7. Lecture 12 – Extension and contraction of ideals in the ring of fractions

Recall that for a quotient ring A/I, we have a bijection between the set of ideals of A/I and the set of ideals of A containing I. If $\phi\colon A \to A/I$, and superscripts e and c denote extension and contraction along ϕ, the correspondence is given by

$$\mathfrak{a} \subseteq A \mapsto \mathfrak{a}^e = \mathfrak{a}/I \subseteq A/I$$

and

$$\mathfrak{a} \subseteq A/I \mapsto \mathfrak{a}^c = \phi^{-1}(\mathfrak{a}) \subseteq A.$$

A similar, but more complicated story, holds for the fractions rings $S^{-1}A$. Recall that given a multiplicatively closed subset $S \subseteq A$, we have a homomorphism $\phi\colon A \to S^{-1}A$, and we let now let superscripts c and e denote contraction and extension along this homomorphism. We begin by analysing concretely what these operations do.

Lemma. *Let $\mathfrak{a} \subset A$ be an ideal. Then*

$$\mathfrak{a}^e = \{\frac{a}{s} \mid a \in \mathfrak{a}\}$$

Proof. By definition, $\mathfrak{a}^e = \{\sum b_i/s_i\phi(a_i) \mid b_i \in A, s_i \in S, a_i \in \mathfrak{a}\}$, where we look at all finite sums. The inclusion $\supseteq$ is then clear. Conversely, $\mathfrak{a}^e$ is generated by elements of the form

$$\frac{b}{s}\phi(a) = \frac{ba}{s},$$

with $a \in \mathfrak{a}$. Then $ba \in \mathfrak{a}$, which gives the inclusion $\subseteq$. □

Lemma. *Let $\mathfrak{a} \subseteq S^{-1}A$ be an ideal. Then $\mathfrak{a}^c = \{a \in A \mid a/1 \in \mathfrak{a}\}$.*

Proof. $a \in \mathfrak{a}^c \Leftrightarrow \phi(a) \in \mathfrak{a}$ by definition, and $\phi(a) = a/1$. □

Proposition. *For any ideal $\mathfrak{a} \subset S^{-1}A$, we have $\mathfrak{a}^{ce} = \mathfrak{a}$.*

Proof. Using the lemmas above, we find

$$a/s \in \mathfrak{a} \Rightarrow (s/1)(a/s) = a/1 \in \mathfrak{a} \Rightarrow a \in \mathfrak{a}^c \Rightarrow a/s \in \mathfrak{a}^{ce},$$

and

$$a/s \in \mathfrak{a}^{ce} \Rightarrow a/s = b/q, b \in \mathfrak{a}^c \Rightarrow b/1 \in \mathfrak{a} \to (1/q)(b/1) = b/q = a/s \in \mathfrak{a}.$$

□

Corollary. *The operation $\mathfrak{a} \to \mathfrak{a}^c$ gives an inclusion of the set of ideals of $S^{-1}A$ into the set of ideals of A.*

Proof. If $\mathfrak{a}^c = \mathfrak{b}^c$, then $\mathfrak{a} = \mathfrak{a}^{ce} = \mathfrak{b}^{ce} = \mathfrak{b}$. □

Corollary. *Every ideal of $S^{-1}A$ is of the form $\mathfrak{a}^e$ for some $\mathfrak{a} \subseteq A$.*

Example. The ideals of $\mathbb{Z}_{(2)}$ are all extensions of ideals from $\mathbb{Z}$. Every ideal of $\mathbb{Z}$ is of the form (n) for some $n \geq 0$, and we have

$$(n)^e = \{\frac{nk}{s} \mid k \in \mathbb{Z}, s \notin (2)\} = (n/1)$$

If $n = 2^f q$ with q odd, then $2^f/1 = n/q \in (n/1)$, so $(2^f/1) \subseteq (n/1)$, while $n/1 = (2^f/1)(q/1) \in (2^f/1)$, so $(n/1) \subseteq (2^f/1)$. It follows that $(n/1) = (2^f/1)$. Thus the complete set of ideals in $\mathbb{Z}_{(2)}$ is

$$(1) \supsetneq (2/1) \supsetneq (2^2/1) \supsetneq \cdots$$

and (0).

