Lecture 13 – Primary ideals

Some rings, such as $\mathbb{Z}$ and $k[x_1, \ldots, x_n]$ where k is a field, are **unique factorisation domains**, meaning every element x can be factored uniquely (up to reordering and multiplication by units) as a product

$$x = p_1 p_2 \cdots p_k$$

with p_i irreducible ring elements.

For most rings A, this is not the case:

Example.
- In $\mathbb{Z}[i\sqrt{5}] = \mathbb{Z}[x]/(x^2+5)$, we have $6 = 2 \cdot 3 = (1+i\sqrt{5})(1-\sqrt{5})$, where $2, 3, 1+i\sqrt{5}$ and $1-i\sqrt{5}$ are all irreducible.
- Let k be a field. In the ring $A = k[x,y]/(y^2 - x^3 + x)$, we have

  $$y^2 = y \cdot y = x(x-1)(x+1),$$

  and y, x, $(x-1)$, $(x+1)$ are all irreducible.

The best we can hope for in general is some kind of factorisation of *ideals*. A reasonable way to formalise this turns out to be that of factoring into **primary ideals**.

Definition. An ideal $\mathfrak{a} \subseteq A$ is primary if $\mathfrak{a} \neq (1)$, and $fg \in \mathfrak{a}$ implies either $f \in \mathfrak{a}$ or $g^n \in \mathfrak{a}$ for some $n \geq 1$.

Remark. An equivalent formulation is: An ideal $\mathfrak{a}$ is primary if $A/\mathfrak{a} \neq 0$, and every 0-divisor in $A/\mathfrak{a}$ is nilpotent.

Example. Every prime ideal is primary.

Example. In $\mathbb{Z}$, the primary ideals are (0) and (p^i) for p a prime.

Example. If a ring A has unique factorisation, then (x^n) is primary for any irreducible x. Here $fg \in (x^n)$ is equivalent to "x^n divides fg". If x divides g, then $g^n \in (x^n)$, and if not, we must have that x^n divides f.

Proposition. *If $\mathfrak{a}$ is primary, then $\sqrt{\mathfrak{a}}$ is prime.*

Proof. Assume $fg \in \sqrt{\mathfrak{a}}$. Then there's some n such that $f^n g^n \in \mathfrak{a}$, so that either $f^n \in \mathfrak{a}$, or there's some m such that $(g^n)^m \in \mathfrak{a}$. In either case one of f and g must lie in $\sqrt{\mathfrak{a}}$, so $\sqrt{\mathfrak{a}}$ is prime. □

We say that $\mathfrak{a}$ is $\mathfrak{p}$-primary if $\sqrt{\mathfrak{a}} = \mathfrak{p}$.

Remark. Let $\mathfrak{a} = (xy, y^2)$. Then $\sqrt{\mathfrak{a}} = (y)$, which is prime, but $\mathfrak{a}$ is not primary, since $yx \in \mathfrak{a}$, but $x \notin \sqrt{\mathfrak{a}}$. So being primary is a stronger condition than having prime radical.

Proposition. *If $\mathfrak{m} \subset A$ is maximal, and $\sqrt{\mathfrak{a}} = \mathfrak{m}$, then $\mathfrak{a}$ is primary.*

Proof. The ring $A/\mathfrak{a}$ has the property that the nilradical $\sqrt{(0)} = \mathfrak{m}/\mathfrak{a}$ is a maximal ideal. Since the nilradical is the intersection of all prime ideals, we must have that $\mathfrak{m}/\mathfrak{a}$ is the unique prime ideal of $A/\mathfrak{a}$. Every element outside of $\mathfrak{m}/\mathfrak{a}$ is then a unit, so every 0-divisor lies in $\mathfrak{m}/\mathfrak{a} = \sqrt{(0)}$, which means $\mathfrak{a}$ is primary. □

Definition. A **primary decomposition** of an ideal $\mathfrak{a} \subseteq A$ is an expression

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

with the $\mathfrak{q}_i$ primary. It is said to be minimal if the $\sqrt{\mathfrak{q}_i}$ are distinct, and if there is no i such that

$$\mathfrak{q}_i \subseteq \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \mathfrak{q}_{i-1} \cap \mathfrak{q}_{i+1} \cap \cdots \cap \mathfrak{q}_n.$$

Remark. A nonminimal primary decomposition easily gives a minimal one. If $\sqrt{\mathfrak{q}_i} = \sqrt{\mathfrak{q}_j}$, just replace $\mathfrak{q}_i$ and $\mathfrak{q}_j$ by $\mathfrak{q}_i \cap \mathfrak{q}_j$, which is primary by the lemma below.

If

$$\mathfrak{q}_i \subseteq \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \cdots \mathfrak{q}_{i-1} \cap \mathfrak{q}_{i+1} \cap \cdots \cap \mathfrak{q}_n,$$

just remove $\mathfrak{q}_i$.

Lemma. *If $\mathfrak{q}, \mathfrak{q}'$ are $\mathfrak{p}$-primary, then so is $\mathfrak{q} \cap \mathfrak{q}'$.*

There are two natural questions to ask of these primary decompositions

(1) Do they exist, i.e. does every ideal have such a decomposition?
(2) Are they unique?

