Suggested exercises week 5

(1) Let B be an A-algebra, and let $I \subset B$ be an ideal. Prove that if B is a finite A-algebra, then so is B/I.

(2) Let B be an A-algebra, and let $I \subset B$ be an ideal. Prove that if B is a finite type A-algebra, then so is B/I.

(3) Let A be a ring, and let $I \subseteq A$ an ideal. Prove that if A/I is a flat A-module, then $I = I^2$. *Hint:* Consider the inclusion of A-modules $I \to A$.

(4) Prove that $\mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Q}$ is isomorphic to $\mathbb{Q}$ as a ring.

(5) Let B be an A-algebra. Prove that

$$A[x] \otimes_A B \cong B[x]$$

(6) Let A be a ring and $I, J \subseteq A$ ideals. Prove that we have an isomorphism of A-algebras

$$A/I \otimes A/J \to A/(I + J).$$

(7) Let A be a ring, let B and C be A-algebras, with A-algebra structure given by

$$\psi_B \colon A \to B \text{ and } \psi_C \colon A \to C.$$

Let $\phi_B \colon B \to B \otimes_A C$ and $\phi_C \colon C \to B \otimes_A C$ be given by $\phi_B(b) = b \otimes 1$ and $\phi_C(c) = 1 \otimes c$. Verify that the following diagram of ring homomorphisms is commutative.

$$\begin{array}{ccc} A & \xrightarrow{\psi_B} & B \\ \downarrow{\scriptstyle \psi_C} & & \downarrow{\scriptstyle \phi_B} \\ C & \xrightarrow{\phi_C} & B \otimes_A C, \end{array}$$

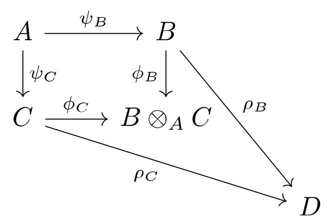
that is, that $\phi_B \circ \phi_B = \phi_C \circ \psi_C$.

(8) (*) With notation as above, let D be a ring, and let $\rho_B \colon B \to D$ and $\rho_C \colon C \to D$ be ring homomorphisms such that $\rho_B \circ \psi_B = \rho_C \circ \psi_C$. Prove that there is a unique ring homomorphism

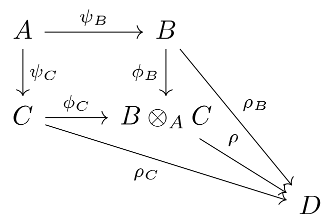
$$\rho \colon B \otimes_A C \to D$$

such that $\rho_B = \rho \circ \phi_B$ and $\rho_C = \rho \circ \phi_C$.

In diagrams: Prove that the commutative diagram of ring homomorphisms



can be extended uniquely to a commutative diagram



(9) Let $S \subseteq A$ be a multiplicatively closed subset. Show that the kernel of the homomorphism $\phi\colon A \to S^{-1}A$ is the set

$$\bigcup_{s \in S} \operatorname{Ann}(s) \subseteq A$$

(10) Let A be a ring and $f \in A$. Prove that $A[x]/(fx - 1) \cong A_f$.

(11) Let A be a ring and let $S \subseteq A$ be a multiplicatively closed subset. Prove that if $S^{-1}A = 0$, then $0 \in S$.