Lecture 20 – Artinian rings

Recall a ring A is **Artinian** if every sequence of ideals

$$\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$$

of A stabilises.

Example. Let k be a field, and let A be a k-algebra which is finite-dimensional as a k-module. Then A is both Artinian and Noetherian as a k-module, since every chain of k-submodules has at most $\dim_k A$ distinct k-modules.

Moreover, since every chain of ideals

$$\mathfrak{a}_1 \supseteq \mathfrak{a}_2 \supseteq \cdots$$

is a chain of A-submodules of A, these are also k-submodules of A, so A is Artinian and Noetherian as a ring.

Take for instance $f = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in k[x]$. Then $A = k[x]/(f)$ has a basis as a k-module given by

$$1 + (f), x + (f), \cdots, x^{n-1} + (f),$$

so $\dim_k A = n$, and A is Artinian and Noetherian.

Example. Let $A = k[x, y]/(x^m, y^n)$. Then A has a k-basis given by $x^i y^j + (x^m, y^n)$, with $0 \leq i \leq m, 0 \leq j \leq n$, and so A is Artinian and Noetherian.

Example. For any $n \geq 1$, the ring $\mathbb{Z}/n$ is Artinian and Noetherian.

Lemma. *Let A be an Artinian ring. Then A has finitely many maximal ideals.*

Proof. Suppose not, then we can find an infinite sequence $\mathfrak{m}_1, \mathfrak{m}_2, \ldots$ of distinct maximal ideals. The descending sequence

$$A \supseteq \mathfrak{m}_1 \supseteq \mathfrak{m}_1 \cap \mathfrak{m}_2 \supseteq \cdots$$

must stabilise, so for some N we must have

$$\bigcap_{i=1}^{N} \mathfrak{m}_i = \bigcap_{i=1}^{N+1} \mathfrak{m}_i,$$

which means

$$\mathfrak{m}_{N+1} \supseteq \bigcap_{i=1}^{N} \mathfrak{m}_i.$$

But this implies $\mathfrak{m}_i \subseteq \mathfrak{m}_{N+1}$, which is impossible since these are maximal and distinct. □

Lemma. *In an Artinian ring, every prime ideal is maximal.*

Proof. If A is Artinian and $\mathfrak{p} \subset A$ is prime, then also $A/\mathfrak{p}$ is Artinian, and moreover an integral domain. For any $x \in A/\mathfrak{p}$, we have a descending chain

$$1 \supseteq (x) \supseteq (x^2) \cdots,$$

which must stabilise, so $(x^N) = (x^{N+1})$ for some N. This implies $x^N = yx^{N+1}$, and since $A/\mathfrak{p}$ is an integral domain, we can cancel to get $xy = 1$. Hence x is a unit, and since this holds for all x, $A/\mathfrak{p}$ is a field, so $\mathfrak{p}$ is maximal. □

Definition. Let A be a ring. Its **dimension** (or **Krull dimension**) is the maximum length n of a chain of prime ideals in A

$$\mathfrak{p}_0 \supsetneq \mathfrak{p}_1 \supsetneq \cdots \supsetneq \mathfrak{p}_n.$$

Example. A field k has one prime ideal, so $\dim k = 0$.

Example. In $\mathbb{Z}$, the chains of maximal length look like $(p) \supsetneq (0)$, so $\dim \mathbb{Z} = 1$.

Similarly $\dim k[x] = 1$, since a maximal length chain looks like $(f) \supsetneq (0)$ with f irreducible.

Example. We have shown that every Artinian ring has dimension 0.

Proposition. *Every Artinian ring is Noetherian.*

Proof. We don't prove this; the main steps are as follows.

(1) Let $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \subset A$ be the maximal ideals of A. For some $e \geq 0$, we have

$$\mathfrak{m}_1^e \cdots \mathfrak{m}_n^e = (0).$$

(2) In the chain

$$A \supseteq \mathfrak{m}_1 \supseteq \cdots \supseteq \mathfrak{m}_1^e \supseteq \mathfrak{m}_1^e \mathfrak{m}_2 \cdots \supseteq \mathfrak{m}_1^e \mathfrak{m}_2^e \cdots \mathfrak{m}_n^e = (0),$$

the quotients

$$\mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_n^{i_n} / \mathfrak{m}_1^{i_1} \cdots \mathfrak{m}_j^{i_j+1} \cdots \mathfrak{m}_n^{i_n}$$

are all Artinian A-modules, since A is Artinian.

