Lecture 9 – Exactness of the tensor product and algebras

Let A be a ring, let M and N be A-modules. Recall we then have a module

$$M \otimes_A N,$$

the **tensor product** of M and N over A, which has the property that for an A-module P, we have

$$\mathrm{Hom}(M \otimes_A N, P) \leftrightarrow \{A - \text{bilinear maps } M \times N \to P\}.$$

Elements of $M \otimes_A N$ are of the form

$$\sum_{i=1}^{n} x_i \otimes y_i, \qquad x_i \in M, y_i \in N.$$

These satisfy relations, for all $x, x' \in M, y, y' \in N, a \in A$, that

$$(x + x') \otimes y = x \otimes y + x' \otimes y$$
$$x \otimes (y + y') = x \otimes y + x \otimes y'$$
$$ax \otimes y = x \otimes ay$$

Two expressions $\sum x_i \otimes y_i$ and $\sum x_i' \otimes y_i'$ are equal if and only if one of them can be rewritten into the other by using these relations.

Recall also that if I is an ideal in A and M is an A-module, then

$$A/I \otimes_A M \cong M/IM,$$

where the isomorphism is given by

$$(a + I) \otimes m \mapsto am + IM$$

As a special case of $I = (0)$, we get $A \otimes_A M \cong M$.

6.2. Flat A-modules.

Proposition. *If M is an A-module and*

$$N' \to N \to N'' \to 0$$

is an exact sequence of A-modules, then the induced sequence

$$N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$$

is exact.

In categorical language, this proposition says that the functor of modules $N \mapsto M \otimes_A N$ is **right exact**.

Example. Consider the $\mathbb{Z}$-module $\mathbb{Z}/(2)$, and the exact sequence

$$0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/(2) \to 0.$$

We have $\mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z}/(2) \cong \mathbb{Z}/(2)$, and $\mathbb{Z}/(2) \otimes \mathbb{Z}/2 \cong \mathbb{Z}/2$. The sequence

$$\mathbb{Z} \otimes \mathbb{Z}/(2) \to \mathbb{Z} \otimes \mathbb{Z}/(2) \to \mathbb{Z}/(2) \otimes \mathbb{Z}/(2) \to 0$$

then becomes, after replacing each tensor product with its isomorphic module

$$\mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2) \to \mathbb{Z}/(2) \to 0.$$

In particular, since the leftmost map is not injective, the sequence

$$0 \to \mathbb{Z}/(2) \xrightarrow{0} \mathbb{Z}/(2) \to \mathbb{Z}/(2) \to 0$$

is not exact.

To be even more concrete, note that the element $1 \otimes 1 \in \mathbb{Z} \otimes \mathbb{Z}/(2)$ is mapped to

$$2 \otimes 1 = 1 \otimes 2 = 1 \otimes 0 = 0 \otimes 0 = 0,$$

and so lies in the kernel of $\mathbb{Z} \otimes \mathbb{Z}/2 \to \mathbb{Z} \otimes \mathbb{Z}/2$.

Definition. An A-module M is **flat** if for all injective homomorphisms $N' \to N$, the induced homomorphism $M \otimes N' \to M \otimes N$ is injective.

Remark. Equivalently, M is flat if for every short exact sequence

$$0 \to N' \to N \to N'' \to 0,$$

the sequence

$$0 \to N' \otimes M \to N \otimes M \to N'' \otimes M \to 0$$

is exact.

Example. The previous example shows that $\mathbb{Z}/(2)$ is not flat as a $\mathbb{Z}$-module.

Example. For any ring A, and A-module N, we have that $A \otimes N \cong N$. If $N' \to N$ is injective, then clearly $A \otimes N' \to A \otimes N$ is injective, so A is flat as an A-module.

Example. Given a collection of modules $\{M_i\}_{i \in S}$ and a module N, we have natural isomorphism

$$(\oplus_{i \in S} M_i) \otimes N \cong \oplus_{i \in S} M_i \otimes N.$$

Now if $N' \to N$ is injective and all the M_i are flat, then the morphism

$$\oplus_{i \in S} M_i \otimes N' \to \oplus_{i \in S} M_i \otimes N$$

is injective. This means $\oplus_{i \in S} M_i$ is also flat.

Example. If k is a field, then every k-module M is isomorphic to $\bigoplus_{i \in S} k$, which we know is flat. In other words, every k-module is flat.

Example. The $\mathbb{Z}$-module $\mathbb{Q}$ is flat, but we won't show this quite yet.

6.3. **Algebras.**

Definition. A pair (B, ϕ) of a ring B and a ring homomorphism $\phi\colon A \to B$ is called an A-**algebra**.

If (B, ϕ) and (C, ψ) are A-algebras, then a ring homomorphism $\chi\colon B \to C$ is a **homomorphism of** A**-algebras** if $\chi \circ \phi = \psi$.

We usually omit the homomorphism $\phi\colon A \to B$ from the notation, and just say "B is an A-algebra".

Example. For any ring A, the polynomial ring $A[x_1, \ldots, x_n]$ is naturally an A-algebra, as are all its quotients $A[x_1, \ldots, x_n]/I$.

Example. For any ring A, there is a unique ring homomorphism $\phi\colon \mathbb{Z} \to A$ given by $\phi(n) = n1_A$, so every ring is a $\mathbb{Z}$-algebra in precisely one way.

Definition. Let B be an A-algebra. We say B is a

- **finite** A-algebra if B is finitely generated as an A-module, i.e. if there exist finitely many elements $b_1, \ldots, b_n \in B$ such that every element of B can be written as

$$\sum_{i=1}^{n} a_i b_i \qquad a_i \in A, b_i \in B$$

- **finite type** A-algebra if there exist a finite set of elements $b_1, \dots, b_n \in B$ such that every element in B can be written on the form

$$\sum_{i_j \geq 0} a_{i_1,\dots,i_n} b_1^{i_1} \cdots b_n^{i_n}$$

Remark. An A-algebra B is of finite type if and only if it is isomorphic as an A-algebra to

$$A[x_1, \dots, x_n]/I$$

for some ideal $I \subseteq A[x_1, \dots, x_n]$.

Proof: If B is isomorphic as an A-algebra to $A[x_1, \cdots, x_n]/I$ via $\phi\colon A[x_1, \dots, x_n]/I$, then we also have a surjection of A-algebras

$$\psi\colon A[x_1, \dots, x_n] \to A[x_1, \dots, x_n]/I \to B.$$

Taking $b_i = \psi(x_i)$, we have that every element $b \in B$ can be written

$$b = \psi(\sum a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}) = \sum a_{i_1,\dots,i_n} b_1^{i_1} \cdots b_n^{i_n},$$

proving that B is finitely generated.

Conversely, if B is generated by $b_1, \dots, b_n$, then there is a surjective A-algebra homomorphism $\phi\colon A[x_1, \dots, x_n] \to B$ given by

$$\phi\left(\sum_{i_j \geq 0} a_{i_1,\dots,i_n} x_1^{i_1} \cdots x_n^{i_n}\right) = \sum_{i_j \geq 0} a_{i_1,\dots,i_n} b_1^{i_1} \cdots b_n^{i_n}$$

Remark. A finite A-algebra is finite type, but not vice versa, e.g. $A[x]$ is not a finite A-algebra.