UiO : Department of Mathematics
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MAT4200 Autumn 2023

Commutative algebra, half term revision

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Rings

A **ring** is a commutative ring with unity: a set A equipped with addition $a+b$ and multiplication ab such that

- $(A,+)$ is an abelian group
- Multiplication is associative and commutative
- The distributive law $a(b+c) = ab + ac$ holds
- There is an element $1 \in A$ such that $1a = a$ for all a.

Basic examples

- $\mathbb{Z}, \mathbb{Z}/n, \mathbb{Q}, \mathbb{R}, \mathbb{C}$.
- Given a ring A, the ring $A[x_1, \dots, x_n]$ of polynomials in n variables.

Homomorphisms

A **homomorphism** of rings is a map $\varphi\colon A \to B$ preserving addition, multiplication and 1, i.e. for all $x, y \in A$ we have

$$\varphi(x+y) = \varphi(x) + \varphi(y),\ \varphi(xy) = \varphi(x)\varphi(y),\ \varphi(1) = 1.$$

If φ is bijective it is an **isomorphism**, write $A \cong B$.

Basic examples

- The natural maps $\mathbb{Z} \to \mathbb{Q} \to \mathbb{R} \to \mathbb{C}$ and $\mathbb{Z} \to \mathbb{Z}/n$ are homomorphisms.
- For a ring A and elements $b_1, \ldots, b_n \in A$, a homomorphism $\varphi\colon A[x_1, \ldots, x_n] \to A$ given by

$$\varphi(\sum a_{i_1 \cdots i_n} x_1^{i_1} \cdots x_n^{i_n}) = \sum a_{i_1 \cdots i_n} b_1^{i_1} \cdots b_n^{i_n}.$$

Ideals and quotient rings

- An **ideal** is a subgroup $\mathfrak{a}$ of $(A, +)$ such that $ax \in \mathfrak{a}$ for all $a \in A, x \in \mathfrak{a}$.
- The **quotient ring** $A/\mathfrak{a}$ is the set of cosets of $\mathfrak{a}$ in $(A, +)$, with addition and multiplication defined by

$$(x + \mathfrak{a}) + (y + \mathfrak{a}) = (x + y) + \mathfrak{a}$$
$$(x + \mathfrak{a})(y + \mathfrak{a}) = xy + \mathfrak{a}$$

- The **quotient homomorphism** $\varphi \colon A \to A/\mathfrak{a}$ is given by

$$\varphi(x) = x + \mathfrak{a},$$

Ideals, examples

- For any homomorphism of rings $\varphi\colon A \to B$, the **kernel** $\ker \varphi = \varphi^{-1}(0) \subseteq A$ is an ideal, and $a + \ker \varphi \mapsto \varphi(a)$ gives an isomorphism
$$A/\ker \varphi \cong \mathrm{im}\ \varphi$$
- Given $f \in A$, the ideal
$$(f) = \{xf \mid x \in A\}$$
is the **principal ideal** generated by f.
- Given $f_1, \ldots, f_n \in A$, the ideal **generated by** the f_i is
$$(f_1, f_2, \ldots, f_n) = \{\sum_{i=1}^{n} x_i f_i \mid x_i \in A\}.$$

Quotient rings, examples

- $\mathbb{Z}/(n) =$ ring of integers mod n.
- For $n \geq 1$ such that n is not a square number,

$$\mathbb{Z}[x]/(x^2 - n) = \mathbb{Z}[\sqrt{n}] \cong \{a + b\sqrt{n} \mid a, b \in \mathbb{Z}\} \subset \mathbb{Q}.$$

- For $n \leq -1$,

$$\mathbb{Z}[x]/(x^2 - n) = \mathbb{Z}[i\sqrt{n}] \cong \{a + ib\sqrt{n} \mid a, b \in \mathbb{Z}\} \subset \mathbb{C}.$$

- Given $a_1, \ldots, a_n \in A$, $A[x_1, \ldots, x_n]/(x_1 - a_1, \ldots, x_n - a_n) \cong A$.
- If k is a field, $f \in k[x]$ irreducible, then $k[x]/(f)$ is a finite extension field of k.

Prime and maximal ideals

- $x \in A$ is **unit** if there is $y \in A$ such that $yx = 1$.
- An ideal $\mathfrak{p} \subseteq A$ is **prime** if $x, y \notin \mathfrak{p}$ implies $xy \notin \mathfrak{p}$.
- An ideal $\mathfrak{m} \subseteq A$ is **maximal** if there is no ideal $\mathfrak{m} \subsetneq \mathfrak{a} \subsetneq (1)$.
- A ring A is an **integral domain** if $x, y \neq 0$ implies $xy \neq 0$.
- A ring A is a **field** if $x \neq 0$ implies x is a unit.

