Lecture 16 – More on integral dependence + chain conditions

Recall the notions of integral dependence and integral closure from last week. We round out the section on integral dependence with the claim that for an integral domain A, being integrally closed is a local property.

Theorem. *Let A be an integral domain. Then the following are equivalent:*

(1) *A is integrally closed.*
(2) *$A_\mathfrak{p}$ is integrally closed for all prime ideals $\mathfrak{p} \subset A$.*
(3) *$A_\mathfrak{m}$ is integrally closed for all maximal ideals $\mathfrak{m} \subset A$.*

The proof goes via understanding how integrality behaves with respect to taking fraction rings more generally.

Proposition. *Let $A \subseteq B$ be rings, and let $C \subseteq B$ be the integral closure of A in B. Let $S \subseteq A$ be a multiplicatively closed subset. Then $S^{-1}C \subseteq S^{-1}B$ is the integral closure of $S^{-1}A$ in $S^{-1}B$.*

Proof. Let $D \subseteq S^{-1}B$ be the integral closure of $S^{-1}A$ in $S^{-1}B$. We want $D = S^{-1}C$.

$D \subseteq S^{-1}C$: If $b/s \in D$, then b/s is integral over $S^{-1}A$, and so we can find $a_i \in A$, $s_i \in S$, such that

$$(b/s)^n + (a_{n-1}/s_{n-1})(b/s)^{n-1} + \cdots + \frac{a_0}{s_0} = 0.$$

Multiplying by $(ss_{n-1}\cdots s_0)^n$ gives us a relation

$$\frac{(bs_{n-1}\cdots s_0)^n + d_{n-1}(bs_{n-1}\cdots s_0)^{n-1} + \cdots + d_0 bs_{n-1}\cdots s_0}{1} = 0,$$

in $S^{-1}A$. This means that there is some $t \in S$ such that the relation

$$t((bs_{n-1}\cdots s_0)^n + d_{n-1}(bs_{n-1}\cdots s_0)^{n-1} + \cdots + d_0 bs_{n-1}\cdots s_0) = 0$$

holds in A. Multiplying by t^{n-1} we get that $bs_{n-1}\cdots s_0 t$ is integral over A, and so $bs_{n-1}\cdots s_0 t \in C$, which implies that $b/s \in S^{-1}C$.

$S^{-1}C \subseteq D$: Given $c \in C$ and $s \in S$, we have a relation

$$c^n + a_{n-1}c^{n-1} + \cdots + a_0 = 0 \qquad a_i \in A,$$

which implies that

$$\left(\frac{c}{s}\right)^n + sa_{n-1}\left(\frac{c}{s}\right)^{n-1} + \cdots + s^n a_0 = 0,$$

hence $\frac{c}{s}$ is integral over $S^{-1}A$. It follows that $S^{-1}C \subseteq D$. □

Corollary. *If A is an integral domain and C is the integral closure of A in the fraction field K, then for any prime ideal $\mathfrak{p} \subset A$, we have that $C_\mathfrak{p} \subset K$ is the integral closure of $A_\mathfrak{p}$ in K.*

Proof of theorem. Let K be the fraction field of A, and let C be the integral closure. If $A = C$, then also $A_\mathfrak{p} = C_\mathfrak{p}$, so we get (1) $\Rightarrow$ (2).

(2) $\Rightarrow$ (3) is obvious since maximal ideals are prime.

To get (3) $\Rightarrow$ (1), we know that being surjective is a local property. The inclusion map $\phi\colon A \to C$ is an A-module homomorphism. By assumption (3), all $A_\mathfrak{m}$ are integrally closed, which means $\phi_\mathfrak{m}\colon A_\mathfrak{m} \to C_\mathfrak{m}$ is surjective. Since being surjective is a local property, it follows that ϕ is surjective, and hence A is integrally closed. □

Example. Recall that $k[x^2, x^3] \subset k[x]$ is an integral domain which is not integrally closed. The fraction field of A is identified with $k(x)$, so we have

$$k[x^2, x^3] \subset k[x] \subseteq k(x),$$

Now $x \in k(x) \setminus k[x^2, x^3]$ is integral over $k[x^2, x^3]$, since x is a zero of the polynomial

$$t^2 - x^2 \in k[x^2, x^3][t].$$

Since x is integral over $k[x^2, x^3]$, it is also integral over all the bigger rings $k[x^2, x^3]_\mathfrak{p} \subseteq k(x)$ for various primes $\mathfrak{p}$. Letting $\mathfrak{m} = (x^2, x^3)$, one can check that $x \notin k[x^2, x^3]_\mathfrak{m}$, and therefore $k[x^2, x^3]_\mathfrak{m}$ is not integrally closed.

Chain conditions

Our theory so far has mostly been developed for arbitrary rings. The motivation for the field of commutative algebra, both historically and in practice, is mostly drawn from number theory and algebraic geometry, where the rings which appear are "reasonably small". In order to develop the theory further, we now begin introducing these smallness conditions. The elegant formulation of these conditions is in terms of chains of subobjects.

