Lecture 11 – Modules on rings of fractions

Let A be a ring and let $S \subseteq A$ be a multiplicatively closed subset.

Definition. If M is an A-module, then define an $S^{-1}A$-module $S^{-1}M$ as the set of equivalence classes of pairs

$$(m, s) \qquad m \in M, s \in S,$$

under the equivalence relation

$$(m, s) \sim (n, q) \text{ if } \exists u \in S \mid (qm - sn)u = 0.$$

The module structure is

$$(m, s) + (n, q) = (qm + sn, sq)$$

and, if $a \in A$

$$(a, s)(m, q) = (am, sq).$$

Remark. We always write m/s instead of (m, s).

Example. Take $A = \mathbb{Z}$, $S = \mathbb{Z} \setminus \{0\}$, so that $S^{-1}\mathbb{Z} = \mathbb{Q}$.
Let $M = \mathbb{Z}$, then $S^{-1}M = \mathbb{Q}$.
Let $M = \mathbb{Z}/n$, then we have the following equalities in $S^{-1}M$:

$$a/s = na/ns = 0/s = 0,$$

so $S^{-1}M = 0$.

Remark. As special notational cases:

- If $S = A \setminus \mathfrak{p}$ for some prime ideal $\mathfrak{p}$, so that $S^{-1}A = A_\mathfrak{p}$, we write $M_\mathfrak{p}$ for $S^{-1}M$.
- If $S = \{1, f, f^2, \cdots$, so that $S^{-1}A = A_f$, we write M_f for $S^{-1}M$.

Remark. Given a homomorphism of A-modules $\phi\colon M \to N$, there is an induced homomorphism of $S^{-1}A$-modules $S^{-1}\phi\colon S^{-1}M \to S^{-1}N$ given by

$$S^{-1}\phi(m/s) = \phi(m)/s.$$

Proposition. *The $S^{-1}A$-module $S^{-1}M$ is isomorphic to $M \otimes_A S^{-1}A$, under the homomorphism ϕ given by*

$$a/s \otimes m \mapsto am/s$$

Proof. To see that ϕ is well-defined, check (which is easy) that $(a/s, m) \mapsto am/s$ is A-bilinear. The map ϕ is clearly surjective since for any $m/s \in S^{-1}M$ we have $\phi(1/s \otimes m) = m/s$. To see that it is injective, note that there is a well-defined inverse

$$\psi\colon S^{-1}M \to S^{-1}A \otimes_A M,$$

given by

$$\psi(m/s) = 1/s \otimes m.$$

Here if $m/s = n/q$, we have an $u \in S$ such that $u(qm - sn) = 0$, which implies that

$$1/s \otimes m = 1/uqs \otimes uqm = 1/uqs \otimes usn = 1/q \otimes n,$$

so ψ is well-defined. One checks that it is a homomorphism of $S^{-1}A$-modules, and that $\psi \circ \phi$ and $\phi \circ \psi$ are both the identities. □

Example. In the examples above, $S^{-1}M = \mathbb{Q} \otimes_{\mathbb{Z}} M$, so our computations recover previous computations of these tensor products.

Proposition. *If $M' \to M \to M''$ is exact, then so is $S^{-1}M' \to S^{-1}M \to S^{-1}M''$.*

Proof. Let m/s be such that $S^{-1}\psi(m/s) = 0$. We must show that there exists an m'/s' such that $S^{-1}\phi(m'/s') = m/s$.

We have
$$S^{-1}\psi(m/s) = \psi(m)/s = 0/1,$$
which by definition means that there is an $u \in S$ such that $u\psi(m) = 0$.

Then $\psi(um) = 0$, and exactness implies there is an n such that $\phi(n) = um$. It follows that
$$S^{-1}\phi(n/su) = \phi(n)/su = um/su = m/s.$$
□

Corollary. *The ring $S^{-1}A$ is flat as an A-module.*

Proof. The operation $S^{-1}-$ preserves exactness, and so preserves injections, hence the operation $S^{-1} \otimes_A -$ preserives injections, which by definition means that $S^{-1}A$ is flat. □

Example. For any integral domain A, $K(A)$ is flat as an A-module.