Theorem

Let $\mathfrak{a} \subseteq A$ be ideal.

- $\mathfrak{a}$ *prime* $\Leftrightarrow A/\mathfrak{a}$ *is an integral domain.*
- $\mathfrak{a}$ *maximal* $\Leftrightarrow A/\mathfrak{a}$ *is a field.*
- $\mathfrak{a}$ *maximal* $\Rightarrow \mathfrak{a}$ *prime.*

Prime and maximal ideals, examples

- In $\mathbb{Z}$, the prime ideals are (0) and (p) for all primes p. The maximal ideals are the primes except for (0).
- If k is a field, then in $k[x]$, the prime ideals are (0) and (f) with f irreducible. The maximal ideals are the prime ideals except for (0).
- Let k be a field, $a_1, \ldots, a_n \in k$. Then $(x_1 - a_1, x_2 - a_2, \ldots, x_n - a_n) \subset k[x_1, \ldots, x_n]$ is a maximal ideal.
- If $f \in k[x_1, \ldots, x_n]$ is irreducible, then (f) is prime.

Maximal ideals exist

Theorem

If $A \neq 0$, then there is a maximal ideal $\mathfrak{m} \subset A$.

Strengthening

If $\mathfrak{a} \subsetneq A$, then there is a maximal ideal $\mathfrak{m} \supseteq \mathfrak{a}$.

Corollary

$f \in A$ is a unit $\Leftrightarrow (f) = (1) \Leftrightarrow f$ lies in no maximal ideal.

Operations on ideals

Given ideals $\mathfrak{a}_1, \ldots, \mathfrak{a}_n \subseteq A$, we define new ideals by:

- Ideal sum: $\mathfrak{a}_1 + \cdots + \mathfrak{a}_n = \{x_1 + \cdots + x_n \mid x_i \in \mathfrak{a}_i\}$
- Intersection: $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n$
- Product:

$$\mathfrak{a}_1 \cdots \mathfrak{a}_n = \{\text{all finite sums of expressions } x_1 x_2 \cdots x_n \mid x_i \in \mathfrak{a}_i\}$$

- Ideal quotient

$$(\mathfrak{a}_1 : \mathfrak{a}_2) = \{x \in A \mid x\mathfrak{a}_2 \subseteq \mathfrak{a}_1\}.$$

The **radical** of an ideal $\mathfrak{a}$ is given by

$$\sqrt{\mathfrak{a}} = \{x \in A \mid x^n \in \mathfrak{a} \text{ for some n} \geq 0\}$$

Finitely generated ideals

An ideal $\mathfrak{a} \subseteq A$ is **finitely generated** if we can find $f_1, \dots, f_n \in A$ such that

$$\mathfrak{a} = (f_1, f_2, \dots, f_n).$$

We say that $f_1, \dots, f_n$ generate $\mathfrak{a}$.

Rewriting finitely generated ideals

Given $a, f_1, \dots, f_n \in A$

$$(f_1, \dots, f_n) = (f_1, f_2 + af_1, f_3, \dots, f_n).$$

Given a unit $u \in A$, we have

$$(f_1, \dots, f_n) = (uf_1, f_2, \dots, f_n).$$

The nilradical

An element $f \in A$ is **nilpotent** if there is an $n \geq 1$ such that $f^n = 0$. The **nilradical** of A is

$$\mathfrak{N} = \sqrt{(0)} = \{f \in A \mid f \text{ is nilpotent}\}.$$

Theorem

The nilradical of A equals the intersection of all prime ideals of A.

Extension and contraction of ideals

Let $\varphi\colon A \to B$ be homomorphism. We can move ideals between A and B as follows.

- If $\mathfrak{b} \subseteq B$ is an ideal, its **contraction** is $\mathfrak{b}^c = \varphi^{-1}(\mathfrak{b}) \subseteq A$.
- If $\mathfrak{a} \subseteq A$ is an ideal, its **extension**, denoted $\mathfrak{a}^e \subseteq B$, is the smallest ideal of B containing $\varphi(\mathfrak{a})$. Concretely

$$\mathfrak{a}^e = \{\sum_{i=1}^{n} b_i\varphi(x_i) \mid b_i \in B, x_i \in \mathfrak{a}\}$$

Theorem

If $\mathfrak{p} \subset B$ is a prime ideal, then so is $\mathfrak{p}^c$.

