Lecture 10 – Tensor product of algebras

Definition. Let B and C be A-algebras. The A-module $B \otimes_A C$ is an A-algebra, with multiplication defined by

$$(b \otimes c)(b' \otimes c') = bb' \otimes cc',$$

and more generally by

$$(\sum_i b_i \otimes c_i)(\sum_j b'_j \otimes c'_j) = \sum_{i,j} b_i b'_j \otimes c_i c'_j.$$

Remark. The unit element in the ring $B \otimes_A C$ is $1 \otimes 1$.

Remark. It is not obvious that this multiplication is well-defined. One way to see this is to observe that any time we rewrite the sums by using the relations

$$b \otimes (c + c') = b \otimes c + b \otimes c',$$

and

$$ab \otimes c = b \otimes ac,$$

the expression on the right hand side can also be rewritten using these relations.

The other way is to use the defining property of the tensor product, see the textbook for details.

Example. Let A be a ring, consider $B = A[x]$ and $C = A[y]$. Then, as A-modules, we have isomorphisms

$$B = \bigoplus_{i \geq 0} Ax^i, \qquad C = \bigoplus_{i \geq 0} Ay^i,$$

or in words, B (resp. C) is the free A-module generated by $1, x, x^2 \ldots$ (resp. by $1, y, y^2, \ldots$). We then find a module isomorphism

$$B \otimes_A C \cong (\bigoplus_{i \geq 0} Ax^i) \otimes (\bigoplus_{j \geq 0} Ay^j) = \bigoplus_{i,j \geq 0} Ax^i \otimes y^j.$$

For the multiplication, we have

$$(x^{i_1} \otimes y^{j_1})(x^{i_2} \otimes y^{j_2}) = (x^{i_1} x^{i_2} \otimes y^{j_1} y^{j_2}) = x^{i_1+i_2} \otimes y^{j_1+j_2}.$$

From this we see that there is an isomorphism of A-algebras $B \otimes_A C \to A[x, y]$, given by $\sum a_{ij} x^i \otimes y^j \mapsto \sum a_{ij} x^i y^j$.

Example. Consider $\mathbb{C}$ as an $\mathbb{R}$-algebra. We have a natural basis $1, i \in \mathbb{C}$ as an $\mathbb{R}$-module, so $\mathbb{C} \cong \mathbb{R}1 \oplus \mathbb{R}i$. We have

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = (\mathbb{R}1 \oplus \mathbb{R}i) \otimes (\mathbb{R}1 \oplus \mathbb{R}i) = (\mathbb{R}1 \otimes 1 \oplus \mathbb{R}i \otimes 1 \oplus \mathbb{R}1 \otimes i \oplus \mathbb{R}i \otimes i).$$

In other words $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is an $\mathbb{R}$-vector space with basis $1 \otimes 1, 1 \otimes i, i \otimes 1, i \otimes i$. The multiplication table is easy to write down in this basis, if $x = 1 \otimes i$, $y = i \otimes 1$, $z = i \otimes i$, then we have

$$x^2 = y^2 = z^2 = -1$$

and

$$xy = z, yz = -x, zx = -y.$$

6.4. **Rings of fractions.** Recall from a previous course (hopefully), that given any integral domain A, we can produce a field $K(A)$, whose elements are written a/b, with $a, b \in A$, the **fraction field** of A.

Example. We have $K(\mathbb{Z}) = \mathbb{Q}$ and, if k is a field, $K(k[x]) = k(x)$, the field of rational functions.

Formally, the field $K(A)$ is constructed as follows

(1) Consider all pairs $(a, b) \in A \times A$ with $b \neq 0$.
(2) Declare that $(a, b) \simeq (a', b')$ if
$$ab' - ba' = 0$$
(3) Check that this is an equivalence relation.
(4) Let $k(A)$ be the set of equivalence classes of pairs $(a, b) \in A \times A$.
(5) Define addition and multiplication, check all's well defined and gives a field.

Example. It is essential that A is an integral domain, otherwise $\sim$ is not an equivalence relation. E.g. in $\mathbb{Z}/(4)$, we find
$$(2, 2) \sim (1, 1) \qquad 2 \cdot 1 - 1 \cdot 2 = 0$$
and
$$(2, 2) \sim (0, 2) \qquad 2 \cdot 2 - 0 \cdot 2 = 0$$
but
$$(1, 1) \not\sim (0, 2) \qquad 2 \cdot 1 - 1 \cdot 0 = 2 \neq 0.$$

We may think of the construction as follows: Given an integral domain A, we build a "smallest" ring $K(A)$ from A such that every non-zero element in A has an inverse.

Generalised question: Given a ring A, and $S \subseteq A$, construct a new ring $S^{-1}A$ where elements of s have inverses.

Note that, if $s_1, s_2 \in S$, and s_1 and s_2 has inverses, then also $s_1 s_2$ must have an inverse. The element $1 \in A$ already has an inverse, so we can always add it in to S. We will therefore assume S is **multiplicatively closed**, meaning $1 \in S$ and $s_1, s_2 \in S \Rightarrow s_1 s_2 \in S$.

Definition. Given a ring A and a multiplicatively closed subset $S \subseteq A$, define the **ring of fractions of** A with respect to S as follows.

