UiO : Department of Mathematics
University of Oslo

MAT4200 Autumn 2023

Commutative algebra revision

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Primary ideals

Definition

An ideal $\mathfrak{q} \subseteq A$ is **primary** if $xy \in \mathfrak{q}$ implies either $x \in \mathfrak{a}$ or $y \in \sqrt{\mathfrak{q}}$.

Equivalent definition

An ideal $\mathfrak{q}$ is primary if every 0-divisor in $A/\mathfrak{q}$ is nilpotent.

Sufficient conditions

- Every prime ideal is primary.
- If $\sqrt{\mathfrak{q}}$ is maximal, then $\mathfrak{q}$ is primary.

Properties of primary ideals

Proposition

If $\mathfrak{q}$ is primary, then $\sqrt{\mathfrak{q}}$ is prime.

$\mathfrak{p}$-primary ideals

If $\mathfrak{p}$ is a prime ideal, we say $\mathfrak{q}$ is $\mathfrak{p}$-**primary** if it is primary and $\sqrt{\mathfrak{q}} = \mathfrak{p}$.

Theorem ("Primary ideals under ring operations")

- Let $\varphi\colon A \to B$ be a ring homomorphism. If $\mathfrak{q} \subset B$ is primary, then so is $\mathfrak{q}^c \subset A$.
- Let $S \subset A$ be a multiplicatively closed subset. Contraction and extension along $A \to S^{-1}A$ gives a bijection between the primary ideals of $S^{-1}A$ and the primary ideals of A contained in $A \setminus S$.

Primary decompositions

Definition

A **primary decomposition** of an ideal $\mathfrak{a}$ is an equality

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

with the $\mathfrak{q}_i$ primary ideals. The primary decomposition is **minimal** if

1. The prime ideals $\sqrt{\mathfrak{q}_i}$ are distinct, and
2. For all i, $\mathfrak{q}_i$ does not contain $\bigcap_{j \neq i} \mathfrak{q}_j$.

Every primary decomposition gives a minimal one after removing some $\mathfrak{q}_i$ or replacing some pairs $\mathfrak{q}_i, \mathfrak{q}_j$ by $\mathfrak{q}_i \cap \mathfrak{q}_j$.

Uniqueness theorem I

Theorem ("The primes of a primary decomposition are unique")

Let $\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i$ be a minimal primary decomposition. Then the set

$$\mathrm{Ass}(\mathfrak{a}) := \{\sqrt{\mathfrak{q}_i}\} \subseteq \operatorname{Spec} A$$

is *uniquely determined* by $\mathfrak{a}$, and equals the set

$$\{\sqrt{(x:\mathfrak{a})} \text{ prime} \mid x \in A\}.$$

Minimal/embedded primes

The prime ideals in $\mathrm{Ass}(\mathfrak{a})$ are called the **associated primes of $\mathfrak{a}$**.
An element $\mathfrak{p} \in \mathrm{Ass}(\mathfrak{a})$ which is minimal with respect to inclusion is called a **minimal** prime of $\mathfrak{a}$, the other ones are called **embedded**.

Minimal = minimal

Theorem

Let $\mathfrak{a}$ be an ideal with a primary decomposition. The minimal elements of the set

$$\{\mathfrak{p} \supseteq \mathfrak{a} \mid \mathfrak{p} \text{ prime}\}$$

are the same as the minimal elements of $\operatorname{Ass}(\mathfrak{a})$.

Uniqueness II

Theorem ("Primary ideals whose primes are minimal are unique")

Let $\mathfrak{a}$ be an ideal with minimal primary decomposition

$$\mathfrak{a} = \bigcap_{i=1}^{n} \mathfrak{q}_i.$$

Then for every i such that $\sqrt{\mathfrak{p}_i}$ is a minimal prime of $\mathfrak{a}$, the ideal $\mathfrak{q}_i$ is determined by $\mathfrak{a}$.

Zerodivisors and associated primes

Theorem

Assume that A is a ring such that (0) has a primary decomposition. Then $x \in A$ is a 0-divisor if and only if x lies in some associated prime of (0).

Integral dependence

Definition

Let $A \subseteq B$ be a ring inclusion. An element $b \in B$ is **integral over** A if we can find $a_0, \ldots, a_{n-1} \in A$ such that

$$b^n + a_{n-1}b^{n-1} + \cdots + a_0 = 0.$$

Definition

We say A **is integrally closed in** B if every $b \in B$ which is integral over A lies in A.

Integral closures

Theorem/definition

Let $A \subseteq B$ be a ring inclusion. The set

$$C = \{b \in B \mid b \text{ integral over } A\}$$

is a subring of B, with $A \subseteq C \subseteq B$.
The ring C is called the **integral closure of** A **in** B.
C is integrally closed in B.

Integral closure in fraction fields

Definition

If A is an integral domain with fraction field K, we say A **is integrally closed** if it is integrally closed in K.

