Lecture 18 – Finite length modules, Noetherian rings

Recall a **composition series** for a module M is a chain

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_n = 0,$$

such that every quotient M_i/M_{i+1} is **simple**, that is admits only 0 and M_i/M_{i+1} as submodules.

We stated and almost proved

Proposition. *If M admits a composition series of length n, then every composition series of M has length n, and every chain*

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0$$

satisfies

(1) $k \leq n$,
(2) *if $k < n$, then the chain can be extended to a composition series by adding modules.*

Definition. The **length** of a module M, denoted $l(M)$, is the length of any composition series of M, and ∞ if M admits no composition series.

Remark. Worth knowing, but not something we will prove or focus on, is the **Jordan–Hölder theorem**. This says that given two composition series M_i and M_i' of a module M, the isomorphism classes of modules appearing in $\{M_i/M_{i+1}\}_{i=1}^{l(M)}$ and $\{M_i'/M_{i+1}'\}_{i=1}^{l(M)}$ are the same. A given isomorphism class appears the same number of times in each of the two sets.

Example. Given distinct primes p and q, the two composition series for $\mathbb{Z}/(pq)$ are

$$\mathbb{Z}/(pq) \supsetneq (p)/(pq) \supsetneq 0$$

and

$$\mathbb{Z}/(pq) \supsetneq (q)/(pq) \supsetneq 0.$$

We have

$$(\mathbb{Z}/(pq))/((p)/(pq)) \cong \mathbb{Z}/p, \quad ((p)/(pq))/0 \cong \mathbb{Z}/q$$

and

$$(\mathbb{Z}/(pq))/((q)/(pq)) \cong \mathbb{Z}/q, \quad ((q)/(pq))/0 \cong \mathbb{Z}/p.$$

Proposition. *Let M be a module. Then M has finite length $\Leftrightarrow$ M is Noetherian and Artinian.*

Proof. $\Rightarrow$: Any increasing sequence has at most $l(M)$ distinct terms, similarly for a decreasing sequence.

$\Leftarrow$: Define a descending chain as follows: Let $M_0 = M$, and let M_1 be a maximal submodule of M not equal to M. This exists because M is Noetherian. Inductively define M_{i+1} as a maximal submodule of M_i among those not equal to M_i. The sequence $M_0 \supsetneq M_1 \supsetneq M_2 \supsetneq \cdots$ cannot be extended indefinitely, since M is Artinian, hence we eventually have $M_n = 0$. Then M_i defines a composition series for M. $\square$

Proposition. *Given a short exact sequence of modules*

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0,$$

we have $l(M) = l(M') + l(M'')$.

Proof. The case where one of $l(M), l(M')$ or $l(M'')$ is ∞ can be handled by the previous proposition and the fact that M is Noetherian (resp. Artinian) if and only if M' and M'' are.

Hence we can assume that M', M and M'' are all of finite length. Take a composition series M'_i for M' and M''_j for M''. These induce the following sequence of submodules of M:

$$\begin{aligned} M = p^{-1}(M''_0) \supsetneq p^{-1}(M''_1) \supsetneq \cdots p^{-1}(M''_{l(M'')}) = p^{-1}(0) = i(M') \\ = i(M'_0) \supsetneq i(M'_1) \supsetneq \cdots i(M'_{l(M')}) = 0. \end{aligned}$$

Since

$$p^{-1}(M''_i)/p^{-1}(M''_{i+1}) \cong M''_i/M''_{i+1}$$

and

$$i(M'_i)/i(M'_{i+1}) \cong M'_i/M'_{i+1}$$

this gives a composition series of length $l(M') + l(M'')$ for M. □

Noetherian rings

Recall a ring A is Noetherian if either of the following equivalent conditions hold

(1) Every ascending chain of ideals stabilises.
(2) Every set of ideals has a maximal element.
(3) Every set of ideals is finitely generated.

We have shown that the class of Noetherian rings is closed under quotients, i.e. if A is Noetherian and $\mathfrak{a} \subseteq A$ is an ideal, then so is $A/\mathfrak{a}$.

