MAT4210—Algebraic geometry I: Notes 5

The dimension of varieties

16th March 2018

Hot themes in notes 5: Krull dimension of spaces—finite maps and Going Up—Noethers Normalization Lemma—transcendence degree and dimension—the dimension of $\mathbb{A}^n$—the dimenion of a product—Krull's Principal Ideal Theorem—Intersections in projective space.

Very preliminary version 0.2 as of 16th March 2018 at 9:46am—Prone to misprints and errors.
Changes 2018-02-26 10:08:34: Rewritten section about Krull's Principal Ideal Theorem. Added a section about the dimension of a product.
Proposition 5.2was wrong as stated; has bee corrected.

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Introduction

Dimension

For general topological spaces there is no good notion of dimension. Manifolds of course, have a dimension (or at least each connected component has). They are locally isomorphic to open sets in some euclidean space, and the dimension of that space is constant along connected components, and is the dimension of the component.

Noetherian topological spaces as well, have a *dimension*, and in many instances it is very useful, although it can be infinite. The definition is inspired by the concept of the *Krull dimension* of a Noetherian ring; and resembles vaguely one of our naive conception of dimension. For example, in three dimensional geometric gadgets, called threefolds, we may imagine increasing chains of subgadgets of length three; points in curves, curves in surfaces and surfaces in the threefold. The definition below works for any topological space, but the ensuing dimension carries not much information unless the topology is "Zariski-like".

For varieties there is another good candidate for the dimension, namely the transcendence degree of the fraction field $K(X)$ over the ground field. This may be motivated by the fact the Krull dimension of the polynomial ring $k[x_1, \ldots, x_n]$ equals n, and obviously the transcendence degree of $k(x_1, \ldots, x_n)$ is n. That the two coincide, follows from the Normalization lemma which states that every variety is a finite cover of an affine space.

This materializes in the following definition. In the topological space X we consider strictly increasing chains of non-empty *closed* *and irreducible* subsets:

$$X_0 \subset X_1 \subset \ldots \subset X_r, \qquad (1)$$

and we call r (that is, the number of inclusions in the chain) *the length* of such a chain. The *dimension* $\dim X$ of X is to be the supremum of the set of r's for which there is a chain like in (1).

ChainDim

The dimension of a topological space

One says that the chain is *saturated* if there is no closed irreducible subset in between any two of the X_i's; that is, if $X_i \subsetneq Z \subsetneq X_{i+1}$ and Z is closed and irreducible, then $Z = X_i$ or $Z = X_{i+1}$. Clearly, the supremum over the lengths of saturated chains will be equal to the dimension.

Saturated chains

Possibly the dimension of X can be equal to ∞, and in fact, there are Noetherian spaces for which $\dim X = \infty$, although we shall not meet many. There are even Noetherian rings whose Krull dimension is infinite; the first example was constructed by Masayoshi Nagata, the great master of counterexamples in algebra. For convenience we put $\dim \emptyset = -\infty$.

EXAMPLE 5.1 One do not need to go far to find Noetherian spaces of infinite dimension. The following weird topology on the set $\mathbb{N}$ of natural numbers is one example. The closed sets of this topology apart from the empty set and the entire space, are the sets defined by $Z_a = \{ x \in \mathbb{N} \mid x \leq a \}$ for $a \in \mathbb{N}$. They form a strictly ascending infinite chain and are irreducible, hence the dimension is infinite. On the other hand, any strictly descending chain is finite so the space is Noetherian. We leave it as an exercise for the interested student to check these assertions. ☆

PROBLEM 5.1 The notion of dimension we introduced is only useful for "Zariski-like" topologies. Show that any Hausdorff space is of dimension zero. HINT: What are the irreducible subsets? ★

PROBLEM 5.2 Show that the only irreducible and finite topological space of dimension one is the so called Sierpiński space. It has two points η and x with $\{\eta\}$ open and $\{x\}$ closed. ★

PROBLEM 5.3 Assume that $Y = Y_1 \cup \ldots \cup Y_r$ is the decomposition of the Noetherian space Y into irreducible components. Show that $\dim Y = \max \dim Y_i$. ★

ONE IMMEDIATELY establishes the following basic properties of the dimension

Lemma 5.1 *Assume that X is a topological space and that $Y \subseteq X$ is a closed subspace. If $Y \subseteq X$, them $\dim Y \leq \dim X$. Assume furthermore that Y is ireducible and that X is of finite Krull dimension. If $\dim Y = \dim X$, then Y is a component of X.*

