Lecture 21 – Discrete valuation rings

Recall from last time the notion of **dimension** of a ring A, the maximal length of any chain of prime ideals

$$\mathfrak{p}_0 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

Proposition. *An integral domain A has dimension 1 if and only it is not a field, and every non-zero prime ideal is maximal.*

Proof. If A has dimension 1, there must be a chain

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1$$

of prime ideals, which implies that A is not a field. Assume for a contradiction that there is a prime ideal $\mathfrak{p} \neq (0)$ which is not maximal. Then we can find a maximal $\mathfrak{m}$ containing $\mathfrak{p}$, and so find the chain

$$(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m},$$

contradicting $\dim A = 1$.

Conversely, if A is not a field, there is a maximal ideal $\mathfrak{m} \neq (0)$, and so we have at least one chain

$$(0) \subsetneq \mathfrak{m}.$$

On the other hand, there can be no chain

$$(0) \subsetneq \mathfrak{p} \subsetneq \mathfrak{m},$$

so $\dim A = 1$. □

Proposition. *Let A be a Noetherian integral domain of dimension 1. Then every ideal $\mathfrak{a}$ can be written as a product of primary ideals.*

Proof. If $\mathfrak{a} = (0)$, then $\mathfrak{a}$ is prime and so primary.

Otherwise, the Lasker–Noether theorem asserts that we can write

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_n,$$

where the $\mathfrak{q}_i$ are primary and have distinct radicals $\sqrt{\mathfrak{q}_i}$. These are all maximal, and we have that

$$\sqrt{\mathfrak{q}_i + \mathfrak{q}_j} \supseteq \sqrt{\mathfrak{q}_i} + \sqrt{\mathfrak{q}_j} = (1),$$

hence $1 \in \mathfrak{q}_i + \mathfrak{q}_j$, and these are pairwise coprime. Thus we can replace the intersection by a product and find

$$\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_n.$$

□

Discrete valuation rings

Definition. Let K be a field. A **discrete valuation** on K is a surjective function $v\colon K \setminus \{0\} \to \mathbb{Z} \cup \{\infty\}$ satisfying three properties

(1) For all $x, y \in K$, we have $v(xy) = v(x) + v(y)$.
(2) For all $x, y \in K$, we have $v(x + y) \geq \min(v(x), v(y))$.
(3) $v(x) = \infty \Leftrightarrow x = 0$.

Example. The field $\mathbb{Q}$ admits an valuation v_p, defined as follows. Every rational number x admits a prime factorisation $x = p^a p_1^{e_1} \cdots p_n^{e_n}$, where the primes $p, p_1, \cdots p_n$ are distinct and $a, e_1, \cdots, e_n \in \mathbb{Z}$. We define $v_p(x) = a$.

E.g. $v_2(2) = 1$, $v_2(3/2) = -1$.

Example. Let k be a field. The field $k(x)$ admits a valuation defined by the "order of vanishing at 0". Every element of $k(x)$ can be written as $x^n \frac{f}{g}$, where f and g are polynomials such that $f(0), g(0) \neq 0$, and $n \in \mathbb{Z}$. We define $v(x^n \frac{f}{g}) = n$.

Definition. Let A be an integral domain, with fraction field K. We say that A is a **discrete valuation ring** (or DVR), if there exists some valuation v on K such that for $x \in K$ we have

$$x \in A \Leftrightarrow v(x) \geq 0.$$

Remark. If v is a valuation on a field K, then the set $\{x \in K \mid v(x) \geq 0\}$ is easily seen to be a subring of K. In other words, every field equipped with a valuation contains a DVR determined by the valuation.

Example. For the valuation v_p on $\mathbb{Q}$, the associated discrete valuation ring consists of fractions of the form $p^a \frac{m}{n}$ where $a \geq 0$ and p divides neither m nor n. Equivalently, setting $m' = p^a m$, we see that it consists of all fractions m'/n such that p does not divide n, which is precisely the ring $\mathbb{Z}_{(p)} \subset \mathbb{Q}$.

Example. For the valuation $v(x^n f/g) = n$, the associated DVR is $k[x]_{(x)} \subset k(x)$.

Example. Let k be a field, and let $k((x))$ be the ring of formal Laurent series, i.e. whose elements are formal sums

$$\sum_{i \geq n} a_i x^i,$$

where $n \in \mathbb{Z}$ (so finitely many terms $a_i x^i$ with $i < 0$ are allowed). One can check that this is a field. Setting $v(f) = i$, where i is the smallest integer such that $a_i \neq 0$, we get a discrete valuation on $k((x))$, with associated DVR the ring of formal power series $k[[x]] \subset k((x))$.

Discrete valuation rings have excellent properties.

Theorem. *Let A be a discrete valuation ring with fraction field K and discrete valuation v. Then*

(1) *The ring A is local, with maximal ideal*

$$\mathfrak{m} = \{x \in A \mid v(x) > 0\}.$$

(2) *For any element $x \in A$ such that $v(x) = 1$, we have $\mathfrak{m} = (x)$.*
(3) *With x as in the previous point, every ideal in A is either (0) or equal to (x^k) for some $k \geq 0$.*
(4) *A has dimension 1.*
(5) *A is integrally closed.*

Proof. (1) Let $x \in A$, and consider $x^{-1} \in K$. The element x is a unit in A if and only if $x^{-1} \in A$, which is if and only if $v(x^{-1}) = -v(x) \geq 0$. But we know that $v(x) \geq 0$, so x is a unit if and only if $v(x) = 0$. Thus the set of non-units is precisely the set described in the proposition, which it's easy to see is an ideal.

