8. Lecture 17 – Chain conditions II

Theorem. *Let*

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

be a short exact sequence of A-modules. Then

$$M \textit{ Noetherian} \Leftrightarrow M' \textit{ and } M'' \textit{ Noetherian.}$$

and

$$M \textit{ Artinian } \Leftrightarrow M' \textit{ and } M'' \textit{ Artinian.}$$

Proof. We only do the statement for te Noetherian condition, the Artinian case is exactly the same.

$\Rightarrow$: If

$$M_1' \subseteq M_2' \subseteq \cdots$$

is a chain of submodules of M', then

$$i(M_1') \subseteq i(M_2') \subseteq \cdots .$$

is a chain of submodules of M. Since M is Noetherian, the latter stabilises, so the first one must as well.

If

$$M_1' \subseteq M_2' \subseteq \cdots$$

is a chain of submodules of M'', then

$$p^{-1}(M_1') \subseteq p^{-1}(M_2') \subseteq \cdots .$$

is a chain of submodules of M. Since M is Noetherian, the latter stabilises, so the first one must as well.

$\Leftarrow$: If

$$M_1 \subseteq M_2 \subseteq \cdots$$

is a chain of submodules, then we get chains

$$i^{-1}(M_1) \subseteq i^{-1}(M_2) \subseteq \cdots$$

and

$$p(M_1) \subseteq p(M_2) \subseteq \cdots .$$

Both of these stabilise, so for some N we have that for all $i \geq N$, then

$$i^{-1}(M_i) = i^{-1}(M_{i+1})$$

and

$$p(M_i) = p(M_{i+1}).$$

Claim: It follows that $M_i = M_{i+1}$. It is not hard to prove this directly,[8] but for fun we can use the snake lemma on this:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & i^{-1}(M_i) & \xrightarrow{i} & M_i & \xrightarrow{p} & p(M_i) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} & & \\
0 & \longrightarrow & i^{-1}(M_{i+1}) & \xrightarrow{i} & M_{i+1} & \xrightarrow{p} & p(M_{i+1}) & \longrightarrow & 0.
\end{array}$$

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[8]If $m \in M_{i+1}$, then $p(m) \in p(M_{i+1}) = p(M_i)$, so there is some $m' \in M_{i+1}$ such that $p(m) = p(m')$. But then $p(m - m') = 0$, so there is some $m'' \in M'_{i+1}$ such that $i(m'') = m - m'$. Since $i^{-1}M_{i+1} = i^{-1}M_i$, we have that $m'' \in M'_i$, and therefore $m = m' + i(m'') \in M_i$.

The snake lemma gives an exact sequence

$$0 \to \ker f' \to \ker f \to \ker f'' \to \operatorname{cok} f' \to \operatorname{cok} f \to \operatorname{cok} f'' \to 0,$$

and since $\operatorname{cok} f' = \operatorname{cok} f'' = 0$, we get that $\operatorname{cok} f = 0$, so f is surjective. We've shown that the sequence M_i stabilises. □

Corollary. *If $M_1, \ldots, M_n$ are Noetherian (resp. Artinian) A-modules, then so is*

$$\bigoplus_{i=1}^{n} M_i.$$

Proof. Inductively prove that $\oplus_{i=1}^{j} M_i$ is Noetherian, using the exact sequence

$$0 \to M_{j+1} \to \bigoplus_{i=1}^{j+1} M_i \to \bigoplus_{i=1}^{j} M_i \to 0.$$

□

Proposition. *Let A be a Noetherian (resp. Artinian) ring, and let M be a finitely generated A-module. Then M is Noetherian (resp. Artinian).*

Proof. A is Noetherian $\Rightarrow$ A^n is Noetherian. There is some surjective homomorphism $\phi\colon A^n \to M$, and the short exact sequence

$$0 \to \ker \phi \to A^n \to M \to 0$$

shows that M is Noetherian. □

Proposition. *Let A be a Noetherian (resp. Artinian) ring, and let $\mathfrak{a} \subsetneq A$ be an ideal. Then A is Noetherian (resp. Artinian).*

Proof. The ring $A/\mathfrak{a}$ has structure as an $A/\mathfrak{a}$-module and an A-module. A set $M \subset A/\mathfrak{a}$ is an $A/\mathfrak{a}$-submodule if and only if it is an A-submodule, since

$$a(x + \mathfrak{a}) \in M \quad \forall a \in A, x + \mathfrak{a} \in M$$

is the same condition as

$$(a + \mathfrak{a})(x + \mathfrak{a}) \quad \forall a + \mathfrak{a} \in A/\mathfrak{a}, x + \mathfrak{a} \in M.$$

Since $A/\mathfrak{a}$ is a Noetherian A-module, it is then also a Noetherian $A/\mathfrak{a}$-module, i.e. Noetherian as a ring. □

Composition series

Definition. A module M is **simple** if it has no proper nontrivial submodules.