PROOF: Any chain like in (1) in Y will be one in X as well; hence $\dim Y \leq \dim X$. Assume that $\dim Y = \dim X = r$, and let

$$Y_0 \subset Y_1 \subset \ldots \subset Y_r = Y$$

be a maximal chain in Y. In case Y is not a component of X, there is a closed and irreducible subset Z of X strictly bigger than Y, and we can extend the chain to

$$Y_0 \subset Y_1 \subset \ldots \subset Y_r \subset Z.$$

Hence $\dim X \geq r + 1$, and we have a contradiction. ❑

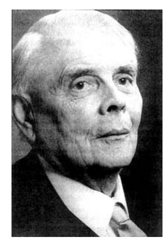
OUR CONCEPT OF DIMENSION coincides, when X is a closed irreducible subset of $\mathbb{A}^m$, with the Krull dimension of the coordinate ring $A(X)$. Indeed, the correspondence between closed irreducible subsets of X and prime ideals in $A(X)$ implied by the Nullstellensatz, yields a bijective correspondence between chains

$$X_0 \subset X_1 \subset \ldots \subset X_r$$

of closed irreducible subsets; like the one (1), and chains

$$I(X_r) \subset \ldots \subset I(X_1) \subset (X_0)$$

of prime ideals.Hence the supremum of the lengths in the two cases are the same, and we have



Krull (1951–)
German mathematician

Proposition 5.1 *Let* $X \subseteq \mathbb{A}^n$ *be a closed algebraic subset. Then* $\dim X = \dim A(X)$.

Given that the polynomial ring $k[x_\bullet]$ is of Krull dimension equal to n we know that $\dim \mathbb{A}^n = n$. This is of course what it shoud be, but it is astonishingly subtle to show, and may be this reflects the fact that if R is not Noetherian the polynomial ring $R[t]$ may have a Krull dimension other than $\dim R + 1$. We give a proof; see theorem 5.2 on page 8 below.

DENSE OPEN SUBSETS do not nessecarily have the same dimension as the surrounding space even when the sourrounding space is irreducible, but it must be less:

DensOpen

Proposition 5.2 *Assume that X is an irreducible topological space and that U is an non-empty open dense subset. Then* $\dim U \leq \dim X$.

PROOF: We establish first that $\dim U \leq \dim X$. So let

$$U_0 \subset U_1 \subset \ldots \subset U_r$$

be a chain of closed irreducible subsets of U. By lemma **??** on page **??** the closures $\overline{U_i}$ are irreducible closed subsets of X and they satisfy $\overline{U_i} \cap U = U_i$. Hence the chain $\{\overline{U_i}\}$ form a strictly ascending chain of closed irreducible sets in X, and it follows that $r \leq \dim X$. ❑

PROBLEM 5.4 Find an example of an irreducible topological space X having an irreducible open subset U so that $\dim U < \dim X$. HINT: Take a look at the Sierpiński space. ★

RECALL THAT IF A is a ring and $\mathfrak{p} \subseteq A$ is a prime ideal, the *height* of $\mathfrak{p}$ is the length r of the longest strictly increasing chain of prime ideals

The height of an ideal

$$\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}$$

ending at $\mathfrak{p}$. It equals the Krull dimension of the localization $\dim A_{\mathfrak{p}}$.

For any ring of finite Krull diension one has the inequality

$$\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} \leq \dim A;$$

It holds true since any satureted chain of prime ideqals in A where $\mathfrak{p}$ occurs, can be split into two chains. The first consists of the ideals in the cahin contained in $\mathfrak{p}$ (this includes $\mathfrak{p}$ itself); its length is $\dim A_{\mathfrak{p}}$. The second chain consists of the remaining ideals, that is, those strictly containing $\mathfrak{p}$, and the length of that chain equals $A/\mathfrak{p}$.

In many case there is even an equality

$$\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim A. \tag{2}$$

DimPlusCodim

However this is slightly subtle—it requires that all saturated chains of prime ideals in A where $\mathfrak{p}$ occurs, are of the same length. Later on we shall see that this holds true for the coordinate rings $A(X)$ of affine varities. However, if the closed algebraic set X has two irreducible components of different dimensions, the equality (2) does trivially not hold for all prime ideals in $A(X)$

PROBLEM 5.5 Let $X = Z(zx, zy) \subseteq \mathbb{A}^3$. Describe X and determine $\dim X$. Exhibit two saturated chains of irreducible subvarities of different lengths. Exhibit a hypersurface Z so that $Z \cap X$ is of dimension zero. ★

Finite morphisms and Noethers Normalization Lemma

A very useful tool when establishing the basic theory of dimension is the Normalization Lemma. Combined with the Going up Theorem of Cohen and Seidenberg, it leads to the result that the dimension of a variety X and the transcendence degree of the function field $K(X)$ coincide. We formulate and prove the Normalization Lemma in the geometric context we work; that is, over an algebraically closed field. However the Normalization Lemma remains true, and the poof is *mutatis mutandis* the same, over any field.