Example

Every unique factorisation domain is integrally closed.

Theorem ("Being integrally closed is a local property")

Let A be an integral domain. The following are equivalent:

1. A is integrally closed.
2. $A_{\mathfrak{p}}$ is integrally closed for every prime $\mathfrak{p} \subset A$
3. $A_{\mathfrak{p}}$ is integrally closed for every maximal $\mathfrak{m} \subset A$.

Chain conditions

Definition

Let M be an A-module.

- The module M is **Noetherian** if every sequence of submodules

$$M_1 \subseteq M_2 \subseteq \cdots \subseteq M$$

  stabilises, that is there is some $N \geq 0$ such that $M_n = M_N$ for $n \geq N$.

- The module M is **Artinian** if every sequence of submodules

$$M \supseteq M_1 \supseteq M_2 \supseteq \cdots$$

  stabilises.

Alternative definition Noetherian/Artinian

Definition through maximal/minimal elements

An A-module M is Noetherian (resp. Artinian) if in every set S of submodules of M we can find a maximal (resp. minimal) submodule.

Theorem

An A-module M is Noetherian if and only if every submodule $M' \subseteq M$ is finitely generated.

Chain conditions and short exact sequences

Theorem

Let

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of A-modules. Then

$$M \text{ Noetherian} \Leftrightarrow M' \,\& M'' \text{ Noetherian}$$

and

$$M \text{ Artinian} \Leftrightarrow M' \,\& M'' \text{ Artinian}.$$

Chain conditions on rings

Definition

A ring A is Noetherian (resp. Artinian) if and only if it is so as an A-module.

Theorem ("Chain conditions preserved under ring change operations")

Let A be Noetherian (resp. Artinian).

- If $\mathfrak{a} \subseteq A$ is an ideal, then $A/\mathfrak{a}$ is Noetherian (resp. Artinian).
- If $S \subseteq A$ is multiplicatively closed, then $S^{-1}A$ is Noetherian (resp. Artinian).

Noetherian modules over Noetherian rings

Theorem

Let A be a Noetherian ring, and let M be an A-module. Then M is Noetherian if and only if it is finitely generated.

Simple modules

Definition

An A-module M is simple if its only submodules are 0 and M.

Definition

Let M be an A-module. We say a sequence of submodules

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_l = 0$$

is a **composition series** if every quotient M_i/M_{i+1} is simple. Its **length** is l.

Equivalently the sequence is a composition series if it maximal in the sense that there is no submodule $M' \subset M$ such that

$$M_i \supsetneq M' \supsetneq M_{i+1}$$

Length

Theorem ("Length is independent of choice of composition series")

If M admits a composition series, then

- Every composition series of M has the same length, and
- Every sequence

$$M = M_0 \supsetneq M_1 \supsetneq \cdots \supsetneq M_k = 0$$

can be extended to a composition series.

Definition

If M admits a composition series, then we say it has **finite length**, and its **length** is $l(M)$, defined as the length of any of its composition series.

Additivity of length

Theorem

A module M has finite length if and only if it is both Noetherian and Artinian.

Theorem ("Additivity of length")

Length is additive, that is if

$$0 \to M' \to M \to M'' \to 0$$

is an exact sequence of finite length modules, then $l(M) = l(M') + l(M'')$.

Hilbert's basis theorem

Hilbert's basis theorem

If A be a Noetherian ring, then $A[x]$ is Noetherian.

Corollary ("Lots of rings are Noetherian")

Every localisation of a finite type $\mathbb{Z}$-algebra is Noetherian.
If k is a field, then every localisation of a finite type k-algebra is Noetherian.

Hilbert's Nullstellensatz

Hilbert's Nullstellensatz

Let k be a field.

- If $\mathfrak{m} \subset k[x_1, \ldots, x_n]$ is a maximal ideal, then $k[x_1, \ldots, x_n]/\mathfrak{m}$ is of finite dimension as a k-module.
- If k is algebraically closed, then the maximal ideals of $k[x_1, \ldots, x_n]$ are all of the form

$$(x_1 - a_1, \ldots, x_n - a_n)$$

for some $a_1, \ldots, a_n \in k$.

Primary decomposition in Noetherian rings

Lasker–Noether theorem

Let A be a Noetherian ring, and let $\mathfrak{a} \subseteq A$ be an ideal. Then there exists a primary decomposition of $\mathfrak{a}$.

Krull dimension

Definition

Let A be a ring. The Krull dimension of A is the supremum (possibly ∞) of the length n of chains of prime ideals

$$\mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n.$$

Artinian rings

Theorem

Let A be a ring. Then A is Artinian if and only if A is Noetherian and $\dim A = 0$.