Let's focus now on the case of prime ideals.

Proposition. *The operations of extension and contraction give a bijection between the prime ideals of $S^{-1}A$ and the prime ideals $\mathfrak{p}$ of A such that $\mathfrak{p} \cap S = \varnothing$.*

Proof. Let $\mathfrak{p} \subset S^{-1}A$ be a prime ideal. Then $\mathfrak{p}^c \subset A$ is a prime ideal (*Contraction always preserves prime ideals*). Moreover, $\mathfrak{p}^c \cap S = \varnothing$, since if $a \in \mathfrak{p}^c \cap S$, then

$$a/1 \in \mathfrak{p} \Rightarrow a/a = 1 \in \mathfrak{p} \Rightarrow \mathfrak{p} = (1),$$

contradicting primality of $\mathfrak{p}$.

If $\mathfrak{q} \subset A$ is a prime ideal with $\mathfrak{q} \cap S = \varnothing$, then we claim (1) $\mathfrak{q}^e$ is a prime ideal, and (2) $\mathfrak{q}^{ec} = \mathfrak{q}$.

For (1), if $\mathfrak{q}^e$ is not a prime ideal, we can find $a/s, b/q$ such that $a, b \notin \mathfrak{q}$ with

$$\frac{ab}{qs} = \frac{c}{t},$$

with $c \in \mathfrak{q}$. That implies $(abt - qsc)u = 0$ for some $u \in S$, but $a, b, t \notin \mathfrak{q}, c \in \mathfrak{q}$ and $u \notin \mathfrak{q}$ makes this impossible, since $\mathfrak{q}$ is prime.

For (2), $\mathfrak{q}^{ec} \supseteq \mathfrak{q}$ always holds, so we only need to prove that $\mathfrak{q}^{ec} \subseteq \mathfrak{q}$. An element of $\mathfrak{q}^{ec}$ is an $a \in A$ such that

$$a/1 = b/s$$

with $b \in \mathfrak{q}$ and $s \in S$. This implies that $(as - b)u = 0$ for some $u \in S$, which since $b \in \mathfrak{q}$, $s, u \notin \mathfrak{q}$, can only happen if $a \in \mathfrak{q}$.

We have now shown that the operations $(-)^e$ and $(-)^c$ gives maps between the sets

$$\{\text{prime ideals in } A \text{ not intersecting } S\}$$

and

$$\{\text{prime ideals in } S^{-1}A\},$$

such that $\mathfrak{p}^{ec} = \mathfrak{p}$ and $\mathfrak{q}^{ce} = \mathfrak{q}$, which means that these maps are bijections. □

Corollary. *Let $\mathfrak{p} \subseteq A$ be a prime ideal. The prime ideals of $A_\mathfrak{p}$ are precisely the ideals $\mathfrak{q}^e$, where $\mathfrak{q} \subseteq \mathfrak{p}$ is a prime ideal.*

Proof. The condition $\mathfrak{q} \cap S \setminus \mathfrak{p}$ is equivalent to $\mathfrak{q} \subseteq \mathfrak{p}$. □

Example. The prime ideals of $\mathbb{Z}_{(2)}$ are exactly $(2)^e = (2/1)$ and $(0)^e = (0/1)$.

Note that the bijection proposition above fails here for non-prime ideals, i.e. $(6) \subset (2) \subset \mathbb{Z}$, but $(6)^e = (2)^e = (2/1)$, and there is no ideal $I \subseteq \mathbb{Z}_{(2)}$ such that $I^c = (6)$.