Images and fibres

Let $\phi\colon X \to Y$ be a polynomial map between closed algebraic sets (the notion *morphism* is reserved for varieties for some unclear reason). To understand a map it is of course important to understand the fibres, and the following lemma gives a simple criterion for a point x to lie in a given fibre

InFibre

Lemma 5.2 *Let $\phi\colon X \to Y$ be a polynomial map beetween the two closed algebraic subsets X and Y and let $x \in X$ and $y \in Y$ be two points. Then $\phi(x) = y$ if and only if $\phi^*\mathfrak{m}_y \subseteq \mathfrak{m}_x$.*

PROOF: One has $\phi(x) = y$, if and only if $f(\phi(x)) = 0$ for all $f \in \mathfrak{m}_y$; that is, if and only if $\phi^*(f) = f \circ \phi \in \mathfrak{m}_x$ for all $f \in \mathfrak{m}_y$. ❑

In orher words, the fibre $\phi^{-1}(y)$ of ϕ over the a point $y \in Y$ is a closed algebtraic set given in X by the ideal $\phi^*\mathfrak{m}_y$. It can of course be empty and in which case $\in \phi^*\mathfrak{m}_y = A(X)$, and the ideal $\phi^*\mathfrak{m}_y$ need not be radical.

PROBLEM 5.6 Let $\phi\colon \mathbb{A}^1 \to \mathbb{A}^1$ be the map $\phi(t) = t^n$. For each point $a \in \mathbb{A}^1$ determine the ideal $\phi^*\mathfrak{m}_a$ and the fibre $\phi^{-1}(a)$. ★

PROBLEM 5.7 Let $\psi\colon \mathbb{A}^2 \to \mathbb{A}^2$ be the map $\psi(x,y) = (x, xy)$. Determine then ideals $\psi^*\mathfrak{m}_{(a,b)}$ for all points $(a,b) \in \mathbb{A}^2$. ★

PROBLEM 5.8 Let $\mathbb{A}^3 \to \mathbb{A}^3$ be given as $(x,y,z) \mapsto (yz, xz, xy)$. Find all fibres. ★

MORPHISMS WHOSE IMAGE is *dense* in the target, are called *dominant*. They are a little easier to handle than general polynomial maps, and several proofs are reduced to this case.

Dominant maps

Suppose that X and Y are varieties and that $\phi\colon X \to Y$ is a dominant morphism. Let f be a regular function defined on some open set in U in Y and shrinking U if necessary, we may assume that f does not have any zeros in U. Since $\phi(X)$ is dense in Y, the intersection $U \cap \phi(X)$ is non empty, and hence $f \circ \phi$ does not vanish identically on $\phi^{-1}U$. In other words, the composition map $\phi^*\colon A(Y) \to A(X)$ is *injective*. This leads to

Lemma 5.3 *A morphism $\phi\colon X \to Y$ between affine varieties is dominant if and only if the corresponding homomorphism $\phi^*\colon A(Y) \to A(X)$ is injective.*

PROOF: Half the proof is already done. For the remaining part, suppose the image $\phi(X)$ is not dense. Then its closure Z in Y is a proper closed subset, and $I(Z)$ is a non-zero ideal. Any function f in $I(Z)$ vanishes along $\phi(X)$, and hence $\phi^*(f) = f \circ \phi = 0$. ❑

The Going up Theorem

A polynomial map $\phi\colon X \to Y$ between two closed algebraic sets X and Y is said to be *finite* if the composition map $\phi^*\colon A(Y) \to A(X)$ makes $A(X)$ into a finitely generated $A(Y)$-module.

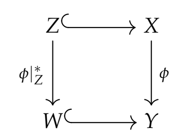
Finite polynomial maps

Finite morphisms have virtue of being closed and they are surjective when they are dominating. This is one part of the Going up Theorem, for finite maps it merely relies on Nakayama's lemma.

