Lecture 14 – Primary ideals II

Recall $(\mathfrak{a} : x) = \{y \in A \mid yx \in \mathfrak{a}\}$.

Theorem (Uniqueness 1). *Let $\mathfrak{a}$ be a decomposable ideal, with minimal decomposition*

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

The prime ideals $\sqrt{\mathfrak{q}_i}$ are precisely the prime ideals which can be written as $\sqrt{(\mathfrak{a} : x)}$.

Corollary. *Every minimal primary decomposition of $\mathfrak{a}$ gives the same set of prime ideals.*

Definition. The ideals of the form $\sqrt{\mathfrak{q}_i}$ are the prime ideals **associated with** $\mathfrak{a}$.

Proof of Uniqueness theorem. The proof requires two reasonably simple lemmas.

Lemma (AM, Prop. 1.11). *If $\mathfrak{p}$ is a prime ideal, and $\mathfrak{a}_1, \mathfrak{a}_2, \ldots, \mathfrak{a}_n \subset A$ are prime ideals such that*

$$\mathfrak{p} \subseteq \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n,$$

then $\mathfrak{a}_i \subseteq \mathfrak{p}$ for some i. If moreover

$$\mathfrak{p} = \mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n,$$

then $\mathfrak{p} = \mathfrak{a}_i$ for some i.

To motivate the second one, note that if

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i,$$

then

$$\sqrt{(\mathfrak{a} : x)} = \sqrt{(\cap \mathfrak{q}_i : x)} = \sqrt{\cap (\mathfrak{q}_i : x)} = \bigcap_{i=1}^{n} \sqrt{(\mathfrak{q}_i : x)},$$

so we need to analyse $(\mathfrak{q}_i : x)$.

Lemma. *Let $\mathfrak{q}$ be a $\mathfrak{p}$-primary ideal. Then*

$$\sqrt{(\mathfrak{q} : x)} = \begin{cases} \mathfrak{p} & \text{if } x \notin \mathfrak{q} \\ (1) & \text{if } x \in \mathfrak{q} \end{cases}$$

Proof. If $x \in \mathfrak{q}$, then $(\mathfrak{q} : x) = (1)$.

If $x \notin \mathfrak{q}$ and $y \in (\mathfrak{q} : x)$, then $xy \in \mathfrak{q}$. Since $\mathfrak{q}$ is primary, this implies that $y^n \in \mathfrak{q}$ for some n, so $y \in \mathfrak{p}$. Conversely, if $x \notin \mathfrak{q}$ and $y \in \mathfrak{p}$, then for some $n \geq 1$ we have

$$y^n \in \mathfrak{q} \Rightarrow xy^n \in \mathfrak{q} \Rightarrow y^n \in (\mathfrak{q} : x) \Rightarrow y \in \sqrt{(\mathfrak{q} : x)}.$$

□

Now to prove the Uniquness theorem, assume that $\mathfrak{a} = \cap \mathfrak{q}_i$ is a minimal primary decomposition and that $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$. Now assume that $\mathfrak{p} = \sqrt{\mathfrak{a} : x)}$ is a prime ideal – we must show that $\mathfrak{p} = \mathfrak{p}_i$ for some i. We have, using Lemma 2, that

$$\mathfrak{p} = \bigcap_{i=1}^{n} \sqrt{(\mathfrak{q}_i : x)} = \mathfrak{p}_{i_1} \cap \mathfrak{p}_{i_2} \cap \cdots \cap \mathfrak{p}_{i_k},$$

where the set $\{i_1, \ldots, i_k\}$ is the set of i for which $x \notin \mathfrak{q}_i$. Using Lemma 1, then $\mathfrak{p} = \mathfrak{p}_{i_j}$ for some j.

We must also show that for every prime ideal $\mathfrak{p}_i$, we can find an $x \in A$ such that $\mathfrak{p}_i = \sqrt{(\mathfrak{a} : x)}$. Taking $x \in \mathfrak{q}_j$ for all $j \neq i$ but $x \notin \mathfrak{q}_i$ (which is possible since the decomposition is minimal), we get that $\sqrt{(\mathfrak{a} : x)} = \mathfrak{p}_i$. □

Example. The ideal $(xy, y^2) \subset k[x, y]$ can be given minimal primary decompositions

$$(xy, y^2) = (x^2, xy, y^2) \cap (y) = (x^n, xy, y^2) \cap (y).$$

The associated primes are $\sqrt{(x^2, xy, y^2)} = (x, y)$ and $\sqrt{(y)} = (y)$, and for each $n \geq 1$ we have that $\sqrt{(x^n, xy, y^2)} = (x, y)$.