Extension and contraction for the quotient homomorphism

Let $\mathfrak{a} \subseteq A$ be an ideal, and let $\varphi \colon A \to A/\mathfrak{a}$ be the quotient homomorphism.

- For an ideal $\mathfrak{b} \subseteq A$, $\mathfrak{b}^e = (\mathfrak{a} + \mathfrak{b})/\mathfrak{a} \subseteq A/\mathfrak{a}$.
- For an ideal $\mathfrak{c} \subseteq A/\mathfrak{a}$ has $\mathfrak{c}^{ce} = \mathfrak{c}$.
- This gives a bijection between the set of ideals of $A/\mathfrak{a}$ and the ideals in A containing $\mathfrak{a}$, which sends prime (resp. maximal) ideals to prime (resp. maximal) ideals.

Extension and contraction, further examples

- If $\varphi\colon A \to B$ is the inclusion of a subring and $\mathfrak{b} \subseteq B$ an ideal, then $\mathfrak{b}^c = \mathfrak{b} \cap A$.
- For $\varphi\colon A \to A[x]$ the standard inclusion, and $\mathfrak{a} \subseteq A$, we have

$$\mathfrak{a}^e = \{\sum_{i=1}^{n} a_i x^i \mid a_i \in \mathfrak{a}\} \subseteq A[x].$$

Coprime ideals

Two ideals $\mathfrak{a}, \mathfrak{b} \subseteq A$ are **coprime** if $\mathfrak{a} + \mathfrak{b} = (1)$.

Theorem

If $\mathfrak{a}_1, \ldots, \mathfrak{a}_n$ are pairwise coprime, then $\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n = \mathfrak{a}_1 \cdots \mathfrak{a}_n$ and

$$A/(\mathfrak{a}_1 \cap \cdots \cap \mathfrak{a}_n) = A/\mathfrak{a}_1 \times \cdots \times A/\mathfrak{a}_n.$$

Example

In $\mathbb{Z}$, ideals (m) and (n) are coprime if and only if integers m and n are. If $k_1, \ldots, k_n$ are pairwise coprime integers, get

$$\mathbb{Z}/(k_1 \cdots k_n) \cong \mathbb{Z}/k_1 \times \cdots \times \mathbb{Z}/k_n.$$

Local rings

A ring A is **local** if it has a *unique* maximal ideal $\mathfrak{m}$.

Theorem

A ring A is local if and only if its set of non-units form an ideal. If A is local, then the set of non-units is the maximal ideal.

Examples

- All fields are local.
- $k[x]/(x^n)$ is local, with maximal ideal $(x)/(x^n)$.
- $\mathbb{Z}_{(2)} = \{m/n \mid m \in \mathbb{Z}, n \in \mathbb{Z} \setminus (2)\} \subset \mathbb{Q}$ is local, with maximal ideal $\{m/n \mid m \in (2), n \in \mathbb{Z} \setminus (2)\}$.

Modules

A **module** of a ring A is an abelian group M equipped with an operation of multiplication from A

$$A \times M \to M \tag{1}$$
$$(a, m) \mapsto am \tag{2}$$

satisfying a short list of axioms.

Example

If k is a field, then a k-module is the same thing as a k-vector space (*the axioms are the same*).

Example

A $\mathbb{Z}$-module is the same thing as an abelian group.

Modules, examples

- Every ring A is an A-module in a natural way.
- Every ideal of A is an A-module in a natural way.
- Given A-modules M, N, get A-module

$$M \oplus N = \{(m, n) \mid m \in M, n \in N\},$$

with $(m, n) + (m', n') = (m + m', n + n'), a(m, n) = (am, an)$.

- Given a ring homomorphism $\varphi \colon A \to B$, B is an A-module via

$$ab = \varphi(a)b \qquad a \in A, b \in B$$

Homomorphism

A **homomorphism of** A-modules is a map $\varphi\colon M \to N$ respecting multiplication from A and addition, so for all $a \in A$, $m, n \in M$, we have

$$\varphi(am) = a\varphi(m), \quad \varphi(m+n) = \varphi(m) + \varphi(n).$$

It is an **isomorphism** if it is bijective.

