28th September, 2023

MAT4200 — Commutative algebra

Mandatory assignment 1 of 1

Submission deadline

Thursday 12 October 2023, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

Problem 1. Let $A = \mathbb{Z}_2$, the fraction ring of $\mathbb{Z}$ with respect to the multiplicative system $\{2^i \mid i \geq 0\}$. Prove that A is a finite type $\mathbb{Z}$-algebra and is not a finite $\mathbb{Z}$-algebra.

Problem 2. Let A be a ring, and let $\mathfrak{p}$ be a prime ideal such that there are no prime ideals $\mathfrak{q} \subsetneq \mathfrak{p}$. Prove that if $f \in A_{\mathfrak{p}}$ is not a unit, then f is nilpotent.

Problem 3. Let k be a field, and let $m, n \geq 1$ be integers.

(a) Let $f \in k[x, y]$ be given by

$$f = \sum_{i,j} a_{ij} x^i y^j \qquad a_{ij} \in k.$$

Give a condition on the coefficients a_{ij} such that $f \in (x^m, y^n)$ if and only if the condition holds.

(b) Let

$$I_1 = (x^{m_1}, y^{m_1}), I_2 = (x^{m_2}, y^{n_2})$$

with $m_1, n_1, m_2, n_2 \geq 1$. Consider the four ideals

- $I_1 + I_2$
- $I_1 I_2$
- $I_1 \cap I_2$
- $\mathfrak{r}(I_1)$ (the radical of I_1).

Write each of these in the form

$$(f_1, \ldots, f_n) \qquad f_i \in k[x, y].$$

Problem 4. Let A be a ring, and let M be an A-module. Prove that for every $m \in M$ there is an injective A-module homomorphism

$$A/\mathrm{Ann}(m) \to M.$$

Problem 5. Let $\phi \colon A \to B$ be a homomorphism of rings, and let M be an A-module. Prove that if M is flat as an A-module, then $M_B = B \otimes_A M$ is flat as a B-module.

Problem 6. For a finitely generated module M over a ring A, define $r(A, M)$ as the minimal number of generators of M, i.e. the minimal n such that we can find elements $m_1, \ldots, m_n \in M$ generating M as an A-module.

Let B be a local integral domain with maximal ideal $\mathfrak{m}$, and let N be a finitely generated B-module.

(a) Prove that $r(B, N) = r(B/\mathfrak{m}, N/\mathfrak{m}N)$.

(b) Prove that $r(B, N) \geq r(B_{(0)}, N_{(0)})$.

(c) Find a pair (B, N) such that $r(B, N) \neq r(B_{(0)}, N_{(0)})$.[1]

[1] *Hint:* You can find an example for any B with $\mathfrak{m} \neq 0$, e.g. $B = \mathbb{Z}_{(2)}$.