24th October, 2023

MAT4200 — Commutative algebra

Mandatory assignment 1 of 1

Submission deadline

Thursday 12 October 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Problem 1. Let $A = \mathbb{Z}_2$, the fraction ring of $\mathbb{Z}$ with respect to the multiplicative system $\{2^i \mid i \geq 0\}$. Prove that A is a finite type $\mathbb{Z}$-algebra and is not a finite $\mathbb{Z}$-algebra.

Solution: Every element of $\mathbb{Z}_2$ is of the form $a/2^i$ for $a \in \mathbb{Z}$ and $n \geq 0$. This can be written as

$$a\left(\frac{1}{2}\right)^n \qquad a \in Z, i \geq 0.$$

This shows that $\mathbb{Z}_2$ is finitely generated (by $\frac{1}{2}$) as a $\mathbb{Z}$-algebra. Hence $\mathbb{Z}_2$ is a finite type $\mathbb{Z}$-algebra.

To show that $\mathbb{Z}_2$ is not a finite $\mathbb{Z}$-algebra, assume for a contradiction that it is. This means that we can find finitely many elements

$$\frac{b_1}{2^{n_1}}, \cdots \frac{b_k}{2^{n_k}} \in \mathbb{Z}_2$$

such that every element $a/2^n \in \mathbb{Z}_2$ can be written

$$a/2^n = \sum_{i=1}^{k} c_i \frac{b^i}{2^{n_i}}.$$

for some $c_i \in \mathbb{Z}$. Let $n = \max(n_i) + 1$, and consider the element $1/2^{n+1} \in \mathbb{Z}_2$. We can then find $c_i \in \mathbb{Z}$ such that

$$\frac{1}{2^{n+1}} = \sum_{i=1}^{k} c_i \frac{b^i}{2^{n_i}}.$$

Multiplying both sides by 2^{n+1} gives

$$1 = \sum_{i=1}^{k} c_i b_i 2^{n-n_i},$$

where $n - n_i \geq 1$ for all i. The right hand side is even and the left hand side odd, which gives a contradiction.

Problem 2. Let A be a ring, and let $\mathfrak{p}$ be a prime ideal such that there are no prime ideals $\mathfrak{q} \subsetneq \mathfrak{p}$. Prove that if $f \in A_{\mathfrak{p}}$ is not a unit, then f is nilpotent.

Solution: The prime ideals of $A_{\mathfrak{p}}$ are all of the form $\mathfrak{q}^e$ with $\mathfrak{q} \subseteq \mathfrak{p}$ a prime ideal. Since the only prime ideal contained in $\mathfrak{p}$ is $\mathfrak{p}$, it follows that the only prime ideal of $A_{\mathfrak{p}}$ is $\mathfrak{p}^e$. If $f \in A_{\mathfrak{p}}$ is not a unit, then it must lie in some maximal ideal. Since $\mathfrak{p}^e$ is the unique maximal ideal of $A_{\mathfrak{p}}$, it follows that $f \in \mathfrak{p}^e$. But then f lies in the intersection of all the prime ideals of $A_{\mathfrak{p}}$, which implies that f is nilpotent.

Problem 3. Let k be a field, and let $m, n \geq 1$ be integers.

(a) Let $f \in k[x, y]$ be given by

$$f = \sum_{i,j} a_{ij} x^i y^j \qquad a_{ij} \in k.$$

Give a condition on the coefficients a_{ij} such that $f \in (x^m, y^n)$ if and only if the condition holds.

(b) Let

$$I_1 = (x^{m_1}, y^{m_1}), I_2 = (x^{m_2}, y^{n_2})$$

with $m_1, n_1, m_2, n_2 \geq 1$. Consider the four ideals

- $I_1 + I_2$
- $I_1 I_2$
- $I_1 \cap I_2$
- $\mathfrak{r}(I_1)$ (the radical of I_1).

Write each of these in the form

$$(f_1, \ldots, f_n) \qquad f_i \in k[x, y].$$

Solution:

(a) We claim that $f \in (x^m, y^n)$ if and only if the following condition holds:

$$a_{ij} = 0 \text{ if } i < m \text{ and } j < n.$$

If the condition holds, we can write

$$\begin{aligned} f &= \sum_{i \geq m} \sum_{j \geq 0} a_{ij} x^i y^j + \sum_{i=0}^{m-1} \sum_{j \geq n} a_{ij} x^i y^j \\ &= x^m \Big(\sum_{i,j \geq 0} a_{(i+m)j} x^i y^j \Big) + y^n \sum_{i=0}^{m} \Big(\sum_{j \geq 0} a_{i(j+n)} x^i y^j \Big), \end{aligned}$$

so that $f \in (x^m, y^n)$.