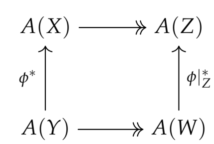
Moreover, two affine varieties which are related by a dominant finite morphism, have the same dimension. In the circles of ideas round the the Going Up and Going Down Theorems, these results are what we need. The proofs we give are formulated in our geometric context, but there are other variants valid in a more general setting, which you probably have seen in the course on commutative algebra.

EndeligLukket

Proposition 5.3 *Let $\phi\colon X \to Y$ be a finite polynomial map. Then ϕ closed. If it is dominating, it is surjective.*



PROOF: We begin with proving that ϕ is surjective when it is dominating. So assume there is a y in Y not by belonging to the image of ϕ. Then by lemma 5.2 above, it holds true that $\mathfrak{m}_y A(X) = A(X)$. Now $A(X)$ being finite, it follows from Nakayama's lemma that $A(X)$ is killed by element of the shape $1 + a$ with $a \in \mathfrak{m}_y$. The assumption that ϕ be dominant ensures that ϕ^* is injective, hence $0 = (1 + a) \cdot 1 = \phi^*(1 + a)$. It follows that $a = -1$ which is absurd since $a \in \mathfrak{m}_y$ which is a proper ideal.



To see that ϕ is a closed map, let $Z \subseteq X$ be closed, and decompose Z into its irreducible components $Z = Z_1 \cup \ldots \cup Z_r$. Then the image $\phi(Z)$ satisfies $\phi(Z) = \phi(Z_1) \cup \ldots \cup \phi(Z_r)$, and it suffices to show that each $\phi(Z_i)$ is closed. That is, we may assume that Z is irreducible. Putting W equal to the closure of $\phi(Z)$, the restriction $\phi|_Z\colon Z \to W$ is a dominating and finite map. Hence by the first part of the proof, it is surjective! In other words $\phi(Z) = W$ and is closed. ❑

LemmaGoingUp

Lemma 5.4 *Let $X \to Y$ be a dominating finite morphism between affine varieties, and suppose that $Z \subset X$ is proper and closed subset. Then $\phi(Z)$ is a proper subset of Y, that is $\phi(Z) \neq Y$.*

PROOF: Assume that $\phi(Z) = Y$ and let f be any regular function on X vanishing along Z. Since ϕ^* makes $A(X)$ a finitely generated module over $A(Y)$, there is a relation

$$f^r + \phi^*(a_{r-1})f^{r-1} + \ldots + \phi^*(a_1)f + \phi^*(a_0) = 0$$

where the a_i are functions on Y and where r is the least integer for which there is a such a relation. Obviously the relation implies that

$a_0 \circ \phi$ vanishes along Z, but since $\phi(Z)$ is equal to Y, the composition map $\phi|_Z^*$ is injective, and hence $a_0 = 0$. The integer r being minimal and $A(X)$ being an integral domain, we conclude that $f = 0$, and $Z = X$. ❑

Proposition 5.4 *Let $\phi\colon X \to Y$ be a finite and dominating morphism between affine varieties. Then* $\dim X = \dim Y$.

PROOF: We proceed by induction on $\dim Y$. Let $W \subseteq Y$ be a closed subset of codimension one, and let Z be an irreducible component of the inverse image $\phi^{-1}W$. It suffices to show that Z is of codimension one in X. To that end, assume that Z' is a proper, closed and irreducible subset of X containing Z; that is $Z \subseteq Z' \subset X$. By lemma 5.4 above, the image $\phi(Z')$ is an irreducible and proper subset of Y, and since it contains W, and W is of codimension one, it holds that $\phi(Z') = W$. Hence $Z' \subseteq \phi^{-1}W$ and therefore $Z' = Z$ since Z is a component of $\phi^{-1}W$. ❑

PROBLEM 5.9 Let $\phi\colon X \to Y$ be a dominating and finite morphism between two affine varieties. Show that if $Z_1 \subset Z_2 \subset \ldots \subset Z_r$ is a strictly ascending chain of irreducible closed subsets, the same is true for $\phi(Z_1) \subset \phi(Z_2) \subset \ldots \subset \phi(Z_r)$. ★

PROBLEM 5.10 A variety X is called *catenary* when it has the following property: Suppose $Z \subseteq Z'$ are two closed irreducible subsets. Then any two chains of irreducible and closed subsets of X connecting Z and Z' have the same length. Show that if $\phi\colon X \to Y$ is a dominating and finite morphism between two affine varieties, then one is catenary if and only if the other one is. ★

