5. Suggested problems week 2

(*) means I suspect a problem is difficult (and you're not likely to miss anything important by skipping it).

(1) Let A be a ring, let $a_1, \dots, a_n \in A$ and $b_1, \dots, b_m \in A$. Prove that

$$(a_1, \dots, a_n)(b_1, \dots, b_m) = (a_1 b_1, a_1 b_2, \dots, a_n b_m),$$

where the sequence in the rightmost brackets contains every product $a_i b_j$ with $1 \leq i \leq n$ and $1 \leq j \leq m$.

(2) Let A be a ring, and let $a \in A$. Prove that $A[x]/(x-a) \cong A$.

(3) Let A be a ring, let $a, b \in A$. Prove that $A[x]/(x-a, b) \cong A/(b)$.

(4) Let k be a field, and let $f, g \in k[x]$. Prove that if (f) and (g) are coprime if and only if f and g have no common irreducible factor.[3]

(5) Let k be a field, and $0 \neq f \in k[x]$. Prove that $k[x]/(f)$ is isomorphic to a direct product of rings of the form $k[x]/(g^n)$, where g is irreducible.

(6) Let $(a_1, b_1) \neq (a_2, b_2) \in \mathbb{C}^2$. Prove that the ideals $(x - a_1, y - b_1)$ and $(x - a_2, y - b_2)$ in $\mathbb{C}[x, y]$ are coprime.

(7) Show that the ideals $(x-2)$ and $(2x^2 - 2)$ in $\mathbb{Z}[x]$ are not coprime.

(8) Let $a_1, \dots, a_n, b \in A$, and let $1 \leq i, j \leq n$ with $i \neq j$. Prove that

$$(a_1, \dots, a_n) = (a_1, \dots, a_{j-1}, a_j + b a_i, a_{j+1}, \dots, a_n)$$

(9) Let $a_1, \dots, a_n \in A$, let $1 \leq i \leq n$, and let $c \in A$ be a unit. Prove that

$$(a_1, \dots, a_n) = (a_1, \dots, a_{i-1}, c a_i, a_{i+1}, \dots, a_n).$$

(10) (*) Let $a_1, \dots, a_n \in A$, and let $(b_{ij})_{1 \leq i,j \leq n}$ be an invertible matrix with coefficients in A.[4] Prove that

$$(a_1, \dots, a_n) = \left(\sum_{j=1}^n b_{1j} a_j, \sum_{j=1}^n b_{2j} a_j, \dots, \sum_{j=1}^n b_{nj} a_j \right).$$

(11) (*) Let $A = \mathbb{Z}[x]/(x^2+1)$, and let p be a prime. Show that the extension $(p)^e$ of (p) along $\phi\colon \mathbb{Z} \to A$ is a prime if and only if -1 is a quadratic residue modulo p, that is if and only if the equation x^2+1 admits solutions in $\mathbb{Z}/(p)$. *Hint:* Show that $A/(p)^e \cong \mathbb{Z}/(p)[x]/(x^2+1)$ and consider what it means for this ring to be an integral domain.

(12) Let $\mathfrak{a}, \mathfrak{p} \subseteq A$ be ideals with $\mathfrak{p}$ prime $\mathfrak{a}$ not contained in $\mathfrak{p}$. Prove that $(\mathfrak{p} : \mathfrak{a}) = \mathfrak{p}$.

(13) Let G be an abelian group, and let $\phi\colon G \to G$ be a group homomorphism. Prove that there is a unique structure of $\mathbb{Z}[x]$-module on G such that $xg = \phi(g)$ for all $g \in G$.

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[3] Recall that in $k[x]$ every ideal is principal, every polynomial can be factored into irreducible polynomials, and $h \in k[x] \setminus \{0\}$ is irreducible if and only if (h) is prime.

[4] This means that there exist $(c_{ij})_{1 \leq i,j \leq n}$ with $c_{ij} \in A$ such that

$$\sum_{k=1}^n b_{ik} c_{kj} = \sum_{k=1}^n c_{ik} b_{kj} = \begin{cases} 1 \text{ if } i = j \\ 0 \text{ if } i \neq j \end{cases}.$$

(14) Let G be an abelian group and $n \geq 1$. Show that G admits a structure of $\mathbb{Z}/(n)$-module if and only if $\overbrace{g + \cdots + g}^{n} = 0$ for all $g \in G$. Show that if this condition holds, then the structure of $\mathbb{Z}/(n)$-module on G is unique.

(15) Let A be an integral domain and $0 \neq a \in A$. Show that A and (a) are isomorphic as A-modules.

(16) Let M be an A-module. Show that the map $F\colon \mathrm{Hom}_A(A, M) \to M$ given by

$$F(\phi) = \phi(1)$$

is an isomorphism of A-modules.