- The set $S^{-1}A$ are equivalence classes of pairs
$$(a, s) \qquad a \in A, s \in S,$$
under the equivalence relation that
$$(a, s) \sim (b, t)$$
if and only if there exists a $u \in S$ such that
$$(at - sb)u = 0.$$
- Addition and multiplication is defined by
$$(a, s) + (b, t) = (at + bs, st)$$
$$(a, s)(b, t) = (ab, st).$$

Remark. The proof that this is a ring is almost exactly the same as the construction of the fraction field of an integral domain.

Remark. We will always write a/s instead of (a, s). The formulas for addition and multiplication of these fractions are the same as the usual ones. The only thing that is harder is the new criterion for when two fractions are equal, i.e.

$$\frac{a}{s} = \frac{b}{t} \Leftrightarrow (at - bs)u = 0 \text{ for some } u \in S.$$

Remark. The unit element in $S^{-1}A$ is $1/1$, and the zero element is $0/1$.

Remark. There is a ring homomorphism $\phi\colon A \to S^{-1}A$ defined by $\phi(a) = \frac{a}{1}$.

Proposition. *Let $\psi\colon A \to B$ be a ring homomorphism such that every $s \in S$, we have $\psi(s)$ is a unit in B. Then there exists a unique homomorphism $\rho\colon S^{-1}A \to B$ such that $\psi = \rho \circ \phi$.*

Proof. *Uniqueness:* Let $a \in A$. We must have

$$\rho(a/1) = \rho(\phi(a)) = \psi(a).$$

Let $s \in S$. We must have

$$\rho(1/s) = \rho((s/1)^{-1}) = \rho(\phi(s)^{-1}) = (\rho(\phi(s)))^{-1} = \psi(s)^{-1}.$$

Then

$$\rho(a/s) = \rho(a/1)\rho(1/s) = \psi(a)\psi(s)^{-1}.$$

Existence: Define $\rho(a/s) = \psi(a)\psi(s)^{-1}$, and check that this is well-defined. □

Example. Let A be an integral domain, and let $S = A \setminus \{0\}$. Then $S^{-1}A = K(A)$. To see this, observe that the equivalence relation

$$(a, s) \sim (b, t) \Leftrightarrow (as - bt)u = 0 \text{ for some } u \in S$$

used in the construction of $S^{-1}A$, and

$$(a, s) \sim (b, t) \Leftrightarrow as - bt = 0$$

used in the construction of $K(A)$, are in fact the same ones, since A is an integral domain.

Example. Let A be a ring $S \subseteq A$ multiplicatively closed, and assume that $0 \in S$. Then we have, for all $a, b \in A$, $s, t \in S$, that

$$\frac{a}{s} = \frac{b}{t},$$

since

$$(ta - bs)0 = 0.$$

Thus $S^{-1}A$ has one element and is the zero ring.

In particular, the homomorphism $\phi\colon A \to S^{-1}A$ is not necessarily injective.

Example. Let $f \in A$, and let $S = \{f^n \mid n \geq 0\} \subseteq A$. We then write $A_f = S^{-1}A$, the ring A with f inverted.

Example. Let $\mathfrak{p} \subset A$ be a prime ideal. Let $S = A \setminus \mathfrak{p}$. Then S is multiplicatively closed, and we write $A_{\mathfrak{p}} = S^{-1}A$.

Proposition. *The ring $A_{\mathfrak{p}}$ is local, with maximal ideal*

$$\mathfrak{m} = \{a/s \mid a \in \mathfrak{p}, s \notin \mathfrak{p}\}.$$

Proof. Recall a ring A is local if and only if its non-units form an ideal, and the set of non-units are then the maximal ideal of the ring.

One checks the set above $\mathfrak{m}$ is an ideal. It does not contain $1/1$, since

$$1/1 = a/s \Rightarrow (s-a)u = 0 \text{ for some u} \notin \mathfrak{p},$$

If $a \in \mathfrak{p}$ and $s \notin \mathfrak{p}$, then $(s-a) \notin \mathfrak{p}$, and so $(s-a)u \notin \mathfrak{p}$, and in particular $(s-a)u \neq 0$. Therefore the elements of $\mathfrak{m}$ are non-units.

Assume next that $a/s \notin \mathfrak{m}$. Then $a \notin \mathfrak{p}$, so a/s has inverse s/a, and hence a/s is a unit. Thus $\mathfrak{m}$ is precisely the set of non-units in $A_\mathfrak{p}$, so $A_\mathfrak{p}$ is local with maximal ideal $\mathfrak{m}$. □

Example. Let $\mathfrak{p} = (p) \subseteq \mathbb{Z}$ for some prime number p. Then

$$\mathbb{Z}_{(p)} = \{m/n \mid p \text{ does not divide } n\},$$

and the maximal ideal is

$$\mathfrak{m} = \{pm/n \mid p \text{ does not divide } n\}.$$

Example. Take $\mathfrak{p} = (x) \subset \mathbb{R}[x]$. Then

$$\mathbb{R}[x]_{(x)} = \{f/g \mid g \notin (x)\},$$

the local ring from the lecture on Nakayama's lemma.