5. Lecture 5 – Direct sums, submodules and quotient modules

Let A be a ring, and recall that an A-module is an abelian group M equipped with a multiplication map $A \times M \to M$ denoted $(a, m) \to am$, satisfying some axioms. Further, a map $\phi\colon M' \to M$ is a homomorphism if it respects addition and multiplication from A, meaning $\phi(x+y) = \phi(x) + \phi(y)$, and $\phi(ax) = a\phi(x)$.

Important special cases are $\mathbb{Z}$-modules, which are the same things as abelian groups, and k-modules for k a field, which are the same things as vector spaces over k.

5.1. Direct sums.

Given a sequence of abelian groups $G_1, \ldots, G_n$, the product set $G_1 \times \cdots \times G_n$ is naturally an abelian group. This generalises directly to modules:

Definition. Let M_1, M_2 be A-modules. The direct sum of the M_1 and M_2 is the module

$$M_1 \oplus M_2 = \{(m_1, m_2) \mid m_1 \in M_1, m_2 \in M_2\},$$

with

$$(m_1, m_2) + (m_1', m_2') = (m_1 + m_1', m_2 + m_2'), \quad a(m_1, m_2) = (am_1, am_2).$$

Definition. Given A-modules $M_1, \ldots, M_n$, we have the direct sum

$$\bigoplus_{i=1}^{n} M_i = M_1 \oplus \cdots \oplus M_n = \{(m_1, \ldots, m_n) \mid m_i \in M_i\},$$

with addition and A-multiplication similar. If $M_1 = \cdots = M_n = M$, we may write $M^{\oplus n}$ instead.

Given a set of A-modules $\{M_i\}_{i \in S}$, their direct sum is

$$\oplus_{i \in S} M_i = \{(m_i)_{i \in S} \mid m_i \in M_i, \text{ only finitely many } m_i \neq 0\},$$

while their direct product is

$$\prod_{i \in S} M_i = \{(m_i)_{i \in S} \mid m_i \in M_i\},$$

If S is finite, then the direct sum and direct product are the same, but in general they differ.

Example. Let k be a field. Every vector space V over k has a basis, meaning there is a set $\{v_i\}_{i \in S}$ such that every $v \in V$ can be expressed uniquely as a sum

$$\sum_{i \in S} a_i v_i \qquad a_i \in k,$$

with only finitely many $a_i \neq 0$.

Define a homomorphism

$$\phi\colon \bigoplus_{i \in S} k \to V$$

by

$$\phi((a_i)) = \sum_{i \in S} a_i v_i.$$

Since $\{v_i\}_{i \in S}$ is a basis for V, every v equals $\phi((a_i))$ for a unique $(a_i) \in \bigoplus_{i \in S} k$, meaning ϕ is an isomorphism, and $\oplus_{i \in S} k \cong V$.

Example. Consider $\mathbb{R}[x,y]$, and define $T = \mathbb{R}[x,y] \oplus \mathbb{R}[x,y]$. Thus elements of T are pairs (f_1, f_2) with $f_1, f_2 \in \mathbb{R}[x,y]$. We may think of elements of T as vector fields on $\mathbb{R}^2$ with components given by polynomials.

Example. Let A be a ring, and consider $A[x]$ as an A-module, i.e. if $f = a_n x^n + \cdots + a_0 \in A[x]$ and $a \in A$, we have

$$af = aa_n x^n + \cdots aa_1 x + aa_0.$$

We have a homomorphism of A-modules

$$\phi \colon \bigoplus_{i \in \mathbb{N}} A \to A[x],$$

Note that this is just a module isomorphism; in fact the left hand side does not have a natural ring structure.

5.2. **Submodules.** If G is an abelian group, a subset $G' \subseteq G$ which is closed under addition and inverses is a subgroup. We can then form the quotient group $G'' = G/G'$, whose elements are the cosets of G' in G. This concept and most of the theory generalises neatly from abelian groups to modules, where we defined submodules as follows.

Definition. Let M be an A-module. A subset $M' \subseteq M$ is a **submodule** if it is a subgroup and for all $a \in A, m \in M'$, we have $am \in M'$.

Example.
- A submodule of A is the same thing as an ideal in A.
- A submodule of a $\mathbb{Z}$-module M is the same thing as a subgroup of M, since if $M' \subseteq M$ is a subgroup, $n \in \mathbb{Z}$ and $m' \in M'$, we automatically have $nm' = m' + \cdots + m' \in M'$ (when n is positive, similar arguments work when n is negative).

