Lecture 22 – Graded rings and Hilbert polynomials

Definition. A **graded ring** is a ring A together with subgroups $A_i \subseteq A$ for each $i \geq 0$ such that

$$A = \bigoplus_{i=0}^{\infty} A_i,$$

and such that for every $i, j \geq 0$, we have

$$a \in A_i, b \in A_j \Rightarrow ab \in A_{i+j}.$$

Remark. The condition that $A = \bigoplus_{i=0}^{\infty} A_i$ is equivalent to requiring that for every $a \in A$, we can write

$$a = \sum_{i=0}^{\infty} a_i \qquad a_i \in A_i$$

in a unique way (with only finitely many $a_i \neq 0$).

Example. For any ring A, the ring $A[x]$ is graded, by setting

$$A[x]_i = \{ax^i \mid a \in A\} \subset A[x]$$

More generally, the ring $A[x_1, \ldots, x_n]$ is graded by setting

$$f \in A[x_1, \ldots, x_n]_i$$

if and only if f is a sum of terms of the form $ax^{i_1} \cdots x_n^{i_n}$ with $\sum i_k = i$.

Definition. If A is a graded ring and $a \in A_i$, we say that a is **homogeneous of degree** i.

Remark. A given ring A may be considered as a graded ring in different ways, e.g. for $k[x, y]$, we can for instance define a grading by saying $x^i y^j$ is homogeneous of degree $i + j$, or we can say it is homogeneous of degree i.

Remark. Since $A_0 A_0 \subseteq A_0$ and more generally $A_0 A_i \subseteq A_i$ for every i, we have that $A_0 \subseteq A$ is a subring, so A is an A_0-algebra, and every A_i is naturally an A_0-module.

Example. If $f \in A$ is a homogeneous element, then $B = A/(f)$ is also a graded ring, with graded pieces

$$B_i = A_i/(f) \cap A_i.$$

It is an easy exercise to check that this defines a valid grading of B, but note that it is necessary that f is homogeneous.

More generally, if $f_1, \ldots, f_n \in A$ are homogeneous, the ring $A/(f_1, \ldots, f_n)$ inherits a grading from A.

Assumption: We will for the rest of these lectures assume of our graded ring A that $A_0 = k$ is a field, and that A is generated as a k-algebra by homogeneous elements $x_1, \ldots, x_n$ of degree 1. In particular, this implies that we have

$$A \cong k[x_1, \ldots, x_n]/\mathfrak{a},$$

for some ideal $\mathfrak{a} = (f_1, \ldots, f_k)$, where $f_i \in k[x_1, \ldots, x_n]$ are homogeneous elements. The grading of elements in A is inherited from that in $k[x_1, \ldots, x_n]$, i.e. in A we have that

$$ax_1^{i_1} \cdots x_n^{i_n} + \mathfrak{a}$$

is homogeneous of degree $\sum i_k$.

Definition. Let A be a graded ring as above. The **Hilbert function of** A is the function $H_A\colon \mathbb{N} \to \mathbb{N}$ given by $H_A(i) = \dim_k(A_i)$.

Example. Consider $k[x]$. We have $k[x]_n = \{ax^n \mid a \in k\}$ for all n, so $H_{k[x]}(n) = 1$ for all n.

Example. For $k[x_1, \ldots, x_n]$, we have that a basis for $k[x_1, \ldots, x_n]_d$ as a k-module is given by elements

$$x_1^{i_1} \cdots x_n^{i_n}$$

with $i_1 + \cdots + i_n = n$. One can compute the number of such to be

$$\binom{n+d-1}{n-1} = \frac{(n+d-1)(n+d-2)\cdots(d)}{(n-1)!},$$

which gives $H_{k[x_1,\ldots,x_n]}(d) = \binom{n+d-1}{n-1} = \frac{d^{n-1}}{(n-1)!}+$ lower order terms in d.

Example. Let $f \in k[x_1, \ldots, x_n]_i$, and let $A = k[x_1, \ldots, x_n]/(f)$. We have a short exact sequence

$$0 \to k[x_1, \ldots, x_n] \xrightarrow{\cdot f} k[x_1, \ldots, x_n] \to A \to 0,$$

which gives short exact sequences

$$0 \to k[x_1, \ldots, x_n]_{d-i} \xrightarrow{\cdot f} k[x_1, \ldots, x_n]_d \to A_d \to 0,$$

and so

$$\dim A_d = \dim k[x_1, \ldots, x_n]_d - \dim k[x_1, \ldots, x_n]_{d-i}.$$

In particular

$$H_A(d) = \begin{cases} \binom{n+d-1}{n-1} \text{ if } d < i \\ \binom{n+d}{n} - \binom{n+d-i-1}{n-1} \text{ if } d \geq i. \end{cases}$$

Note in particular that for $d \geq i$, we have $H_A(d)$ is a polynomial in d, of the form

$$\frac{id^{n-1}}{(n-2)!} + \text{ lower order terms}$$

Proposition. *Let A be a graded ring as above. There exists a rational polynomial g and an integer N such that, for $n \geq N$, we have $H_A(n) = f(n)$.*

Lemma. *Let $F\colon \mathbb{N} \to \mathbb{N}$ be a function. Assume that there is an $N \geq 0$ and a rational polynomial g such that*

$$F(n+1) - F(n) = g(n)$$

for all $n \geq N$. Then there exists a polynomial f, with $\deg f = \deg g + 1$, such that $F(n) = f(n)$ for all $n \geq N$.

Proof. Let V_d be the space of degree d rational polynomials. We have a $\mathbb{Q}$-linear map $\phi\colon V_d \to V_{d-1}$ given by $f(x) \mapsto f(x+1) - f(x)$. The kernel of ϕ is the set of constant polynomials, and since $\dim V_d = \dim V_{d-1} + 1$, the map ϕ is surjective. We can therefore find an $f \in V_d$ such that $\phi(f) = g$, and by adjusting the constant term of f, we can ensure that $f(N) = F(N)$. By induction on n, starting from N, we then find that $f(n) = F(n)$ for all $n \geq N$. □

Proof of proposition. The proof is by induction on the number of generators of A as a k-algebra. Assume A is generated by elements $x_1, \dots, x_n$, homogeneous of degree 1. If x_1 is not a 0-divisor, we have a short exact sequence

$$0 \to A \overset{\cdot x_1}{\to} A \to A/(x_1) \to 0,$$

which gives, for each i, a short exact sequence

$$0 \to A_i \overset{\cdot x_1}{\to} A_{i+1} \to (A/(x_1))_{i+1} \to 0.$$

We thus have

$$H_A(i+1) = H_A(i) + H_{A/(x_1)}(i+1)$$

or equivalently

$$H_A(i+1) - H_A(i) = H_{A/(x_1)}(i+1).$$

Since $A/(x_1)$ is generated by the elements $x_2, \cdots, x_n$, the induction hypothesis shows that $H_{A/(x_1)}$ is eventually a polynomial, and our lemma shows that the same is then true of H_A.

If x_1 *is* a 0-divisor, the short exact sequence

$$0 \to A/\operatorname{Ann}(x_1) \overset{\cdot x_1}{\to} A \to A/(x_1) \to 0$$

let's us reduce the claim from A to $A/(x_1)$ (handled by induction) and $A/\operatorname{Ann}(x_1)$ (where x_1 is not a 0-divisor, so handled above). □

Definition. We call the polynomial which computes the Hilbert function for large integers the **Hilbert polynomial**.