Example

- If M is A-module and $x \in A$, have a homomorphism $\varphi\colon M \to M$ given by $\varphi(m) = xm$.
- If $m \in M$, get a homomorphism $\varphi\colon A \to M$ by $\varphi(x) = xm$.

Sub- and quotient modules

Let M be an A-module. Then $M' \subseteq M$ is **submodule** if it is closed under addition and multiplication from A, i.e.

$$x, y \in M' \Rightarrow x + y \in M' \quad x \in M' \Rightarrow ax \in M'.$$

The quotient group M/M' is an A-module in a natural way.

Example

Given an ideal $\mathfrak{a} \subset A$ and A-module M, get submodule

$$\mathfrak{a}M = \{\sum_{i=1}^{n} a_i m_i \mid a_i \in \mathfrak{a}, m_i \in M\} \subseteq M.$$

Module isomorphism theorems

- Given homomorphism $\varphi\colon M \to N$, have

$$\mathrm{im}(\varphi) \cong M/\ker\varphi.$$

- Given chain of A-modules $M \subseteq N \subseteq P$, have a submodule inclusion $N/M \subseteq P/M$ and an isomorphism

$$P/N \cong (P/M)/(N/M).$$

- Given submodules $P, N \subseteq M$, have an isomorphism

$$(P + N)/N \cong P/(P \cap N).$$

Kernels, images and cokernels

Given a homomorphism of modules $\varphi\colon M \to N$, there are three main associated modules:

- Kernel: $\ker \varphi = \varphi^{-1}(0) \subseteq M$
- Image: $\operatorname{im} \varphi = \{\varphi(m) \mid m \in M\}$.
- Cokernel: $\operatorname{cok} \varphi = N / \operatorname{im} \varphi$.

We have "φ injective $\Leftrightarrow \ker \varphi = 0$" and φ surjective $\Leftrightarrow \operatorname{cok} \varphi = 0$.

Exact sequences

A sequence of homomorphisms of modules

$$M_1 \xrightarrow{\varphi_1} M_2 \xrightarrow{\varphi_2} \cdots \xrightarrow{\varphi_{n-1}} M_n$$

is **exact** at M_i if $\ker \varphi_i = \mathrm{im}\, \varphi_{i-1}$. It is exact if it is exact at M_i for $2 \leq i \leq n-1$.

Example

- A sequence $M \to M'' \to 0$ is exact if and only if φ is surjective.
- A sequence $0 \to M' \to M$ is exact if and only if φ is injective.
- A sequence $0 \to M \to N \to 0$ is exact if and only if φ is an isomorphism.

Short exact sequences

An exact sequence of the form

$$0 \to M' \xrightarrow{i} M \xrightarrow{p} M'' \to 0$$

is called **short exact**. The sequence being exact is equivalent to

1. i is injective,
2. p is surjective, and
3. $\mathrm{im}(i) = \ker(p)$

Finitely generated modules

A module M is **finitely** generated if equivalently

1. there is a surjective homomorphism of A-modules $A^n \to M$, or
2. M is isomorphic to A^n/K for some submodule $K \subseteq A^n$, or
3. there are elements $m_1, \ldots, m_n \in M$ such that all $m \in M$ can be written as

$$m = a_1 m_1 + \cdots + a_n m_n, \quad a_i \in A.$$

Nakayama's lemma

In all statements let A be a local ring with maximal ideal $\mathfrak{m}$, and let M be a finitely generated A-module.

Theorem

Nakayama's lemma If $\mathfrak{m}M = M$, then $M = 0$.

Theorem

Let N be an A-module. A homomorphism $\varphi\colon N \to M$ is surjective if and only if the composed map $N \to M \to M/\mathfrak{m}M$ is surjective.

Theorem

Elements $x_1, \ldots, x_n \in M$ generate M as an A-module if and only if $x_1 + \mathfrak{m}M, \ldots, x_n + \mathfrak{m}M$ generate $M/\mathfrak{m}M$ as an $A/\mathfrak{m}$-module.

Tensor product

Let M, N and P be A-modules. A map $\varphi\colon M \times N \to P$ is **bilinear** if for $x, x' \in M, y, y' \in N$ and $a \in A$ we have

$$\varphi(x + x', y) = \varphi(x, y) + \varphi(x', y)$$
$$\varphi(x, y + y') = \varphi(x, y) + \varphi(x, y')$$
$$\varphi(ax, y) = \varphi(x, ay) = a\varphi(x, y).$$

Theorem

There exists a module $M \otimes_A N$ together with a bilinear map $\chi\colon M \times N \to M \otimes_A N$ such that every bilinear map $\varphi\colon M \times N \to P$ can be factored uniquely as $\varphi = \psi \circ \chi$ for an A-module homomorphism $\psi\colon M \otimes_A N \to P$. The module $M \otimes_A N$ is uniquely defined by this property (up to isomorphism).