Theorem ("Artinian rings are products of local ones")

Every Artinian ring A has finitely many maximal ideals, and can be written uniquely as a direct product

$$A \cong \prod_{i=1}^{n} A_i$$

where the rings A_i are Artinian and local.

If A is a local Artinian ring, then $\mathfrak{m}^n = 0$ for some $n \geq 1$.

Discrete valuations

Definition

Let K be a field. A **discrete valuation** of K is a surjective function

$$v\colon K \to \mathbb{Z} \cup \{\infty\}$$

such that for $x, y \in K$, we have

1. $v(x) = \infty \Leftrightarrow x = 0$
2. $v(xy) = v(x) + v(y)$
3. $v(x) + v(y) \geq \min(v(x), v(y))$.

Discrete valuation rings

Definition

Let A be an integral domain, with fraction field K. We say A is a **discrete valuation ring** (or **DVR**) if there exists a valuation v of K such that

$$A = \{x \in K \mid v(x) \geq 0\}.$$

The structure of a DVR

Theorem

Let A be a DVR. Then

- A is local, with maximal ideal

$$\mathfrak{m} = \{x \in K \mid v(x) \geq 1\}$$

- For any $x \in A$ with $v(x) = 1$, we have $\mathfrak{m} = (x)$.
- Given such an x, the ideals of A are

$$(1) \supsetneq (x) \supsetneq (x^2) \supsetneq \cdots$$

  and (0).
- A is integrally closed.

Criteria for being a DVR

Theorem

Let A be a Noetherian local integral domain of dimension 1. Then the following are equivalent

1. A is a DVR
2. $\mathfrak{m}$ is principal
3. With $k = A/\mathfrak{m}$, $\dim_k \mathfrak{m}/\mathfrak{m}^2 = 1$.
4. A is integrally closed

Graded rings

Definition

A **graded ring** is a ring A together with subgroups $A_i \subseteq A$ for every $i \geq 0$, such that

$$A = \bigoplus_{i \geq 0} A_i,$$

and such that for all $i, j \geq 0$, we have $A_i A_j \subseteq A_{i+j}$.

Homogeneous elements

If A is a graded ring and $x \in A$, we say x is **homogeneous** of degree d if $x \in A_d$.

An assumption

Let A be a graded ring. Assume from now on that

- A_0 is a field, denoted k.
- There exist $x_1, \ldots, x_n \in A_1$ such that A is generated as a k-algebra by the x_i.

Hilbert function

For a graded ring A with the above assumption, its **Hilbert function**

$$H_A \colon \mathbb{N} \to \mathbb{N}$$

is defined by

$$H_A(d) = \dim_k A_d.$$

Asymptotic behaviour of the Hilbert function

Theorem

Let A be a graded ring satisfying the assumptions above. There exists a rational polynomial f and an integer $N \geq 0$ such that

$$f(n) = H_A(n)$$

when $n \geq N$.

The associated graded ring

Definition

Let A be a local ring with maximal ideal $\mathfrak{m}$. Its **associated graded ring** is the ring

$$G(A) = \bigoplus_{i \geq 0} G(A)_i,$$

where

$$G(A)_i = \mathfrak{m}^i / \mathfrak{m}^{i+1},$$

and the multiplication $G(A)_i \times G(A)_j \to G(A)_{i+j}$ is by

$$(x + \mathfrak{m}^{i+1})(y + \mathfrak{m}^{j+1}) = xy + \mathfrak{m}^{i+j+1}$$

The characteristic polynomial

Theorem–Definition

Let A be a Noetherian local ring. There is a polynomial χ_A and an $N \geq 0$ such that when $n \geq N$, we have

$$\chi_A(n) = l(A/\mathfrak{m}^n).$$

We call χ_A the **characteristic polynomial of** A.

A formula

With A as above, we have $l(A/\mathfrak{m}^n) = \sum_{d=0}^{n-1} H_{G(A)}(d)$, and the Hilbert polynomial of $G(A)$ is

$$\chi_A(n+1) - \chi_A(n).$$

The dimension theorem

Dimension theorem

Let A be a Noetherian local ring. Then A has finite dimension, and the following integers are equal:

- $\dim A$
- $\deg \chi_A$
- The minimal number of generators of an $\mathfrak{m}$-primary ideal.

Corollaries of the dimension theorem

- If k is a field, then $\dim k[x_1, \ldots, x_n] = n$.
- If A is a local, Noetherian ring, with $k = A/\mathfrak{m}$, then

$$\dim_k \mathfrak{m}/\mathfrak{m}^2 \geq \dim A$$

- If A is a local, Noetherian ring and $x \in \mathfrak{m}$ is not a 0-divisor, then $\dim A/(x) = \dim A - 1$.
- **Krull's principal ideal theorem**: If A is a Noetherian ring, $x \in A$, and $\mathfrak{p}$ is an ideal minimal among those containing x, then $\dim A_{\mathfrak{p}} \leq 1$.

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