On the other hand, if $f \in (x^m, y^n)$, we can find $g, h \in k[x, y]$ such that

$$f = g x^m + h y^n.$$

It is clear that the $x^i y^j$-coefficient of $g x^m$ is 0 if $i < m$, and that that $x^i y^j$-coefficient of $h y^n$ is 0 if $j < n$. Hence the $x^i y^j$-coefficient of f, which equals a_{ij}, vanishes if both $i < m$ and $j < n$ hold.

(b) As a general comment, since a presentation of ideals via a list generators is not unique, there are multiple correct answers for each of these.

$I_1 + I_2$: We have

$$I_1 + I_2 = (x^{m_1}, y^{n_1}) + (x^{m_2}, y^{n_2}) = (x^{m_1}, x^{m_2}, y^{n_1}, y^{n_2}).$$

$I_1 I_2$: We have

$$\begin{aligned} I_1 I_2 &= (x^{m_1}, y^{n_1})(x^{m_2}, y^{n_2}) \\ &= (x^{m_1}x^{m_2}, x^{m_1}y^{n_2}, x^{m_2}y^{n_1}, y^{n_1}y^{n_2}) \\ &= (x^{m_1+m_2}, x^{m_1}y^{n_2}, x^{m_2}y^{n_1}, y^{n_1+n_2}) \end{aligned}$$

$I_1 \cap I_2$: We begin by analysing the elements of $I_1 \cap I_2$. Since

$$I_1 = \{\sum a_{ij}x^i y^j \mid a_{ij} = 0 \text{ if } i < m_1 \text{ or } j < n_1\}$$

and similarly

$$I_2 = \{\sum a_{ij}x^i y^j \mid a_{ij} = 0 \text{ if } i < m_2 \text{ or } j < n_2\},$$

we see that a polynomial $f = \sum a_{ij}x^i y^j$ lies in $I_1 \cap I_2$ if the following condition **C** holds: $a_{ij} = 0$ if either "$i < m_1$ and $j < n_1$" or "$i < m_2$ and $j < n_2$".

We now claim that

$$I_1 \cap I_2 = (x^{\max(m_1,m_2)}, x^{m_1}y^{n_2}, x^{m_2}y^{n_1}, y^{\max(n_1,n_2)}).$$

To prove the inclusion $\supseteq$, note that each of the elements

$$x^{\max(m_1,m_2)}, x^{m_1}y^{n_2}, x^{m_2}y^{n_1}, y^{\max(n_1,n_2)}$$

is contained in both I_1 and I_2, and so in $I_1 \cap I_2$. It follows that the ideal which they generate is contained in $I_1 \cap I_2$.

For the inclusion $\subseteq$, let $f = \sum a_{ij}x^i y^j \in I_1 \cap I_2$. If $a_{ij} \neq 0$, this means that condition C above fails, which can happen in 4 ways:

- $i \geq m_1$ and $i \geq m_2$.
- $i \geq m_1$ and $j \geq n_2$.
- $j \geq n_1$ and $i \geq m_2$.
- $j \geq n_1$ and $j \geq n_2$.

In these 4 cases we get, respectively

- $x^i y^j \in (x^{\max(m_1,m_2)})$
- $x^i y^j \in (x^{m_1} y^{n_2})$
- $x^i y^j \in (x^{m_2} y^{n_1})$
- $x^i y^j \in (y^{\max(n_1,n_2)})$.

In each case, we find that

$$a_{ij} x^i y^j \in (x^{\max(m_1,m_2)}, x^{m_1} y^{n_2}, x^{m_2} y^{n_1}, y^{\max(n_1,n_2)}).$$

Since f is a finite sum of such terms $a_{ij} x^i y^j$, we get

$$f \in (x^{\max(m_1,m_2)}, x^{m_1} y^{n_2}, x^{m_2} y^{n_1}, y^{\max(n_1,n_2)}),$$

which concludes the proof.

$\mathfrak{r}(I_1)$: We have $x^{m_1} \in I_1$, so $x \in \mathfrak{r}(I_1)$, and similarly $y^{n_1} \in I_1$ implies $y \in \mathfrak{r}(I_1)$. Thus $(x, y) \subseteq \mathfrak{r}(I_1)$. On the other hand $1 \notin I_1$ implies $1 \notin \mathfrak{r}(I_1)$, so $\mathfrak{r}(I_1) \subsetneq (1)$. Since $k[x, y]/(x, y) \cong k$, the ideal (x, y) is maximal, and so we must have $\mathfrak{r}(I_1) = (x, y)$.