Definition. The set of prime ideals associated with $\mathfrak{a}$ is partially ordered with respect to inclusion, i.e. $\mathfrak{p} \leq \mathfrak{p}'$ if $\mathfrak{p} \subseteq \mathfrak{p}'$.

The minimal elements of this set are called the **isolated** or **minimal** prime ideals associated with $\mathfrak{a}$, while the other ones are called **embedded**.

Example. In the decomposition $(y^2, xy) = (x^2, xy, y^2) \cap (y)$, the we have $(y) \subsetneq (x, y)$, so (y) is an isolated prime ideal, while (x, y) is embedded.

Proposition. *The isolated prime ideals of $\mathfrak{a}$ are exactly the prime ideals minimal over $\mathfrak{a}$, i.e. the prime ideals $\mathfrak{p}$ such that there is no prime ideal $\mathfrak{p}'$ with $\mathfrak{a} \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}$.*

Proof. Let $\mathfrak{r}_1, \ldots, \mathfrak{r}_k$ be the prime ideals which are minimal over $\mathfrak{a}$, let $\mathfrak{p}_1, \ldots, \mathfrak{p}_n$ be the prime ideals associated with $\mathfrak{a}$, ordered in such a way that $\mathfrak{p}_i$ is isolated for $i = 1, \ldots, m$ and embedded for $i = m + 1, \ldots, n$. The claim to be shown is

$$\{\mathfrak{r}_1, \ldots, \mathfrak{r}_k\} = \{\mathfrak{p}_1, \ldots, \mathfrak{p}_m\}.$$

We first show the inclusion $\subseteq$: Let $\mathfrak{p} = \mathfrak{r}_j \supseteq \mathfrak{a}$ be a minimal prime ideal containing $\mathfrak{a}$, and assume $\mathfrak{a} = \cap \mathfrak{q}_i$ is a minimal primary decomposition. We have

$$\mathfrak{a} \subseteq \mathfrak{p} \Rightarrow \sqrt{\mathfrak{a}} = \cap \sqrt{\mathfrak{q}_i} = \cap \mathfrak{p}_i \subseteq \mathfrak{p}.$$

This implies by the lemma above that for some i, $\mathfrak{p}_i \subseteq \mathfrak{p}$. But $\mathfrak{p}$ being minimal over $\mathfrak{a}$ then implies that $\mathfrak{p}_i = \mathfrak{p}$. Hence every prime ideal minimal over $\mathfrak{a}$ is an isolated prime ideal for $\mathfrak{a}$.

Now for the inclusion $\supseteq$, let $\mathfrak{p}_i$ be an isolated prime ideal of $\mathfrak{a}$. Assume for a contradiction that $\mathfrak{p}_i$ is not minimal over $\mathfrak{a}$, then there will be some minimal prime ideal $\mathfrak{p}'$ such that $\mathfrak{a} \subseteq \mathfrak{p}' \subsetneq \mathfrak{p}_i$ (this requires a Zorn's lemma argument). But by the above we know $\mathfrak{p}'$ is associated with $\mathfrak{a}$, so then $\mathfrak{p}_i$ is not isolated, giving a contradiction. □

Example. In our decompositions of $(y^2, xy) \subset k[x, y]$, we will always have two components, of which one is (y) and the other could be (x^2, xy, x^n) for any n. Our uniqueness statement from today says that at least $\sqrt{(y)} = (y)$ and $\sqrt{(x^2, xy, y^n)} = (x, y)$ will be the same for any primary decomposition.

Theorem. *Assume A admits a primary decomposition of (0), and let $\mathfrak{p}_1, \cdots, \mathfrak{p}_n$ be the associated prime ideals of (0). Then*

$$\{x \in A \mid x \text{ is a 0-divisor}\} = \bigcup_{i=1}^{n} \mathfrak{p}_i$$

Proof. Let

$$(0) = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

be a minimal primary decomposition, with $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$.