Lemma. *Let $(S, \geq)$ be a partially ordered set. The following two conditions are equivalent:*

- *Every sequence $s_1 \leq s_2 \leq s_3 \leq \cdots$ stabilises, that is there is some N such that $s_i = s_N$ for all $i \geq N$.*
- *Every nonempty subset $T \subseteq S$ contains a maximal element of T.*

Recall an element $t \in T$ is maximal if there is no $t' \in T$ with $t' > t$.

Proof. (1) $\Rightarrow$ (2): Suppose $T \subseteq S$ contains no maximal element. This means that for every $t \in T$, we can choose an $f(t) \in T$ with $f(t) > t$. Take now the sequence

$$s_1 = t, s_2 = f(t), s_3 = f(f(t)), \cdots$$

which does not stabilise, so contradicts (1).

(2) $\Rightarrow$ (1): Given a sequence $s_1 \leq s_2 \leq \cdots$, let $T = \{s_i\}_{i=1}^{\infty}$. By (2) there is a maximal element, say s_N, and since $s_i \geq s_N$ for $i \geq N$, we have $s_i = s_N$ for $i \geq N$. $\square$

Definition. Let A be a ring and let M be an A-module, and let S be the set of submodules of M.

- We say M is **Noetherian** if the set S, partially ordered by $M' \leq M''$ if $M' \subseteq M''$, satisfies either condition above.
- We say M is **Artinian** if the set S, partially ordered by $M' \leq M''$ if $M' \supseteq M''$, satisfies either condition above.

In concrete terms, M is Noetherian if it satisfies the **ascending chain condition**: Every sequence

$$M_1 \subseteq M_2 \subseteq M_3 \supseteq \cdots$$

of submodules stabilises, or equivalently, every set T of submodules has a maximal element.

The module M is Artinian if it satisfies the **descending chain condition**, every sequence of submodules

$$M_1 \supseteq M_2 \supseteq M_3 \supseteq \cdots$$

stabilises. Equivalently, every set T of submodules has a minimal element.

Definition. A ring A is called Noetherian (resp. Artinian) if it is Noetherian (resp. Artinian) as an A-module.

Example. The ring $\mathbb{Z}$ is Noetherian, but not Artinian. A submodule of $\mathbb{Z}$ is an ideal (n). An ascending chain looks like

$$(n_1) \subseteq (n_2) \subseteq (n_3) \subseteq \cdots .$$

The containment $(n_i) \subseteq (n_{i+1})$ implies that n_{i+1} divides n_i, so $n_{i+1} \leq n_i$. The sequence must then clearly stabilise.

The ring $\mathbb{Z}$ is not Artinian, since

$$(2) \supsetneq (4) \supsetneq (8) \supsetneq \cdots$$

does not stabilise.

Example. Let k be a field, and let M be a k-module (vector space). Then M is Noetherian if and only if M is Artinian if and only if M has finite dimension.

Example. The ring $C(\mathbb{R})$ of smooth functions on $\mathbb{R}$ is neither Artinian nor Noetherian, since

$$(1) \supsetneq (x) \supsetneq (x^2) \cdots ,$$

and

$$(\sin(x)) \subsetneq (\sin(x/2)) \subsetneq (\sin(x/4)) \subsetneq (\sin(x/8)) \subsetneq \cdots .$$

This example is mainly to show that the rings appearing outside of algebra typically satisfy none of the smallness conditions we want.

Proposition. *Let M be an A-module. Then M is Noetherian if and only if every submodule of M is finitely generated.*

Proof. Assume that M is Noetherian, and let $M' \subseteq M$ be a submodule. Let

$$T = \{N \subseteq M' \mid M'' \text{ a finitely generated submodule of } M'\}.$$

By the Noetherian hypothesis, there is a maximal element $N_{max} \in T$. Assume for a contradiction that $N_{max} \neq M'$. Then there is an $m \in M' \setminus N_{max}$, so

$$N_{max} \subsetneq N = N_{max} + Am \subseteq M',$$

and N is still finitely generated, so $N \in T$. This contradicts the maximality of N_{max}, so we have our contradiciton, and $N_{max} = M'$, which means M' is finitely generated.

Assume that every submodule $M' \subseteq M$ is finitely generated. Let

$$M_1 \subseteq M_2 \subseteq M_3 \subseteq \cdots$$

be a chain of submodules, and let

$$M' = \bigcup_{i=1}^{\infty} M_i \subseteq M.$$

Then M' is by assumption finitely generated, say by $m_1, \ldots, m_n$. We must then have $m_i \in M_{k_i}$ for certain k_i, and taking $k = \max(k_1, \ldots, k_n)$, we have $m_1, \ldots, m_n \in M_k$. But then $M_k = M'$, and the chain stabilises at M_k. $\square$