Problem 4. Let A be a ring, and let M be an A-module. Prove that for every $m \in M$ there is an injective A-module homomorphism

$$A/\mathrm{Ann}(m) \to M.$$

Solution: Define a homomorphism of A-modules $\phi\colon A \to M$ by $\phi(a) = am$. The kernel of this map is $\mathrm{Ann}(m)$, and so the fundamental homomorphism theorem says we get an isomorphism A-modules

$$A/\operatorname{Ann}(m) \cong \mathrm{im}(\phi) \subseteq M$$

Problem 5. Let $\phi\colon A \to B$ be a homomorphism of rings, and let M be an A-module. Prove that if M is flat as an A-module, then $M_B = B \otimes_A M$ is flat as a B-module.

We take the following fact as known (see Exercise 2.15 i [AM] for a more general statement).

Lemma Let M be an A-module, N be a B-module. Then there is an isomorphism of A-modules

$$\psi\colon (M \otimes_A B) \otimes_B N \to M \otimes_A (B \otimes_B N),$$

such that

$$\psi((m \otimes b) \otimes n) = m \otimes (b \otimes n).$$

Let now $\rho \colon N' \to N$ be an injective homomorphism of B-modules. Consider the following diagram, which we can easily verify has commutative squares.

$$\begin{array}{ccccc}
M_B \otimes_B N' & \xrightarrow[\cong]{\psi} & M \otimes_A (B \otimes_B N') & \xrightarrow{\cong} & M \otimes_A N' \\
\Big\downarrow{\scriptstyle id_{M_B} \otimes \rho} & & \Big\downarrow{\scriptstyle id_M \otimes (id_B \otimes \rho)} & & \Big\downarrow{\scriptstyle id_M \otimes \rho} \\
M_B \otimes_B N & \xrightarrow[\cong]{\psi} & M \otimes_A (B \otimes_B N) & \xrightarrow{\cong} & M \otimes_A N
\end{array}$$

Now since M is flat as an A-module, the rightmost vertical arrow is an injective, and since the horizontal lines are isomorphisms, it follows that the leftmost vertical arrow is injective. Since this holds for all injective homomorphisms ρ of B-modules, we have shown that M_B is flat as a B-module.

Problem 6. For a finitely generated module M over a ring A, define $r(A, M)$ as the minimal number of generators of M, i.e. the minimal n such that we can find elements $m_1, \dots, m_n \in M$ generating M as an A-module.

Let B be a local integral domain with maximal ideal $\mathfrak{m}$, and let N be a finitely generated B-module.

(a) Prove that $r(B, N) = r(B/\mathfrak{m}, N/\mathfrak{m}N)$.

(b) Prove that $r(B, N) \geq r(B_{(0)}, N_{(0)})$.

(c) Find a pair (B, N) such that $r(B, N) \neq r(B_{(0)}, N_{(0)})$.[1]

Solutions:

(a) If $m_1, \dots, m_n \in N$, generate N as a B-module, then their images generate $N/\mathfrak{m}N$ as a $B/\mathfrak{m}$-module. This shows $r(B, N) \geq r(B/\mathfrak{m}, N/\mathfrak{m}N)$. For the reverse inequality, Nakayama's lemma tells us that if the images of $m_1, \dots, m_n$ generate $N/\mathfrak{m}N$, then $m_1, \dots, m_n$ generate N.

(b) If every element of N can be written as

$$n = \sum_{i=1}^{n} b_i n_i,$$

[1] *Hint:* You can find an example for any B with $\mathfrak{m} \neq 0$, e.g. $B = \mathbb{Z}_{(2)}$.

then an element

$$\frac{n}{s}$$

can be written as

$$\frac{n}{s} = \sum_{i=1}^{n} \frac{b_i}{s} n_i.$$

(c) Let B be a local integral domain which is not a field, e.g. $B = \mathbb{Z}_{(2)}$. Let $\mathfrak{m} \subset B$ be the maximal ideal and let $N = B/\mathfrak{m}$. Then $r(B, N) = 1$. We claim that $N_{(0)} = 0$, and show this as follows. Every element of N has the form $b + \mathfrak{m}$ for some $b \in B$. Every element of $N_{(0)}$ has the form $(b + \mathfrak{m})/s$ for some $b \in B$ ans $s \in B \setminus \{0\}$. Now take $f \in \mathfrak{m} \setminus \{0\}$, and compute

$$\frac{b+\mathfrak{m}}{s} = \frac{f(b+\mathfrak{m})}{fs} = \frac{fb+\mathfrak{m}}{fs} = \frac{0}{fs} = 0.$$

Since $N_{(0)} = 0$, we have $r(B_{(0)}, N_{(0)}) = 0$.