(3) The quotients are Artinian A-modules, hence Artinian $A/\mathfrak{m}_j$-modules, hence finite dimensional $A/\mathfrak{m}_j$-modules, hence Noetherian $A/\mathfrak{m}_j$-modules, hence Noetherian A-modules.

(4) A is a Noetherian A-module, i.e. Noetherian as a ring.

□

Proposition. *If A is Noetherian and every prime ideal is maximal, then A is Artinian.*

Proof. We assume for a contradiction that A is not Artinian, and consider the set of ideals $\mathfrak{a} \subset A$ such that $A/\mathfrak{a}$ is not Artinian. Since A is Noetherian, we can take a maximal ideal $\mathfrak{a}$ in this set, and obtain $B = A/\mathfrak{a}$, with the property that

- B is Noetherian, but not Artinian.
- Every prime ideal of B is maximal
- If $(0) \neq \mathfrak{b} \subseteq B$ is an ideal, then $B/\mathfrak{b}$ is Artinian.

Claim: B is an integral domain.

Proof. If $xy = 0$ in B with $x, y \neq 0$, then we get a short exact sequence of B-modules

$$0 \to B/\operatorname{Ann}(x) \xrightarrow{\cdot x} B \to B/(x) \to 0.$$

The outer two modules are Artinian, by our assumptions, and so B must be, which is a contradiction. □

Now since B is an integral domain and every prime ideal is maximal, it follows that B is a field, which contradicts our assumption that B is not Artinian. □

Summing up, we have shown

Theorem. *Let A be a ring. Then A is Artinian if and only if it is Noetherian and of dimension 0.*

Proposition. *Every Artinian ring A is isomorphic to a product of Artinian local rings.*

More precisely, if $e \geq 1$ is such that $\mathfrak{m}_1^e \cdots \mathfrak{m}_n^e = 0$, then

$$A \cong \prod_{i=1}^{n} A/\mathfrak{m}_i^e.$$

Proof. The ideal $\mathfrak{m}_i^e$ is not contained in $\mathfrak{m}_j$ for $j \neq i$. It follows that $\mathfrak{m}_i^e + \mathfrak{m}_j^e = (1)$ when $j \neq i$, and that $A/\mathfrak{m}_i^e$ is local.

By the Chinese remainder theorem, the natural homomorphism

$$\phi\colon A \to \prod_{i=1}^{n} A/\mathfrak{m}_i^e,$$

is surjective, and $\ker \phi = \mathfrak{m}_1^e \cdots \mathfrak{m}_n^e = (0)$, so it is an isomorphism. □

Theorem. *Every Artinian ring is isomorphic to a product of Artinian local rings.*

Corollary. *A finite type k-algebra A is Artinian if and only if it is a finite k-algebra (i.e. finite-dimensional as a k-module).*

Proof. We have seen the implication $\Leftarrow$.

Since A is Artinian, it is also Noetherian, and we therefore have a composition series

$$A = \mathfrak{a}_0 \supsetneq \mathfrak{a}_1 \supsetneq \cdots \supsetneq \mathfrak{a}_n = 0,$$

where each quotient $\mathfrak{a}_i/\mathfrak{a}_{i+1}$ is a simple A-module. We know that simple A-modules are isomorphic $A/\mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. By the Nullstellensatz, a module of the form $A/\mathfrak{m}$ has finite dimension as a k-module. The short exact sequences

$$0 \to \mathfrak{a}_{i+1} \to \mathfrak{a}_i \to \mathfrak{a}_i/\mathfrak{a}_{i+1} \to 0$$

together with additivity of dimension show that

$$\dim_k A = \sum_{i=0}^{n-1} \dim_k \mathfrak{a}_i/\mathfrak{a}_{i+1},$$

and in particular is finite. □