Suggested problems week 11

From Atiyah–Macdonals: Problem 6.6., 7.4.

(1) Let A be a ring such that every ideal is principal. Prove that $\dim A \leq 1$.

(2) Let v be a discrete valuation on a field K. Prove that if $x, y \in K$ are such that $v(x) < v(y)$, then $v(x+y) = v(x)$.

(3) Let v be a discrete valuation on a field K. Prove that if $x \in K$ is such that $x^n = 1$ for some $n \geq 1$, then $v(x) = 0$.

(4) Let A be an integral domain. Let $v\colon A \to \mathbb{Z} \cup \{\infty\}$ be a function such that
$$v(x) = \infty \Leftrightarrow x = 0.$$
and such that for all $x, y \in A$ we have
$$v(xy) = v(x) + v(y)$$
and
$$v(x) + v(y) \geq \min(v(x), v(y)).$$
Assume further that the elements $\{v(x) \mid x \in A\} \subset \mathbb{Z}$ generate $\mathbb{Z}$ as a group. Prove that v can be extended uniquely to give a discrete valuation on the fraction field of A.

(5) Let k be a field. Prove that there is a unique discrete valuation v on $k(x)$ such that
$$v(f) = -n$$
if $f \in k[x]$ is a polynomial of degree n. Describe the associated discrete valuation ring and find a generator of its maximal ideal.

(6) Let k be a field. Prove that there exists a discrete valuation v on $k(x, y)$ such that
$$v(\sum a_{i,j} x^i y^j) = n,$$
where $n = \min\{i + j \mid a_{i,j} \neq 0\}$. Describe the associated discrete valuation ring and find a generator of its maximal ideal.