Corollary. *Let $f \in A$. The prime ideals of A_f are precisely the ideals $\mathfrak{q}^e$, where $\mathfrak{q} \subset A$ is a prime ideal not containing f.*

Proof. Since $A_f = S^{-1}A$ with $S = \{f^k\}_{k \geq 0}$, the prime ideals of A_f are the $\mathfrak{q}^c$ which don't intersect S. Now $f \in \mathfrak{q} \Leftrightarrow f^k$ for some $k \geq 0$ by primality of $\mathfrak{q}$, so this is the set of prime ideals in A which don't contain f. □

Example. The prime ideals of $\mathbb{Z}_2$ are $(3/1), (5/1), (7/1), (11/1), \ldots$.

We can give better proofs of a few things we've seen before.

Corollary. *The ring $A_\mathfrak{p}$ is local.*

Proof. Every prime ideal $\mathfrak{q} \subset A_\mathfrak{p}$ is contained in $\mathfrak{p}^e$. □

Proposition. *The nilradical of A is the intersection of all the prime ideals of A.*

Proof. If f is nilpotent, then it must lie in every prime ideal. The hard part is to see that if f lies in every prime ideal, then f is nilpotent. Consider the ring A_f. A ring without prime ideals must be the zero ring, and $S^{-1}A$ is the zero ring if and only if $0 \in S$, which is if and only if f is nilpotent. □

Geometric interlude

Let k be an algebraically closed field, and for concreteness we may as well take $k = \mathbb{C}$. We are interested in the ring $k[x_1, \ldots, x_n]$, and want to know what its maximal ideals are. There is a very natural source of such maximal ideals: Let $(a_1, \ldots, a_n) \in k^n$, and let

$$\phi_{(a_1,\ldots,a_n)} \colon k[x_1, \ldots, x_n] \to k$$

be given by

$$\phi_{(a_1,\ldots,a_n)}(f) = f(a_1, \ldots, a_n).$$

This homomorphism is surjective onto k, so $k = \operatorname{im} \phi = k[x_1, \ldots, x_n]/\ker \phi$, which means that $\ker \phi$ is maximal. It's easy to check that $\ker \phi = (x_1 - a_1, \ldots, x_n - a_n)$.

We will see later the following theorem:

Theorem (Nullstellensatz (special case))**.** *The maximal ideals of the ring $k[x_1, \ldots, x_n]$ are precisely the ideals*

$$(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n),$$

where $(a_1, a_2, \ldots, a_n) \in k^n$.

We are in principle interested in subsets of k^n defined as the zero sets of polynomials $f_1, f_2, \ldots, f_m \in k[x_1, \ldots, x_n]$. We write

$$V(f_1, \ldots, f_m) = \{(a_1, \ldots, a_n) \in k^n \mid f_1(a_1, \ldots, a_n) = \cdots = f_m(a_1, \ldots, a_n) = 0\}.$$

A set $V \subseteq k^n$ which can be expressed in this form is called **algebraic**.

Example. The set $\{(a_1, a_2) \mid a_1^2 + a_2^2 = 1\} \subseteq k^2$ is an algebraic subset (which if $k = \mathbb{R}$ is of course a circle).

Lemma. *Let $f \in k[x_1, \ldots, x_n]$. Then $f(a_1, \ldots, a_n) = 0$ if and only if $f \in (x_1 - a_1, \ldots, x_n - a_n)$.*

Proof. $f(a_i) = 0 \Leftrightarrow f \in \ker \phi_{(a_i)} \Leftrightarrow f \in (x_i - a_i)$. □

Lemma. *Given $f_1, \ldots, f_n \in k[x_1, \ldots, x_n]$, we have that $(a_1, \ldots, a_n) \subseteq V(f_1, \ldots, f_n)$ if and only if $(f_1, \ldots, f_n) \subseteq (x_1 - a_1, \ldots, x_n - a_n)$.*

Proof. By definition, we have $V(f_1, \ldots, f_n) = V(f_1) \cap V(f_2) \cap \cdots V(f_n)$. □

Corollary. *The set $V(f_1, \ldots, f_n)$ are in natural bijection with the maximal ideals of the ring $k[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$.*

Proof. The maximal ideals of the quotient ring are in bijection with the maximal ideals of $k[x_1, \ldots, x_n]$ containing $(f_1, \ldots, f_n)$, which is in bijection with $V(f_1, \ldots, f_n)$. □