Catenary

Catenary affine varieties

The Normalization lemma

We shall formulate the Normalization lemma in the context of varieties; that is, in an algebra version this corresponfs to algebras finitely generated over an algebraically closed field. The proof however, works fine over any infinite field, and we shall have the occation during the course to use this more general result, but shall not prove it. There is also a slightly different proof valid over finite fields, which we shall not need. The proof is an inductibe argument, and the basic ingredient is the induction step as formulated in the following lemma:

NoetNormL1

Lemma 5.5 *Let $X \subseteq \mathbb{A}^m$ be an affine variety whose fraction field $k(X)$ has transcendence degree at most $m-1$; then there is a linear projection $\pi\colon \mathbb{A}^m \to \mathbb{A}^{m-1}$ so that $\pi|_X\colon X \to \mathbb{A}^{m-1}$ is a finite morphism.*

PROOF: Let $A(X) = k[x_1, \ldots, x_m]/I(X)$ be the coordinate ring of X and denote by a_i the image of x_i in $A(X)$. Since the transcendence degree of $A(X)$ over k is less than m, the m elements $a_1, \ldots, a_m$ can not be algebraically independent and must satisfy and equation

$$f(a_1, \ldots, a_n) = 0,$$

where f is a polynomial with coefficients in k. Let d be the degree of f and let f_d be the homogenous component of degree d. Put $a'_i = a_i - \alpha_i a_1$ for $i \geq 2$ where the α_i's are scalars to be chosen. This gives[1]

$$0 = f(a_1, \ldots, a_m) = f_d(1, \alpha_2, \ldots, \alpha_m) a_1^d + Q(a'_1, \ldots, a'_m)$$

where Q is a polynomial whose terms all are of degree less that d in a_1. Now, since the ground field is infinite, a generic choice of the scalars α_i implies that $f(1, \alpha_1, \ldots, \alpha_m) \neq 0$ (see exercise 5.11 below). Hence the element a_1 is integral over $k[a'_2, \ldots, a'_m]$ and by conscequence, $A(X)$ is a finite module over the algebra $k[a'_2, \ldots, a'_m]$. The projection $\mathbb{A}^m \to \mathbb{A}^{m-1}$ sending $(a_1, \ldots, a_m)$ to $(a'_2, \ldots, a'_m)$ does the thrick. ❏

[1] Recall that for any polynomial $p(x)$ it holds true that $p(x+y) = p(x) + yq(x,y)$ where q is a polynomial of total degree less than the degree of f.

PROBLEM 5.11 Let k be an infinite field and $f(x_1, \ldots, x_n)$ a non-zero polynomial with coefficients from k. Show that $f(a_1, \ldots, a_n) \neq 0$ for infinitely many choices of a_i from k. HINT: Use induction on n and expand f as $f(x_1, \ldots, x_n) = \sum_i g_i(x_1, \ldots, x_{n-i}) x_n^i$. ★

AlltidForsjNull

By induction on m one obtains the full version of the normalization lemma:

Theorem 5.1 (Noethers Normalization Lemma) *Assume that $X \subseteq \mathbb{A}^m$ is a closed subvariety and that the function field $k(X)$ is of transcendence degree n over k. Then there is a linear projection $\pi\colon \mathbb{A}^m \to \mathbb{A}^n$ such that the projection $\pi|_X\colon X \to \mathbb{A}^n$ is a finite map.*

PROOF: We proceed by induction on m. If $m \leq n$, the elements $a_1, \ldots, a_m$ must be algebraically independent since they generate the field $K(X)$ over k. But any non-zero polynomial in $I(X)$ would give a dependence relation among them, so we infer that $I(X) = 0$, and hence that $X = \mathbb{A}^m$.

Suppose then that $m > n$. By lemma 5.5 above, there is a finite projection $\phi\colon X \to \mathbb{A}^{m-1}$. The image $\phi(X)$ is closed by proposition 5.3 on page 6 and of the same transcendence degree as X since $K(X)$ is a finite extension of $K(\phi(X))$. Applying the induction hypothesis to $\phi(X)$, we may find a finite projection $\pi\colon \phi(X) \to \mathbb{A}^n$. The composed map $X \to \phi(X) \to \mathbb{A}^n$ is finite. ❏

Theorem 5.2 *Let X by any variety. Then* $\dim X = \operatorname{trdeg}_k K(X)$. *Moreover, X is catenary.*

DimLikTransDeg

In particular the theorem states that the affine n-space $\mathbb{A}^n$ is of dimension n since cleary the transcendence degree of $K(\mathbb{A}^n) = k(x_1, \ldots, x_n)$ is n.