Tensor product, description

Elements of $M \otimes_A N$ are formal sums

$$\sum_{i=1}^{n} x_i \otimes y_i \qquad x_i \in M, y_i \in N,$$

subject to relations $(x, x' \in M, y, y' \in N, a \in A)$:

$$(x + x') \otimes y = x \otimes y + x' \otimes y$$
$$x \otimes (y + y') = x \otimes y + x \otimes y'$$
$$a(x \otimes y) = ax \otimes y = x \otimes ay.$$

Given a bilinear map $\varphi\colon M \times N \to P$, the associated homomorphism $\psi\colon M \otimes_A N \to P$ has $\psi(x \otimes y) = \varphi(x, y)$.

Tensor product, applications

If $\varphi\colon A \to B$ is a ring homomorphism, then we can turn A-modules into B-modules and vice versa.

- If N is a B-module, the induced A-module is $N_A = N$, with A-module structure given by $an = \varphi(a)n$.
- If M is an A-module, the induced B-module is $M_B = B \otimes_A M$, with B-module structure given by $b(c \otimes m) = bc \otimes m$.

Tensor product, examples

- For any module M, $A \otimes_A M \cong M$, via the map $a \otimes m \mapsto am$.
- Given M, M', N, we have $(M \oplus M') \otimes N = M \otimes N \oplus M' \otimes N$ ($\otimes$ *behaves as multiplication*).
- For any ideal $\mathfrak{a} \subseteq A$, we have $M \otimes A/\mathfrak{a} \cong M/\mathfrak{a}M$.

Flat modules

The functor of tensor product preserves some exactness.

Theorem

If M is an A-module, and

$$N' \to N \to N'' \to 0$$

is an exact sequence of A-modules, then so is

$$M \otimes_A N' \to M \otimes_A N \to M \otimes_A N'' \to 0.$$

A module M is **flat** if equivalently

- $\varphi \colon N' \to N$ injective $\Rightarrow \varphi \otimes_A \mathrm{id}_M \colon N' \otimes_A M \to N \otimes_A M$ injective, or
- $N' \to N \to N''$ exact $\Rightarrow M \otimes_A N' \to M \otimes_A N \to M \otimes_A N''$ exact.

Flatness, examples

- M and N flat $\Leftrightarrow M \oplus N$ flat.
- A is flat as A-module.
- Every free module is flat.
- Every fraction ring $S^{-1}A$ is flat as an A-module.
- $\mathbb{Z}/2$ is not flat as a $\mathbb{Z}$-module, since the map $\varphi\colon \mathbb{Z} \to \mathbb{Z}$ gives $\varphi \otimes \mathrm{id}_{\mathbb{Z}/2} = 0\colon \mathbb{Z} \otimes \mathbb{Z}/2 \to \mathbb{Z} \otimes \mathbb{Z}/2$.

Algebras

Let A be a ring. An A-algebra is a ring B together with a homomorphism $\varphi\colon A \to B$.

- An A-algebra is an A-module via $ab = \varphi(a)b$.

Example

Every ring B of the form $A[x_1, \dots, x_n]/\mathfrak{a}$ is an A-algebra.

Fraction rings

A subset $S \subseteq A$ is **multiplicatively closed** if $1 \in S$ and $s, t \in S \Rightarrow st \in S$.

Definition

The fraction ring $S^{-1}A$ has elements equivalence classes of symbols a/s with $a \in A$ and $s \in S$, and with equivalence relation defined by

$$a/s = b/t \Leftrightarrow \exists u \in S \mid (at - bs)u = 0.$$

Addition and multiplication is defined by the usual fraction formulas:

$$a/s + b/t = (at + bs)/st, \quad (a/s)(b/t) = ab/st.$$

Example

If A is an integral domain and $S = A \setminus \{0\}$, then $S^{-1}A$ is the fraction field of A.

Fraction rings examples

Example

- If $\mathfrak{p} \subset A$ is a prime ideal, take $S = A \setminus \mathfrak{p}$, and write $A_{\mathfrak{p}} = S^{-1}A$.
- If $f \in A$, take $S = \{f^k\}_{k \geq 0}$, and write $A_f = S^{-1}A$.

