Suggested problems week 3

(1) Let A be a ring and $A[[x]]$ be the ring of power series with coefficients in A. Think of $A[[x]]$ as an A-module via the homomorphism Prove that $\prod_{i\in\mathbb{N}} A$

(2) Let $M_1, \dots, M_n$ and N be A-modules. Prove that $\mathrm{Hom}(M_1 \oplus \cdot \oplus M_n, N)$ is isomorphic as an A-module to $\mathrm{Hom}(M_1, N) \oplus \cdots \oplus \mathrm{Hom}(M_n, N)$.

(3) Let $\{M_i\}_{i\in S}$ and N be A-modules. Prove that

$$\mathrm{Hom}(\bigoplus_{i\in S} M_i, N) \cong \prod_{i\in S} \mathrm{Hom}(M_i, N)$$

and

$$\mathrm{Hom}(N, \prod_{i\in S} M_i) \cong \prod_{i\in S} \mathrm{Hom}(N, M_i)$$

as A-modules.

(4) Let A be a ring, let $M = A \oplus A$, and let $\phi\colon M \to M$ be the homomorphism given by $\phi(x, y) = (0, x)$. Prove that the sequence

$$\cdots \xrightarrow{\phi} M \xrightarrow{\phi} M \xrightarrow{\phi} M \xrightarrow{\phi} \cdots$$

is exact.

(5) Let k be a field, let $n \geq 3$

$$0 \to V_1 \to \cdots \to V_n \to 0$$

be an exact sequence of finite-dimensional vector spaces. Prove that the homomorphism $V_{n-2} \to V_{n-1}$ is surjective if and only if $\sum_{i=1}^{n-1}(-1)^i \dim V_i = 0$.

(6) Let $A \subset \mathbb{R}(x)$ be the ring of rational functions defined at 0 as in the lecture notes, with maximal ideal $\mathfrak{m} \subset A$. Thinking of $\mathbb{R}(x)$ as an A-module, prove that $\mathbb{R}(x) = \mathfrak{m}\mathbb{R}(x)$. Why does this not contradict Nakayama's lemma?

(7) Let A be a local ring with maximal ideal $\mathfrak{m}$, let $\phi\colon N \to M$ be a homomorphism of A-modules with M finitely generated. Use the diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & \mathfrak{m}N & \longrightarrow & N & \longrightarrow & N/\mathfrak{m}N & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow{\scriptstyle\phi} & & \downarrow & & \\
0 & \longrightarrow & \mathfrak{m}M & \longrightarrow & M & \longrightarrow & M/\mathfrak{m}M & \longrightarrow & 0,
\end{array},$$

to show that if $N/\mathfrak{m}N \to M/\mathfrak{m}M$ is an isomorphism, then $\mathfrak{m}M \subseteq \phi(\mathfrak{m}N)$.