PROOF: There are two parts of the proof; the case of $\mathbb{A}^n$ and the general case, and the general case is easily reduced to the case of $\mathbb{A}^n$. Indeed, replacing X by some open dense and affine subset, we may assume that X is affine. Let $n = \operatorname{trdeg}_k K(X)$. By the normalization theorem there is a finite map $X \to \mathbb{A}^n$, hence $\dim X = \dim \mathbb{A}^n = n$.

The case of $\mathbb{A}^n$ is done by induction on n; obviously it holds that $\mathbb{A}^1$ is one dimensional (the ring $k[t]$ is a PID). So assume that $n > 1$ and let $Z \subset \mathbb{A}^n$ be a maximal proper and closed subvariety. Then $\dim Z = \dim \mathbb{A}^n - 1$ and $\operatorname{trdeg}_k K(Z) \leq n - 1$ because $I(Z) \neq 0$. Noethers Normalization Lemma gives us a finite and dominating morphism $Z \to \mathbb{A}^m$, where $m = \operatorname{trdeg}_k K(X)$. By induction it holds true that $\dim \mathbb{A}^m = m$ and hence $\dim Z = \operatorname{trdeg}_k K(Z)$. Hence we find

$$\dim X = \dim \mathbb{A}^n - 1 = \operatorname{trdeg}_k k(Z) \leq n - 1,$$

and therefore $\dim \mathbb{A}^n \leq n$. The other inequallity is trivial; there is an obvious ascending chain of linear subspaces of length n in $\mathbb{A}^n$.

Finally, to show that X is catenary, we may assume that X is affine. This follows from proposition 5.2 on page 3, or rather from the proof of thata proposition.

We proceed by induction on $\dim X$, and it clearly suffices to see that $\mathbb{A}^n$ is catenary, and by induction on $\dim X$, it suffices to see that two maximal saturated chains of subvarieties in $\mathbb{A}^n$ have the same length. So let Z and Z' respectively be the largest members of the two chains not equal to $\mathbb{A}^n$. Their dimensions are less than n and by induction both are catenary, and the rest of two chain must have the same length. ❑

The dimension of a product

The Normalization Lemma also gives an easy proof of the formula for the dimension of a product. It hinges on the fact that the product of to finite maps is finite, and by The Normalization Lemma the proof is reduced to the case of two affine spaces. The formula is stated for affine varieties due to the fact that we merely have defined products of these.

Proposition 5.5 *Let X and Y be two (affine) varieties. Then* $\dim X \times Y = \dim X + \dim Y$. DimProdukt

Lemma 5.6 *Let X, Y, Z and W be affine varieties. Let $\phi\colon X \to Y$ and $\psi\colon Z \to W$ be two finite morphisms. Then the morphism $\phi \times \psi\colon X \times Z \to Y \times W$ is finite.* ProdFinite

PROOF: We first establish the lemma for the special case when $W = Z$ and $\psi = \mathrm{id}_Z$. In that case the map $(\phi \times \mathrm{id}_Z)^*\colon A(Y) \otimes A(Z) \to A(X) \otimes A(Z)$ is just $\phi^* \otimes \mathrm{id}_{A(Z)}$. If $a_1, \ldots, a_r$ are elements in $A(X)$ that generates $A(X)$ as an $A(Y)$-module, the elements $a_i \otimes 1$ generates $A(X) \otimes A(Z)$ as a module over $A(Y) \otimes A(Z)$, and we are through.

One reduces the general case to this special case by observing that $\phi \times \psi$ is equal to the the composition

$$X \times Z \xrightarrow{\phi \times \mathrm{id}_Z} Y \times Z \xrightarrow{\mathrm{id}_Y \times \psi} Y \times W$$

and using that the composition of two finite maps is finite. ❑

PROOF OF PROPOSITION 5.5: Let $\phi\colon X \to \mathbb{A}^n$ and $\psi\colon Y \to \mathbb{A}^m$ be finite and surjective maps. Then $\phi \times \psi\colon X \times Y \to \mathbb{A}^n \times \mathbb{A}^m$ is finite by lemma 5.6 above, and it is clearly surjective. ❑

Krull's Principal Ideal Theorem

This is another german theorem, H*Hauptidealsatz*, but it is mostly refered to by its english name. The simplest version of the theorem concerns the dimension of the intersection of hypersurface with a variety X in $\mathbb{A}^m$, and confirms the intutive belief that a hypersurface cuts out a space in X of dimension one less than $\dim X$. This statement must be taken with a grain of salt since the intersection could be empty, and of course, the variety X could be contained in the hypersurface in which case the intersection equals X, and the dimension does not drop. If X is not irreducible, the situation is somehow more complicated. The different components of X can be of different dimensions and they may or may not meet the hypersurface; the components must be treated one by one.