- $\mathbb{Z}_{(0)} = \mathbb{Q}$.
- More generally, if A is an integral domain, $A_{(0)}$ is the fraction field of A.
- For p a prime, $\mathbb{Z}_{(p)} = \{m/n \mid m \in \mathbb{Z}, n \notin (p)\} \subset \mathbb{Q}$.
- $\mathbb{Z}_n = \{m/n^k \mid m \in \mathbb{Z}, k \geq 0\} \subset \mathbb{Q}$.

Fraction modules

If $S \subset A$ is multiplicatively closed and M is an A-module, define an $S^{-1}A$-module as the set of equivalence classes of symbols m/s, with $m \in M$, $s \in S$ and

$$m/s \sim n/t \Leftrightarrow \exists u \in S \text{ such that } (tm - sn)u = 0$$

Theorem

- *If $M' \to M \to M''$ is exact, then so is $S^{-1}M' \to S^{-1}M \to S^{-1}M''$.*
- *We have $S^{-1}A \otimes_A M \cong S^{-1}M$.*
- *The ring $S^{-1}A$ is flat as an A-module.*

Properties of fraction modules

Theorem ("Taking fraction modules commutes with most things")

Let $S \subset A$ be a multiplicatively closed system. The functor from A-modules to $S^{-1}A$-modules given by $M \mapsto S^{-1}M$

- sends submodules to submodules

$$M' \subseteq M \rightsquigarrow S^{-1}M' \subseteq S^{-1}M$$

- commutes with taking direct sums

$$S^{-1}(M \oplus N) = S^{-1}M \oplus S^{-1}N \cong S^{-1}(M \oplus N)$$

- commutes with taking intersections and sums, that is for $M, N \subseteq P$, we have

$$S^{-1}(M \cap N) = S^{-1}M \cap S^{-1}N \subseteq S^{-1}P$$

and

$$S^{-1}(M + N) = S^{-1}M + S^{-1}N \subseteq S^{-1}P$$

- commutes with tensor product

$$S^{-1}(M \otimes_A N) \cong S^{-1}M \otimes_{S^{-1}A} S^{-1}N.$$

Fraction modules, examples

Notation

Similarly to the case of rings

- For $\mathfrak{p} \subset A$ a prime ideal, write $M_{\mathfrak{p}} = S^{-1}M$ with $S = A \setminus \mathfrak{p}$.
- For $f \in A$, write $M_f = S^{-1}M$ with $S = \{f^k\}_{k \geq 0}$.

Local properties

A property P of modules (rings, homomorphisms, ideals. . .) is **local** if it "P holds for M" is equivalent to "P holds for $M_{\mathfrak{p}}$ for all prime ideals $\mathfrak{p} \subset A$".

Theorem

The following are local properties

- *A module being 0*
- *A homomorphism being injective*
- *A homomorphism being surjective*
- *A homomorphism being an isomorphism*
- *A module being flat*

Ideals in fraction rings

Let $S \subseteq A$ be multiplicatively closed, and consider the standard homomorphism $\varphi\colon A \to S^{-1}A$.

Theorem

Every ideal $\mathfrak{a} \subseteq S^{-1}A$ *satisfies*

$$\mathfrak{a}^{ce} = \mathfrak{a}.$$

Immediate consequences:

- Every ideal of $S^{-1}A$ is of the form $\mathfrak{b}^e$ for some $\mathfrak{b} \subseteq A$.
- Different ideals $\mathfrak{a}, \mathfrak{b} \subseteq S^{-1}A$ contract to distinct ideals $\mathfrak{a}^c, \mathfrak{b}^c \subseteq A$.

Prime ideals in fraction rings

Let $S \subseteq A$ be multiplicatively closed and $\varphi\colon A \to S^{-1}A$ the standard homomorphism.

Theorem

The operations of extension and contraction along φ give a bijection between prime ideals of $S^{-1}A$ and the prime ideals of A contained in $A \setminus S$.

Examples

- The prime ideals of $A_{\mathfrak{p}}$ are of the form $\mathfrak{q}^e$ for some unique prime ideal $\mathfrak{q} \subseteq \mathfrak{p}$. This shows $A_{\mathfrak{p}}$ is local with maximal ideal $\mathfrak{p}^e$.
- The prime ideals of A_f are in bijection with prime ideals $\mathfrak{p}$ such that $f \notin \mathfrak{p}$.

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