The answers are

(1) No in general, but for "Noetherian rings", then yes. In particular the answer is yes for all rings of the form $k[x_1, \ldots, x_n]/I$ and $\mathbb{Z}[x_1, \ldots, x_n]/I$. We'll prove this later.
(2) No in general, but certain aspects are preserved.

Example. In the ring $\mathbb{Z}[i\sqrt{5}]$, one can check that (6) is written as an intersection

$$(6) = (2) \cap (3, 1 - i\sqrt{5}) \cap (3, 1 + i\sqrt{5})$$

The ideal (2) is $(2, 1 - i\sqrt{5})$-primary, and $(3, 1 \pm i\sqrt{5})$ are prime ideals. The ideals involved are coprime, so we also have

$$(6) = (2)(3, 1 - i\sqrt{5})(3, 1 + i\sqrt{5}).$$

This decomposition is (we'll see) unique.

Example. The ideal $(xy) \subseteq k[x, y]$ admits a minimal primary decomposition $(xy) = (x) \cap (y)$, which is also unique.

Example. Let's compute a primary decomposition of $(xy, y^2) \subset k[x, y]$. Firstly, note that (xy, y^2) is not primary, since $yx \in (xy, y^2)$, but $y \notin (xy, y^2)$ and no power of x lies in (xy, y^2).

A reasonable candidate for one primary ideal is (y), which is a prime ideal, and clearly $(xy, y^2) \subseteq (y)$. A second possible ideal is $(x^2, xy, y^2) = (x, y)^2$. This is (x, y)-primary since (x, y) is maximal.

Now to see

$$(xy, y^2) = (y) \cap (x^2, xy, y^2),$$

note that

$$(xy, y^2) = \{\sum a_{ij} x^i y^j \mid a_{i0} = 0 \text{ and } a_{01} = 0\}$$

while

$$(y) = \{\sum a_{ij} x^i y^j \mid a_{i0} = 0\}$$

and

$$(x^2, xy, y^2) = \{\sum a_{ij}x^iy^j \mid a_{00} = a_{10} = a_{01} = 0\}.$$

This is a minimal primary decomposition since $(y) \neq (x, y)$ and neither of the ideals (y) and (x^2, xy, y^2) are contained in the other.

In this case, the primary decomposition is *not* unique, in particular we can also write

$$(xy, y^2) = (y) \cap (y^2, xy, x^n)$$

for any $n \geq 1$, or

$$(xy, y^2) = (y) \cap (y^2, x + ay)$$

for any $a \in k$.

Recall $(\mathfrak{a} : x) = \{y \in A \mid yx \in \mathfrak{a}\}$.

Theorem (Uniqueness 1)**.** *Let $\mathfrak{a}$ be a decomposable ideal, with minimal decomposition*

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

The prime ideals $\sqrt{\mathfrak{q}_i}$ are precisely the prime ideals which can be written as $\sqrt{(\mathfrak{a} : x)}$.

Corollary. *Every minimal primary decomposition of $\mathfrak{a}$ gives the same set of prime ideals.*

Definition. The set of primes $\sqrt{\mathfrak{q}_i}$ the prime ideals **associated with** $\mathfrak{a}$.

Proof. We have

$$\sqrt{(\mathfrak{a} : x)} = \sqrt{(\cap \mathfrak{q}_i : x)} = \bigcap_{i=1}^{n} \sqrt{(\mathfrak{q}_i : x)},$$

so we need to analyse $(\mathfrak{q}_i : x)$.

Lemma. *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Then*

$$\sqrt{(\mathfrak{q} : x)} = \begin{cases} \mathfrak{p} & \text{if } x \notin \mathfrak{q} \\ (1) & \text{if } x \in \mathfrak{q} \end{cases}$$

Picking $x \in \mathfrak{q}_i$ but not in any other $\mathfrak{q}_j$ gives $\sqrt{(\mathfrak{a} : x)} = \mathfrak{p}_i$, so all the prime ideals $\mathfrak{p}_i$ appear as $\sqrt{(\mathfrak{a} : x)}$.

Conversely, if a prime ideal $\mathfrak{p}$ appears as $\sqrt{(\mathfrak{a} : x)}$, we must have that $\mathfrak{p} = \mathfrak{p}_{i_1} \cap \cdots \cap \mathfrak{p}_{i_k}$, which means that $\mathfrak{p} = \mathfrak{p}_i$ for one of these, by a basic lemma for prime ideals (Prop. 1.11 in the book). □

Definition. The minimal elements of the set of prime ideals associated with $\mathfrak{a}$ is called the **minimal** prime ideals belonging to $\mathfrak{a}$. The remaining ones are called **embedded** prime ideals belonging to $\mathfrak{a}$.

Example. In our decompositions of $(y^2, xy) \subset k[x, y]$, we will always have two components, of which one is (y) and the other could be (x^2, xy, x^n) for any n. Our uniqueness statement from today says that at least $\sqrt{(y)} = (y)$ and $\sqrt{(x^2, xy, y^n)} = (x, y)$ will be the same for any primary decomposition.