HERE COMES KRULL'S THEOREM. We shall state it, but not prove it (check your favorit text on commuative algebra to refresh your memory).

Theorem 5.3 *Let A be Noetherian ring and let $f \in A$ be a non-zero element that is not a unit. Then the height of a minimal prime of the principal ideal (f) is a most one.*

The geometric version of Krull's Hauptidealsatz reads as follows. We formulate it for affine *varieties*, that is for irreducible closed algebraic sets. Notice that the coordinate rings of affine vareties are catenary, so the equality (2) on page 4 holds, and the heigh of a prime ideal $A(X)$ equals the codimension of the variety it defines.

KrullGeom

Theorem 5.4 *Let $X \subseteq \mathbb{A}^m$ be an affine variety and let f be a polynomial on $\mathbb{A}^m$. Assume f does not vanish identically along X. Then every component Y of $Z(f) \cap X$ satisfies* $\dim Y = \dim X - 1$.

It can of course very well happen that $Z(f)$ and X have an empty intersection, in which case the theorem says nothing; in the theorem it is implicitly understood that the component Y is non-empty.

One may formulated the theorem in terms of regular functions on a general variety as done below. This version is easily reduced to theorem 5.4. Just replace X by an open affine subset intersecting the component Y, and the regular function by an extension to a surrounding affine space.

Theorem 5.5 *Let X be a variety and let f be a non-zero regular function on X. Then every component Y of $Z(f) \cap X$ satisfies* $\dim Y = \dim X - 1$.

For reducible subsets X of $\mathbb{A}^m$ there is no clear and uniform statement. The result depends on how f is related to the different components of X, and virtually every conclusion is possible.

Proof: The restriction $f|_X$ is a unit if and only if $Z(f) \cap X = \emptyset$, and $f|_X = 0$ if and only if $X \subseteq Z(f)$. So we may assume that $g = f|_X$ is not equal to zero and not a unit. A component Y of $Z(f) \cap X$ corresponds to minimal prime ideal $I(Y)$ of the principal ideal (g) in $A(X)$. By the algebraic version of Krull's Principal Ideal theorem, the height of $I(Y)$ is at most one, but since 0 is a prime ideal in $A(X)$, the height equals one, and $\dim Y = \dim X - 1$.

The Hauptidealsatz generalizes to intersections of a closed algebraic set X with a sequences of hypersurfaces. Since intersecting with each one of the hypersurfaces increases the codimension with at most one, induction on the number of hypersurfaces easily gives the following:

Theorem 5.6 *Suppose that $X \subseteq \mathbb{A}^n$ is a closed algebraic subset and that $f_1, \ldots, f_r$ are polynomials. Then any component Y of $Z(f_1, \ldots, f_r) \cap X$ is of codimension at most r in X.*

Proof: The proof goes by induction on r. Let Y' be a component of $Z(f_1, \ldots, f_{r-1}) \cap X$ containing Y. By induction Y' is of codimension at most $r - 1$ in X; that is, $\dim Y' \geq \dim X - r + 1$. Moreover, Y must be a component of of $Y' \cap Z(f_r)$, and therefore either $\dim Y' = \dim Y - 1$ or f_r vanishes on Y' by theorem The Principal Ideal Theorem (theorem 5.4 above). In the former case obviously $\dim Y \geq \dim X - r$, and in the latter, we find $Y = Y'$ and $\dim Y \dim X - r + 1 \geq \dim X - r$. ❑

PROBLEM 5.12 Let $n > m$ be two natural numbers. Exhibit a (necessarily reducible) closed algebaic subset X of some affine space $\mathbb{A}^N$ and a hypersurface $Z(f)$ such hat $\dim X = n$ and $\dim X \cap Z(f) = m$. HINT: For instance, one can let X be the union of two linear subspaces of dimension n and $m + 1$. ★