(2) If $x, y \in A$ and $v(y) \geq v(x)$, then $v(xy^{-1}) \geq 0$, so $xy^{-1} \in A$, which means that $y \in (x)$. Since a discrete valuation is by definition surjective, there exists at least one such x. In particular, $\mathfrak{m} = (x)$ for any element $x \in A$ with $v(x) = 1$.

(3) Let $\mathfrak{a}$ be an ideal, and let $x \in \mathfrak{a}$ be such that $v(x)$ is minimal. Then for any $y \in \mathfrak{a}$, we have $v(y) \geq v(x)$, so as above we find $y \in (x)$. Thus $\mathfrak{a} \subseteq (x)$. Since obviously $(x) \subseteq \mathfrak{a}$, we have $\mathfrak{a} = (x)$.

(4) By the previous two points, we have that the ideals of A are $(1), (x), (x^2), \cdots$ and (0). It is easy to see that (x) and (0) are the only prime ideals of A, so A has dimension 1.

(5) Let $x \in K$, and assume that x is integral over A. We must show that $x \in A$. Since x is integral over A, we can find a relation.

$$x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0, \qquad a_i \in A$$

so

$$x^n = -a_{n-1}x^{n-1} - \cdots - a_0.$$

If $v(x) = d$, we then get

$$\begin{aligned} v(x^n) = nv(x) = nd &= v(-a_{n-1}x^{n-1} - \cdots - a_0) \\ &\geq \min_i(v(-a_i x^i)) \end{aligned}$$

Thus there exists an $i \leq n-1$ such that

$$nd = v(x^d) \geq v(-a_i x^i) = v(-a_i) + v(x^i) \geq id,$$

This gives $(n-i)d \geq 0$, and so $d \geq 0$. Hence $x \in A$.

□

In fact, most of these properties characterise DVRs (among Noetherian local domains of dimension 1).

Proposition. *Let A be a Noetherian, local integral domain of dimension 1. The following are equivalent:*

(1) *A is a DVR.*
(2) *A is integrally closed.*
(3) *$\mathfrak{m}$ is principal.*
(4) *$\mathfrak{m}/\mathfrak{m}^2$ is a 1-dimensional $A/\mathfrak{m}$-module.*
(5) *Every non-zero ideal of A is a power of $\mathfrak{m}$.*
(6) *There exists an $x \in A$ such that every ideal in A is of the form (x^k).*

Proof. We have already seen that (1) implies all the other conditions.

Let us just do a few of the easier other implications.

(4) ⇒ (3): If $\mathfrak{m}/\mathfrak{m}^2$ is 1-dimensional, there is some $x \in \mathfrak{m}$ such that $x + \mathfrak{m}^2$ generates $\mathfrak{m}/\mathfrak{m}^2$. But $\mathfrak{m}$ is finitely generated, since A is Noetherian, and then Nakayama's lemma says that x generates $\mathfrak{m}$.

(3) ⇒ (6): There is an x such that $\mathfrak{m} = (x)$, so that every non-unit in A is of the form ax for some $a \in A$. Assume for a contradiction that $\mathfrak{a}$ is an ideal which is not of the form (x^k), and let it be maximal among ideals with this property (there is such a maximal one since A is Noetherian). We have

$$\mathfrak{a} = (y_1, \ldots, y_n) = (a_1 x, \cdots, a_n x) = (a_1, \cdots, a_n)(x).$$

Now $\mathfrak{a} = (a_1, \ldots, a_n)(x) \subseteq (a_1, \ldots, a_n)$. If $\mathfrak{a} = (a_1, \ldots, a_n)$, we have $\mathfrak{a} = (x)\mathfrak{a}$, which by Nakayama's lemma implies $\mathfrak{a} = (0)$.

Otherwise $\mathfrak{a} \subsetneq (a_1, \ldots, a_n)$, which by the maximality property of $\mathfrak{a}$ implies that $(a_1, \ldots, a_n) = (x^k)$ for some k. But then $\mathfrak{a} = (x)(x^k) = (x^{k+1})$, contradicting the defining property of $\mathfrak{a}$.

(6) $\Rightarrow$ (1): For every $y \neq 0$, we have $\sqrt{(y)} = \mathfrak{m} = (x)$. It then follows that $y \in (x^k)$ for some k, and we take a minimal such. Then define $v(y) = k$, and extend this multiplicatively to the fraction field of A. □

Example. Consider the domain $A = k[x^2, x^3] \subset k[x]$, and consider the maximal ideal $\mathfrak{m} = (x^2, x^3) \subset A$. Then $\mathfrak{m}^2 = (x^4, x^5, x^6)$, and we find that $\mathfrak{m}/\mathfrak{m}^2$ is spanned by $x^2 + \mathfrak{m}^2, x^3 + \mathfrak{m}^2$, so is 2-dimensional as a k-module.

The ring $A_\mathfrak{m}$ has local ideal $\mathfrak{m}_\mathfrak{m}$, and the quotient is given by

$$\mathfrak{m}_\mathfrak{m}/(\mathfrak{m}_\mathfrak{m})^2 = (\mathfrak{m}/\mathfrak{m}^2)_\mathfrak{m} \cong \mathfrak{m}/\mathfrak{m}^2,$$

so is 2-dimensional as an $A/\mathfrak{m}$-module.

It follows that $A_\mathfrak{m}$ is not a DVR.