Assume $y \notin \mathfrak{p}_i$ for any i, and that $xy = 0$. Then $\mathfrak{q}_i$ being primary implies that $x \in \mathfrak{q}_i$, and since this holds for all i, we have $x \in \cap \mathfrak{q}_i = (0)$, so $x = 0$. This means y is not a 0-divisor, proving the $\subseteq$ inclusion of the theorem.

For the inclusion $\supseteq$, assume $y \in \mathfrak{p}_i$ for some i. There is some $x \in A$ such that

$$\sqrt{\mathrm{Ann}(x)} = \sqrt{((0) : x)} = \mathfrak{p}_i.$$

This means there is some n such that $y^n \in \mathrm{Ann}(x)$, which means there is some n such that $y^n x = 0$. Taking n_0 to be the minimal such n, we have $y^{n_0 - 1} x \neq 0$, and $y(y^{n_0-1}x) = y^{n_0} x = 0$, which means y is a 0-divisor. □

Example. In the ring $A = k[x,y]/(xy, y^2)$, the ideal (0) has a primary decomposition

$$(0) = (\bar{x}^2, \bar{x}\bar{y}, \bar{y}^2) \cap (\bar{y}),$$

with $\bar{x}$ and $\bar{y}$ the images of x and y in A. The associated primes ideals of (0) are $(\bar{x}, \bar{y})$ and $(\bar{y})$, and so the set of 0-divisors in A is

$$(\bar{x}, \bar{y}) \cup (\bar{y}) = (\bar{x}, \bar{y}) \subset A.$$

Theorem (Uniqueness 2)**.** *Let $\mathfrak{a}$ be an ideal with primary decomposition*

$$\bigcap_{i=1}^{n} \mathfrak{q}_i,$$

with associated prime ideals $\mathfrak{p}_i = \sqrt{\mathfrak{q}_i}$, and assume that $\mathfrak{p}_1, \dots, \mathfrak{p}_m$ are the minimal prime ideals.

Then for each i with $1 \leq i \leq m$, we have

$$\mathfrak{q}_i = \mathfrak{a}^{ec},$$

where extension and contraction are along the homomorphism $A \to A_{\mathfrak{p}_i}$.

In particular, these $\mathfrak{q}_i$ are the same in any minimal primary decomposition.

Example. In the case of (xy, y^2) and all its primary decompositions $(x^n, xy, y^2) \cap (y)$, the primary ideal (y) is uniquely determined by (xy, y^2).

Example. The ideal $(xy) = (x) \cap (y) \subset k[x,y]$ can have no other minimal primary decomposition, since both (x) and (y) are minimal over (xy).

Proof. The idea of the proof is the following simple lemma.

Lemma. *Let $\mathfrak{p} \subset A$ be a prime ideal, and assume that $\mathfrak{q}$ is a primary ideal. We consider the extension and contraction of ideals with respect to $A \to A_{\mathfrak{p}}$.*

- *If $\sqrt{\mathfrak{q}} \not\subseteq \mathfrak{p}$, then $\mathfrak{q}^e = (1)$, and $\mathfrak{q}^{ec} = (1)$.*
- *If $\sqrt{\mathfrak{q}} \subseteq \mathfrak{p}$, then $\mathfrak{q}^{ec} = \mathfrak{q}$.*

Now let $\mathfrak{p}_j$ be a minimal prime of $\mathfrak{a}$, and consider extension and contraction with respect to $A \to A_{\mathfrak{p}_j}$. Since $\mathfrak{p}_j$ is minimal, we have for all $i \neq j$ that $\sqrt{\mathfrak{q}_i} = \mathfrak{p}_i \not\subseteq \mathfrak{p}_j$, so $\mathfrak{q}_i^{ec} = (1)$, while $\mathfrak{q}_j^{ec} = \mathfrak{q}_j$.

Recall from the lecture on modules of fractions that localisation (the operation $\mathfrak{a} \mapsto \mathfrak{a}^e$) preserves finite intersections of ideals, as does contraction.

$$\mathfrak{a}^{ec} = \bigcap_{i=1}^{n} \mathfrak{q}_i{}^{ec} = (1) \cap (1) \cap \cdots \cap \mathfrak{q}_j \cap (1) \cap \cdots \cap (1).$$

□