Intersection in projective spaces

The Principal Ideal Theorem has important cosequence for intersections in projectice spaces. The most stiking is that the intersection of two subvarietis will be non-empty once their dimesnions satisfy the following very natural condition. If X and Y are the two subvarieties of $\mathbb{P}^n$ then $X \cap Y \neq \emptyset$ once

$$\operatorname{codim} X + \operatorname{codim} Y \leq n. \tag{3}$$

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One can even say more, for any component Z of the intersection $X \cap Y$, the following inequality holds

$$\operatorname{codim} Z \leq \operatorname{codim} X + \operatorname{codim} Y.$$

It is straight forward to finf examples of projective varieties X and Y not satisfying he inequality (3) and having an empty intersection. Just take two linear subvarieties $\mathbb{P}(V)$ and $\mathbb{P}(W)$ with $\dim \mathbb{P}(V) + \dim \mathbb{P}(W) < n$ (*e.g.*, two skew lines in $\mathbb{P}^3$).

Reduction to the diagonal

These intersection theorms follows by combining Krull's Principal Ideal Theorem with a trick called the "Reduction to the diagonal". It consists of the observation that the following observation. Let X and Y be two subvarieties of $\mathbb{A}^n$. Then of course $X \times Y$ lies as a closed subvariety of the affine space $\mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$. And clearly $X \cap Y$ is isomorphic to the intersection $\Delta \cap X \times Y$, where Δ is the diagonal in $\mathbb{A}^n \times \mathbb{A}^n$.

The salient point is that the diagonal is cut out by a set of very simple equations. If the coordinates on corresponidng to the left factor in $\mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ are $\{x\}_i$ and those of the right factor $\{y_i\}$ the diagonal is given by the vanishing of the n functions $x_i - y_i$. Hence we can conclude by Krull's Principal Ideal Theorem that any (non-empty) component Z of $X \cap Y$ satsfies $\dim Z \geq \dim X \times Y - n$ but $\dim X \times Y = \dim X + \dim Y$ and we find

$$\dim Z \geq \dim X + \dim Y - n.$$

Summing up we formuate the result as a lemma Let X and Y be two subvarieties of $\mathbb{A}^n$ then any (non-empty) component of the intersection $X \cap Y$ satifies

$$\operatorname{codim} Z \leq \operatorname{codim} X + \operatorname{codim} Y.$$

Of course, it might very well happen that $X \cap Y$ is empty, even for hyper surfaces. As well, the strict inequality might hold; for example it could happen that $X = Y$!

The projective case

The proof of thenext intersection theorem below will pass by the afine cones over the projective varities, so we begin with a few observations about them. The natural equality $C(X \cap Y) = C(X) \cap C(Y)$ is obvious, and if Z is a component of the intersection $X \cap Y$ the cone $C(Z)$ will be a component of $C(X \cap Y)$. Going to cones increses the dimensions by one; that is, for any varety X it holds that $\dim C(X) = \dim X + 1$. Then of cpurse, it holds true that $\operatorname{codim}_{\mathbb{P}^n} X = \operatorname{codim}_{\mathbb{A}^{n+1}} C(X)$; that is the codiemsnion of X in $\mathbb{P}^n$ is the same as the codimension of its cone in $\mathbb{A}^{n+1}$.

The following theorem is on of the cornerstones in projective geometry. Wether two varities intersect or not is as much a question of their size as of their relative position: If they are "large enough" they intersect.

Proposition 5.6 *Let X and Y be two projective varieties in the projective space $\mathbb{P}^n$. Assume that $\dim Y + \dim X \geq n$. Then the intersection $X \cap Y$ is non-empty, and any component Z of $X \cap Y$ satisfies*

$$\operatorname{codim} Z \leq \operatorname{codim} X + \operatorname{codim} Y.$$

PROOF: Firstly, if $\dim X + \dim Y \geq n$ then $\dim C(X) + \dim C(Y) \geq n + 2$ and the salient point is that the intersection $C(X) \cap C(Y)$ is always non-empty: The two cones both contain the origin! Moreover, the dimenson of any component W of $C(X) \cap C(Y)$ satisfies $\dim W \geq \dim C(X) + \dim C(Y) - n - 1 = \dim X + \dim Y - n + 1 \geq 1$. One deduces that the intersectiion $C(X) \cap C(Y)$ is not reduced to the origin, and hence is the cone over a non-empty sunset in $\mathbb{P}^n$.

Since X and the cone over X have the same codimension, we deduce directly from xxx that

$$\operatorname{codim} Z \leq \operatorname{codim} X + \operatorname{codim} Y.$$

❑