# Suggested exercises week 4

From Atiyah–Macdonald Chapter 2: Problems 1, 2, 4, 5, 8, 9, 12.

(1) Let $M$ and $N$ be vector spaces over $k$, with $e_1, \ldots, e_m$ a basis for $M$ and $f_1, \ldots f_n$ a basis for $N$. Prove that $M \otimes_k N$ has a basis given by

$$e_1 \otimes f_1, \ldots, e_m \otimes f_n.$$

*Hint:* Use the "distributive law" (Prop. 2.14 (iii)) of $\oplus$ and $\otimes$.

(2) Show that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z} = \mathbb{Q}$ and $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}/(n) = 0$.

(3) An inclusion of $A$-modules

$$i \colon N' \to N$$

is called a **split** inclusion if there exists a homomorphism $p \colon N \to N'$ such that $p \circ i = 1_{N'}$. Prove that if $M$ is an $A$-module and an inclusion $i$ as above is split, then the homomorphism

$$i \otimes 1_M \colon N' \otimes_A M \to N \otimes_A M$$

is injective.

(4) Prove that if $N$ is a free $A$-module and $i \colon N' \to N$ is injective, then $i$ is a split inclusion.

(5) Let $k$ be a field. Prove that every injective map of $k$-modules is a split inclusion, and use this to show (as we saw in the lecture) that every $k$-module is flat.

(6) Let $A$ be an integral domain, and let $k(A)$ be its fraction field. Prove that if $0 \neq I \subseteq A$ is an ideal, then $A/I \otimes_A k(A) \cong 0$.

(7) Let $A$ be an integral domain. Prove that $A/I$ is a flat $A$-module if and only if $I = 0$ or $I = A$. *Hint:* Consider the injection of $A$-modules $A \to k(A)$.

(8) Show that the $A$-module $M \oplus N$ is flat if and only if both $M$ and $N$ are flat.