Example. To find the maximal ideals of the ring $A = k[x, y]/(x^2 + y^2 - 1, x)$, we simply solve the set of equation

$$x^2 + y^2 - 1 = 0$$
$$x = 0,$$

giving $(x, y) = (0, 1)$ and $(x, y) = (0, -1)$. This means that A has two maximal ideals. Letting $\bar{x} = x + (x^2 + y^2 - 1), \bar{y} = y + (x^2 + y^2 - 1, x) \in A$, the maximal ideals of A are

$$(\bar{x}, \bar{y} - 1) \text{ and } (\bar{x}, \bar{y} + 1).$$

Motivated by this, for any ideal $I \subseteq k[x_1, \ldots, x_n]$, we define $V(I) \subseteq k^n$ to be the subset $(a_1, \ldots, a_n)$ such that $I \subseteq (x_1 - a_1, \ldots, x_n - a_n)$. Sets of the form $V(I)$ are called **algebraic subsets** of k^n.

Proposition. *The operation $I \mapsto V(I)$ satisfies the following properties*

(1) $I \subseteq J \Rightarrow V(J) \subseteq V(I)$.
(2) $V(0) = k^n$
(3) $V(1) = \varnothing$.
(4) $V(I + J) = V(I) \cap V(J)$.
(5) $V(IJ) = V(I \cap J) = V(I) \cup V(J)$.
(6) $V(\mathfrak{r}(I)) = V(I)$.

Proof. (1), (2) and (3) are obvious.

(4): A maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$ contains I and J if and only if it contains $I + J$.

(5): From $IJ \subseteq I \cap J$ and (1) follows $V(IJ) \supseteq V(I \cap J)$. If a maximal ideal contains I or J, then it contains $I \cap J$, which gives $V(I \cap J) \supseteq V(I) \cap V(J)$. If a maximal ideal $\mathfrak{m}$ contains I, J, then it contains either I or J, since otherwise we can find $f \in I \setminus \mathfrak{m}$, $g \in J \setminus \mathfrak{m}$, from which we get $fg \in IJ \setminus \mathfrak{m}$, since $\mathfrak{m}$ is prime. It follows that $V(I) \cap V(J) \supseteq V(IJ)$.

(6): From (1) and $I \subseteq \mathfrak{r}(I)$ we get $V(\mathfrak{r}(I)) \subseteq V(I)$. If a maximal ideal $\mathfrak{m}$ contains I, then it also contains $\mathfrak{r}(I)$, since $f^n \in I \Rightarrow f^n \in \mathfrak{m} \Rightarrow f \in \mathfrak{m}$. Hence $V(\mathfrak{r}(I)) \supseteq V(I)$. □

Let now $V = V(I) \subseteq k^n$ be an algebraic subset. We say that V is **irreducible** if there is no way to write $V(I)$ as the union of two strictly smaller algebraic subsets.

Example. For every maximal $\mathfrak{m}$, we have $V(\mathfrak{m})$, so points are irreducible.

Proposition. *If $\mathfrak{p} \subset k[x_1, \ldots, x_n]$ is a prime ideal, then $V(\mathfrak{p})$ is an irreducible subset.*

Moreover, every irreducible algebraic subset of k^n is of the form $V(\mathfrak{p})$ for some prime ideal $\mathfrak{p}$.

Proof. These claims rely on the Nullstellensatz, which we don't know yet, so we won't prove this. □

Example. For every irreducible $f \in k[x_1, \ldots, x_n]$, the set

$$V(f) = \{(a_1, \ldots, a_n) \mid f(a_1, \ldots, a_n) = 0\}$$

is irreducible, since (f) is a prime ideal.

Example. The fact that $xy \in k[x, y]$ is not irreducible is equivalent to (xy) not being a prime ideal, which is equivalent to $V(xy)$ not being irreducible. Concretely, $V(xy) = V(x) \cup V(y)$ shows that $V(xy)$ is not irreducible.