Suggested problems week 7

From Atiyah–Macdonald chapter 4, problems 2 (assume $\mathfrak{a}$ is decomposable), 4, 5.

(1) Let $A = \mathbb{Z}[x]/(x^2+5)$.
  (a) Let p be a prime. Show that the ideal $(p) \subseteq A$ is a prime ideal if and only if the congruence equation $t^2 + 5 \equiv 0 \pmod{p}$ has no solutions.
  (b) Prove that (2) is a primary ideal in A.
  (c) Prove that $(3) = (3, x-1) \cap (3, x+1)$ is a minimal primary decomposition of (3).
  (d) Compute the primary decomposition of (6) as given in the lecture notes.

(2) An integral domain has dimension 1 if it is not a field and every non-zero prime ideal is maximal.
  (a) Prove that $\mathbb{Z}$ and $k[x]$ have dimension 1.
  (b) For the rest of the problem, let A be an integral domain of dimension 1. Prove that an ideal $\mathfrak{a} \subseteq A$ is primary if and only if $\sqrt{\mathfrak{a}}$ is prime.
  (c) Let $\mathfrak{q}, \mathfrak{q}'$ be non-zero primary ideals with different radicals. Prove that $\mathfrak{q}$ and $\mathfrak{q}'$ are coprime.
  (d) Prove that if $\mathfrak{a} \subseteq A$ admits a primary decomposition, then it admits a product decomposition

$$\mathfrak{a} = \mathfrak{q}_1 \cdots \mathfrak{q}_n,$$

  with each $\mathfrak{q}_i$ primary.
  (e) With $\mathfrak{a}$ as above, prove that every prime associated with $\mathfrak{a}$ is minimal.

(3) (*) Let $A = k[x,y]/(y^2 - x^3 + x)$. We want to prove that $y, x, x-1, x+1 \in A$ are irreducible, to show that A does not have unique factorisation.
  (a) Prove that every $f \in A$ can be expressed uniquely as

$$f = g_0 + y g_1 \qquad g_i \in k[x].$$

  (b) Prove that the map $\phi\colon A \to k[x]$ defined by

$$\phi(g_0 + y g_1) = g_0^2 - (x^3 - x) g_1^2$$

  is multiplicative, that is

$$\phi(f f') = \phi(f)\phi(f')$$

  for all $f, f' \in A$.
  (c) Prove that for any $g_0, g_1 \in k[x]$, we have

$$\deg \phi(g_0 + y g_1) \neq 1.$$

  (d) Prove that if $\deg(g_0 + y g_1) = 0$, then $g_0 + y g_1$ is a unit in A.
  (e) Prove that if $y = f f'$, then either f or f' is a unit.
  (f) Prove the same thing for x, $x-1$ and $x+1$ instead of y.