Proposition. *Let A be a ring, and let $S \subseteq A$ be a multiplicative closed subset. The if A is Noetherian, so is $S^{-1}A$.*

Proof. Every ideal in $S^{-1}A$ is of the form $\mathfrak{a}^e$, where $\mathfrak{a} \subseteq A$ is an ideal and extension is along $A \to S^{-1}A$. Since A is Noetherian, we can write $\mathfrak{a} = (a_1, \ldots, a_n)$, and then $\mathfrak{a}^e = (a_1/1, \cdots, a_n/1)$. Hence every ideal of $S^{-1}A$ is finitely generated. □

Theorem (Hilbert's basis theorem)**.** *Let A be a Noetherian ring. Then $A[x]$ is Noetherian.*

Proof. For any ideal $\mathfrak{a} \subseteq A[x]$, define

$$\mathfrak{a}_n = \{a_n \in A \mid a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in \mathfrak{a}\}.$$

In words, $\mathfrak{a}_n$ is the set of leading terms of degree n polynomials in $\mathfrak{a}$.

Easy claim 1: $\mathfrak{a}_n \subseteq A$ is an ideal.

Easy claim 2: $\mathfrak{a}_n \subseteq \mathfrak{a}_{n+1}$ for every n.

Since A is Noetherian, there is an N such that $\mathfrak{a}_n = \mathfrak{a}_\infty$, for all $n \geq N$, i.e. we have

$$\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \cdots \subseteq \mathfrak{a}_N = \mathfrak{a}_{N+1} \subseteq \cdots.$$

Now for each $i = 0, \ldots, N$, we can find a finite set of generators $a_{i,j} \in A$ for $\mathfrak{a}_i$, so that e.g.

$$(a_{i,1}, \ldots, a_{i,k_i}) = \mathfrak{a}_i$$

For each i, j, the fact that $a_{i,j} \in \mathfrak{a}_i$ means there is an $f_{i,j} \in \mathfrak{a}$ such that

$$f_{i,j} = a_{i,j}x^i + \text{lower order terms}.$$

Main claim: We have $\mathfrak{a} = (f_{i,j})_{i,j}$. Let $g \in \mathfrak{a}$, we need to show $g \in (f_{i,j})_{i,j}$. Arguing by induction on $\deg g$, starting from $\deg g = -\infty$ where $g = 0$. There are two cases:

- $\deg g < N$: Writing
$$g = a_i x^i + \text{lower order terms},$$
we have $a_i \in \mathfrak{a}_i$. We can then write
$$a_i = \sum_{j=1}^{k_i} c_i a_{i,j}, \qquad c_i \in A.$$
Considering
$$g' = g - \sum_{j=1}^{k_i} c_i f_{i,j} = (a_i - \sum c_i a_{i,j}) x^i + \text{ lower order terms},$$
we have $\deg g' < \deg g$, and clearly $g' \in \mathfrak{a}$. By induction on degree $g' \in (f_{i,j})$, so $g \in (f_{i,j})$.
- If $\deg g \geq N$, take instead
$$g' = g - \sum_{j=1}^{r_N} c_i f_{N,j} x^{\deg g - N},$$
and conclude similarly.

□

Example. Consider the case of a field k. Then for any ideal $\mathfrak{a} \subseteq k[x]$, we have $\mathfrak{a}_i = (0)$ or $\mathfrak{a}_i = (1)$. We thus get
$$0 = \mathfrak{a}_0 = \mathfrak{a}_1 = \cdots = \mathfrak{a}_{N-1} \subsetneq \mathfrak{a}_N = (1) = \mathfrak{a}_{N+1} = \cdots .$$
In this case the proof above says: Take a generator $a_{N,1}$ $\mathfrak{a}_N$. Choose an $f_{N,1} \in \mathfrak{a}_N$ such that
$$f_{N,1} = a_{N,1} x^N + \text{ lower order terms}.$$
Then $\mathfrak{a} = (f_{N,1})$.

Corollary. *If A is a Noetherian ring, then so is $A[x_1, \ldots, x_n]$.*

Proof. A Noetherian $\Rightarrow A[x_1]$ Noetherian $\Rightarrow A[x_1, x_2] \cong A[x_1][x_2]$ Noetherian and so on. □

Corollary. *If A is Noetherian and B is an A-algebra of finite type, then B is Noetherian.*

Proof. B is of finite type if it is isormorphic (as A-algebra) to $A[x_1, \ldots, x_n]/\mathfrak{a}$. Now A Noetherian $\Rightarrow A[x_1, \ldots, x_n]$ Noetherian $\Rightarrow A[x_1, \ldots, x_n]/\mathfrak{a}$ Noetherian. □