Given $M, M' \subseteq N$, we have their sum defined as $M + M' \subseteq N$, given by

$$M + M' = \{m + m' \mid m \in M, m' \in M\}.$$

This generalises the notion of sum of ideals.

Example. With ring $\mathbb{R}[x,y]$ and $T = \mathbb{R}[x,y]^{\oplus 2}$, we have the submodule $T' \subset T$ given by

$$T' = \{(fx, fy) \mid f \in \mathbb{R}[x,y]\}$$

Informally, this is the submodule of vector fields which point outwards from the origin at all points. We have $\phi \colon \mathbb{R}[x,y] \to T$ given by $\phi(f) = (fx, fy)$, and this is an isomorphism.

Let's take $T'' = \{(g, 0) \mid g \in \mathbb{R}[x,y]\} \subset T$, this is again a submodule, the horizontal vector fields.

We have

$$T' + T'' = \{(fx + g, fy) \mid f, g \in \mathbb{R}[x,y]\} = \{(h, fy) \mid h, f \in \mathbb{R}[x,y]\},$$

vector fields which are horizontal along the x-axis.

5.3. **Quotients.** If M' is a submodule of M, then the group M/M' has a natural structure of A-module such that $M \to M/M'$ is a homomorphism of A-modules. Concretely, we define the A-multiplication on M' by

$$a(m + M') = am + M'$$

In particular, for any ideal $\mathfrak{a}$, the quotient ring $A/\mathfrak{a}$ is an A-module.

5.4. **Kernels, images and cokernels.**

Definition. Let $\phi\colon M \to N$ be a homomorphism of A-modules. We have

- The **kernel** of ϕ,
$$\ker\phi \subseteq M,$$
a submodule of M.
- The **image** of ϕ,
$$\operatorname{im}\phi = \{\phi(m) \mid m \in M\} \subseteq N,$$
a submodule of N.
- The **cokernel** of ϕ,
$$\operatorname{cok}\phi = N/\operatorname{im} M.$$

Example. Let $a_1, \dots, a_n \in A$, and define
$$\phi\colon \bigoplus_{i=1}^{n} A \to A$$
by
$$\phi(x_1, \dots, x_n) = \sum_{i=1}^{n} x_i a_i.$$
Then
$$\operatorname{im}(\phi) = \{x_1a_1 + x_2a_2 + \cdots + x_na_n \mid x_i \in A\} = (a_1, \dots, a_n) \subseteq A.$$

The following statements are "well known" for abelian groups, and the content of this proposition is that the natural isomorphisms respect the module structures as well.

Proposition ("Module isomorphism theorems").
- *Let $\phi M \to N$ be a homomorphism of modules. We have*
$$\operatorname{im} M \cong M/\ker\phi.$$
- *Let $M'' \subseteq M' \subseteq M$ be A-modules and submodules. There is an isomorphism*
$$M/M'' \cong (M/M')/(M''/M')$$
- *Let M, N be submodules of P. We then have*
$$(M+N)/N \cong M/(M \cap N)$$

Definition. A module M is **finitely generated** if either of the following two equivalent conditions hold:

- There exists $m_1, \dots, m_n \in M$ such that every $m \in M$ is of the form $x_1m_1 + \cdots + x_nm_n$, with $x_i \in A$.
- There exists a surjective homomorphism $\phi\colon \bigoplus_{i=1}^{n} A \to M$.

Example.
- An abelian group is finitely generated as a group if and only if it is finitely generated as a $\mathbb{Z}$-module.

  So every finitely generated $\mathbb{Z}$-module is isomorphic to one of the form
$$\mathbb{Z} \oplus \mathbb{Z} \oplus \cdots \mathbb{Z} \oplus \mathbb{Z}/(p_1^{e_1}) \oplus \cdots \oplus \mathbb{Z}/(p_n^{e_n}),$$
while $\mathbb{Q}$ is not a finitely generated $\mathbb{Z}$-module.
- If k is a field, then a finitely generated k-module is the same thing as a finite-dimensional k-vector space.

- An ideal $\mathfrak{a} \subseteq A$ is finitely generated as an A-module if and only if it is finitely generated as an ideal, i.e. it is of the form $(a_1, \ldots, a_n)$.

Main ideas:

- Direct sums and products of modules
- Submodules
- Sums and intersections of modules
- “The module isomorphism theorems”
- Kernels, images and cokernels
- The “module isomorphism theorems”