Example. If A is a ring with a maximal ideal $\mathfrak{m} \subset A$, then $A/\mathfrak{m}$ is a simple A-module: If $0 \subseteq M \subseteq A/\mathfrak{m}$ is a chain of modules, and $p\colon A \to A/\mathfrak{m}$ is the projection, then

$$p^{-1}(0) = \mathfrak{m} \subseteq p^{-1}(M) \subseteq p^{-1}(A/\mathfrak{m}) = A$$

is a chain of submodules (ideals) of A. Since $\mathfrak{m}$ is maximal, then either $p^{-1}(M) = \mathfrak{m}$ or $p^{-1}(M) = A$, which implies $M = 0$ or $M = A/\mathfrak{m}$.

Remark. One can show that every simple A-module is isomorphic to one of the form $A/\mathfrak{m}$.

Definition. A **composition series** of a module M is a finite chain

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \subsetneq M_{n-1} \supsetneq M_n = 0,$$

which is maximal, that is it cannot be extended to a longer chain by inserting

$$M_i \supsetneq M' \subsetneq M_{i+1}.$$

Equivalently, maximality means that M_i/M_{i+1} is simple for each i. The length of a composition series is n, the number of pieces M_i/M_{i+1} appearing.

Example. Let p be a prime, $k \geq 1$, and consider the $\mathbb{Z}$-module $\mathbb{Z}/(p^k)$. This has a composition series of length k, given by

$$\mathbb{Z}/(p^k) \supsetneq (p)/(p^k) \supsetneq (p^2)/(p^k) \supsetneq \cdots \supsetneq (p^k)/(p^k) = 0.$$

The quotients are $((p^i)/(p^k))/(p^{i+1})/(p^k) \cong (p^i)/(p^{i+1}) \cong \mathbb{Z}/p$, so are simple.

Example. Let p and q be primes, and consider the module $\mathbb{Z}/(pq)$. This has two compositions series

$$\mathbb{Z}/(pq) \supsetneq (p)/(pq) \supsetneq (pq)/(pq) = 0$$

and

$$\mathbb{Z}/(pq) \supsetneq (q)/(pq) \supsetneq (pq)/(pq) = 0$$

Proposition. *Let M be a module with a composition series of length n. Then every composition series has length n, and every chain*

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0$$

can be extended to a composition series by adding finitely many modules M' with $M_i \supsetneq M' \supsetneq M_{i+1}$.

Proof. Let $l(N)$ be the function on modules defined as the minimal length of a composition series of N ($+\infty$ if N has no composition series).

Lemma. *If $N \subsetneq M$, then $l(N) < l(M)$.*

Proof. Let

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_{l(M)} = 0$$

be a composition series of minimal length. We claim

$$N = M_0 \cap N \supseteq M_1 \cap N \supseteq \cdots \supseteq M_{l(M)} \cap N = 0$$

contains a composition series of N, in the sense that we can find

$$0 = j_0 < j_1 < \cdots < j_k \leq l(M)$$

such that

$$N = M_{j_0} \cap N \supsetneq M_{j_1} \cap N \supsetneq \cdots \supsetneq M_{j_k} \cap N = 0$$

is a composition series. For every i, we have a homomorphism

$$\phi\colon M_i \cap N \hookrightarrow M_i \to M_i/M_{i+1},$$

with

$$\ker \phi = M_{i+1} \cap N.$$

and with

$$\operatorname{im} \phi = M_i/M_{i+1} \text{ or } \operatorname{im} \phi = 0,$$

since M_i/M_{i+1} is simple. Hence

$$(M_i \cap N)/(M_{i+1} \cap N) = (M_i \cap N)/\ker\phi \cong \operatorname{im}\phi \begin{cases} M_i/M_{i+1}(\text{ Case 1}) \\ 0 \text{ Case 2.} \end{cases}$$

In Case 1, we have

$$N \cap M_i = N \cap M_{i+1}$$

and in Case 2,

$$(N \cap M_i)/(N \cap M_{i+1})$$

is simple. Taking the sequence

$$N \cap M_0 \supsetneq N \cap M_{j_1} \supsetneq N \cap M_{j_2} \supsetneq \cdots \supsetneq N \cap M_{j_k} = 0,$$

where $0 = j_0 < j_1 < \cdots < j_k \leq l(m)$ are the indices such that $N \cap M_{j_k} \neq N \cap M_{j_k - 1}$, we have produced a composition series of N of length $k \leq l(M)$, proving $l(N) \leq l(M)$.

Now as $N \subsetneq M$, we have

$$N = N \cap M_0 \neq M_0 = M,$$

while

$$0 = N \cap M_{l(M)} = M_{l(M)} = 0.$$

Let $i > 0$ be the smallest number such that $N \cap M_i = M_i$. Then we have

$$M_{i-1} \supsetneq N \cap M_{i-1} \supseteq N \cap M_i = M_i,$$

which shows that $N \cap M_{i-1} = N \cap M_i$, so i is not in the set $\{j_l\}_{l=1}^{k}$ Hence $k = l(N) < l(M)$. □

Now if M has a chain of length n, we have

$$l(M) = l(M_0) > l(M_1) > \cdots > l(M_n) = 0,$$

so $l(M) \geq n$. But by definition $l(M) \leq n$, so $l(M) = n$.

If

$$M = M_0 \supsetneq \cdots \supsetneq M_n = 0$$

is a chain of length $n < l(M)$, then by definition of $l(M)$ it cannot be a composition series, so we can extend it. □