made a genuine effort at solving the exercises, are given a second attempt at revising their answers. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) well before the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

Problem 1. Let $A = \mathbb{C}[t^2, t^3] \subseteq \mathbb{C}[t]$ be the subring of $\mathbb{C}[t]$ generated by t^2 and t^3. For each $\alpha \in \mathbb{C}$ let I_α be the ideal $I_\alpha = (t-\alpha)\mathbb{C}[t] \cap \mathbb{C}[t^2, t^3]$. Assume that $\alpha \neq 0$.

a) Prove that I_α is a maximal ideal and that $I_\alpha = (t^2(t-\alpha), (t^2-\alpha^2))$.

a') Hint: Express $t^3(t-\alpha)$ as a combination of $t^2(t^2-\alpha^2)$ and $t^2(t-\alpha)$ and conclude that it lies in I_α. Prove that a polynomial $Q(t) = (t-\alpha)P(t)$ in $(t-\alpha)\mathbb{C}[t]$ lies in I_α if and only if $P(t)$ is on the form $P(t) = a(t+\alpha) + a_2t^2 + a_3t^3 + \ldots + a_nt^n$ (Use that the polynomials in $\mathbb{C}[t^2, t^3]$ are precisely those without a linear term).

b) Prove that I_α is not a principal ideal.

c) Prove that the different I_α together with (t^2, t^3) are all the maximal ideals in A.

d) Prove that $I_\alpha I_{-\alpha}$ is a principal ideal.

Problem 2. Let A be a principal ideal domain with field of fractions K. Show that any ring between A and K is a localization of A. That is, show that if $A \subseteq B \subseteq K$, then $B = A_S$ for an appropriate multiplicative set in A.

Problem 3. Let A be a domain with field of fractions K. For a finitely generated projective module P let the *rank* of P be $\operatorname{rk} P = \dim_K P \otimes_A K$. Let Q be another finitely generated projective A-module.

a) Prove that $P \oplus Q$ has rank $\operatorname{rk} P + \operatorname{rk} Q$.

b) Prove that $P \otimes_A Q$ and $\operatorname{Hom}_A(P, Q)$ are projective A-modules. Show that both equalities $\operatorname{rk} P \otimes_A Q = \operatorname{rk} P \cdot \operatorname{rk} Q$ and $\operatorname{rk} \operatorname{Hom}_A(P, Q) = \operatorname{rk} P \cdot \operatorname{rk} Q$ are true.

Problem 4. La A be a local ring with maximal ideal $\mathfrak{m}$ and assume that $\mathfrak{m}$ is a principal ideal; say $\mathfrak{m} = (x)$. Assume moreover that $\bigcap_i \mathfrak{m}^i = (0)$. Prove that the powers $\mathfrak{m}^n$ are the only non-zero proper ideals in A.

Problem 5. Let $\mathbb{Q}[T, X_1, X_2, \ldots]$ be the ring of polynomials with rational coefficients in T and the infinite number of variables $X_1, X_2, \ldots$. Let $\mathfrak{a}$ be the ideal generated by the elements $X_i - TX_{i+1}$ for $i \geq 1$. Let $A = \mathbb{Q}[T, X_1, X_2, \ldots]/\mathfrak{a}$, and denote by t and x_i the images of T and X_i in A.

a) Prove that $x_i = tx_{i+1}$ and that the principal ideal $\mathfrak{m} = (t)$ is a maximal ideal. Show that $\mathfrak{p} = (x_1, x_2, \ldots)$ is a prime ideal contained in $\mathfrak{m}$.

b) Show that $\bigcap_i \mathfrak{m}^i = \mathfrak{p}$.