Proposition (The operation $M \mapsto S^{-1}M$ commutes with everything)**.** *Let M be an A-module.*

- *If M' is an A-submodule of M, then $S^{-1}M'$ is an $S^{-1}A$-submodule of $S^{-1}M$, and we have $S^{-1}M/S^{-1}M' \cong S^{-1}(M/M')$.*
- *If $M', M'' \subset M$, then $S^{-1}(M' + M'') = S^{-1}M' + S^{-1}M''$, and $S^{-1}M' \cap S^{-1}M'' = S^{-1}(M' \cap M'')$.*
- *If N is an A-module, then*
$$S^{-1}M \otimes_{S^{-1}A} S^{-1}N \cong S^{-1}(M \otimes_A N).$$

Proof. Let's only prove the last one. We use the $S^{-1}A \otimes_A M = S^{-1}M$ to rewrite the left hand side as
$$\begin{aligned} S^{-1}M \otimes_{S^{-1}A} S^{-1}N &\cong M \otimes_A S^{-1}A \otimes_{S^{-1}A} S^{-1}A \otimes_A N \\ &\cong M \otimes_A S^{-1}A \otimes_A N \cong S^{-1}A \otimes_A M \otimes_A N \cong S^{-1}(M \otimes_A N). \end{aligned}$$
□

Remark. All of the above hold more generally for the operation $M \mapsto M \otimes_A B$ whenever B is a flat A-algebra.

6.5. **Local properties.** Let P be a property of module. We say (somewhat informally) that the property P is **local** if

$$P \text{ holds for } M$$
$$\Updownarrow$$
$$P \text{ holds for all localisations } M_{\mathfrak{p}}.$$

Proposition ("Being 0 is local")**.** *Let M be an A-module. The following are equivalent*

(1) $M = 0$

(2) $M_{\mathfrak{p}} = 0$ *for all prime ideals* $\mathfrak{p}$
(3) $M_{\mathfrak{m}} = 0$ *for all maximal ideals* $\mathfrak{m}$*.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) are obvious.

To prove (3) $\Rightarrow$ (1), assume $M \neq 0$, and for a contradiction that $M_{\mathfrak{m}} = 0$ for all maximal ideals $\mathfrak{m}$. Let $0 \neq m \in M$. Then

$$\operatorname{Ann}(m) = \{x \in A \mid xm = 0\} \subset A$$

is an ideal, and $\operatorname{Ann}(m) \neq (1)$. Hence there is a maximal ideal $\mathfrak{m} \supseteq \operatorname{Ann}(m)$. Now, since $M_{\mathfrak{m}} = 0$, we have

$$\frac{m}{1} = 0 \Leftrightarrow \exists u \in A \setminus \mathfrak{m} \mid um = 0,$$

but $\operatorname{Ann}(m) \subseteq \mathfrak{m}$, so this is a contradiction. ☐

Proposition. *Let* $\phi\colon M \to N$ *be a homomorphism of* A*-modules. Then the following are equivalent:*

(1) ϕ *is injective.*
(2) *For all prime ideals* $\mathfrak{p}$*, the map* $\phi_{\mathfrak{p}}\colon M_{\mathfrak{p}} \to N_{\mathfrak{p}}$ *is injective.*
(3) *For all maximal ideals* $\mathfrak{m}$*, the map* $\phi_{\mathfrak{p}}\colon M_{\mathfrak{m}} \to N_{\mathfrak{m}}$ *is injective.*

Proof. The sequence $0 \to \ker\phi \to M \to N$ is exact. Since localisation is exact, we have for every prime $\mathfrak{p}$ that

$$0 \to (\ker\phi)_{\mathfrak{p}} \to M_{\mathfrak{p}} \xrightarrow{\phi_{\mathfrak{p}}} N_{\mathfrak{p}}$$

is exact. But that implies $(\ker\phi)_{\mathfrak{p}} \cong \ker\phi_{\mathfrak{p}}$.

We have ϕ injective if and only if $\ker\phi = 0$. By the above, we have $\phi_{\mathfrak{p}}$ injective if and only if $(\ker\phi)_{\mathfrak{p}} = 0$. Combining with the previous proposition gives what we want. ☐

Remark. The same result holds with "injective" replaced by "surjective" throughout.

Proposition. *Being flat is a local property.*