MAT4210—Algebraic geometry I: Notes 6

Bezout's theorem

6th March 2018

Hot themes in Notes 6: Divisors—Local multiplicities—The Koszul complex—Bezout's theorem in the plane—General Koszul—A little about graded modules

Super-Preliminary version 0.1 as of 6th March 2018 at 9:17pm—Well, still not really a version at all, but better. Improvements will follow!

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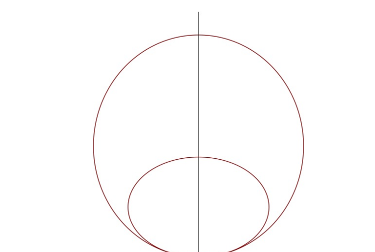
Étienne Bézout (1730–1783)
French Mathematician

Introduction

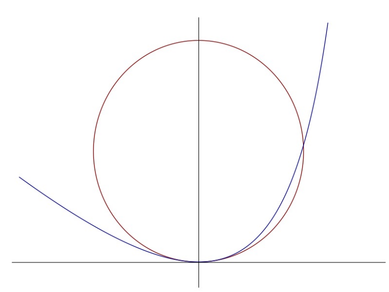
Already sir Isaac Newton seems to have observed that the number of intersection points of two curves in $\mathbb{P}^2$ equals the product of their degrees. If one start looking at examples this pattern emerges almost immediately. Two lines meet in one point and two conics in four—at least if the two conics are what one calls general.

That a line L meets a curve X which is the zero locus of a homogeneous form F of degree n, is a direct consequence the fundamental theorem of algebra. Choosing appropriate coordinates we can parametrize the line as $(u; v; 0)$; and hence the parameter values of the intersection points are the roots of the equation $F(u, v, 0) = 0$, of which there are n, unless, of course x_2 is a factor of F, in which case the line L is a component of X. There is also an issue of multiplicities, all roots need not be simple. This issue persists in the general situation and is inherent part of the problem.



Two ellipses with fourth order contact.

Example 6.1 The two conics $zy - x^2$ and $zy - x^2 - y^2$ only intersect at $(0; 0; 1)$. Indeed, the difference of the two equations being y^2, it must hold that $y = 0$ at a common zero; and then x must vanish there as well. The parabolas have contact oder four at $(0; 0; 1)$; inserting the parametrization (uv, u^2, v^2) of the first into the equation of the second yields, the equation $u^4 = 0$, which has a quadruple root at $u = 0$. ☆



Apparently what now is called Bézout's theorem in the plane, was known a good time before Bézout's published his famous paper *Théorie générale des équations algébriques* in 1779. His original contribution is the generalization to projective n-space $\mathbb{P}^n$. He proved that the number of points n hypersurfaces in $\mathbb{P}^n$ have in common, if finite, equals the product of their degrees. And as usual there is an issue of multiplicities; local multiplicities are part of the accounting.

Curves versus divisors

Like in the section xxx where we spoke about hypersurfaces, we shall give an extended meaning to the word curve in the present context around Bézout's theorem. The proper technical term is an *effective divisor*. Such an animal is just a *formal* linear combination $\sum_i n_i X_i$ where the n_i's are non-negative[1] integers and the X_i's are (irreducible) subvarieties of $\mathbb{P}^2$ of codimension one—that is, irreducible curves. This might look enigmatic at the first encounter, but it is merely a convenient geometric way to keep track of all the irreducible components of a homogeneous form. Indeed, if F is homogeneous of degree n and splits like $F = \prod_i F_i^{n_i}$ into a product of irreducible forms, the associate divisor is $\sum_i n_i Z_+(F_i)$.

Effective divisors

[1] The significance of the attribute *effective* is that the coefficients n_i are non-negative. A *divisor* is a linear combination $\sum_i n_i X_i$ with integral coefficients

And so, by a curve X we shall mean the divisor associated to a homogeneous from F, and we shall allow ourself the slight abuse of the language to write $Z_+(F)$ for that divisor (in case of an imminent danger of confusion, we shall be precise about what Z_+ means). In the benign case that F has no multiple factors, the two interpretations of $Z_+(F)$ coincide.

The degree of the curve X means the degree of F and it holds that $\deg X = \sum_i n_i \deg X_i$.

Bézout's theorem in the plane

To set the scene let X and Y be two curves on $\mathbb{P}^2$ without a common component. The curve X will be the zero locus of a homogeneous form F and Y that of the homogeneous form G ; that is, $X = Z_+(F)$ and $Y = Z_+(G)$. The degrees of F and G will be denoted by m and n respectively.

Since $X \cap Y$ is a finite set we can choose homogeneous coordinates x_0, x_1, x_2 on $\mathbb{P}$ in a way that $X \cap Y$ is contained in the basic affine open set $D = D_+(x_2)$; in other words x_2 does not vanish at any point of $X \cap Y$. The basic open set D is an affine 2-space with coordinates $x_0 x_2^{-1}$ and $x_1 x_2^{-1}$. We shall keep this notation through out Notes 6.

The local picture

A description of the intersection $X \cap Y$ in the local affine piece D is obtain by dehomogenizing the two forms F and G with respect to the variable x_2. This yields two polynomials f and g in $x_0 x_2^{-1}$ and $x_1 x_2^{-1}$ which are related to F and G by the equalities

$$x_2^m f(x_0 x_2^{-1}, x_1 x_2^{-1}) = F(x_0, x_1, x_2)$$
$$x_2^n g(x_0 x_2^{-1}, x_1 x_2^{-1}) = G(x_0, x_1, x_2).$$

The zero locus $Z(f,g)$ in the affine space D equals $X \cap Y$, but of course, if local multiplicities are involved, it will not be radical. Anyhow, we introduce the algebra

$$\mathcal{O}_{X\cap Y} = k[x_0x_2^{-1}, x_1x_2^{-1}]/(f,g).$$

This algebra is supported at the intersection $X \cap Y$, and again since $X \cap Y$ is finite, $\mathcal{O}_{X\cap Y}$ is a finite dimensional algebra over k. Therefore it decomposes as the product

$$\mathcal{O}_{X\cap Y} = \prod_{p\in X\cap Y} \mathcal{O}_{X\cap Y,p}$$

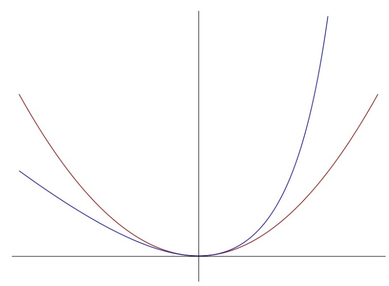
where $\mathcal{O}_{X\cap Y,p}$ denotes the localization of $\mathcal{O}_{X\cap Y}$ at the maximal ideal corresponding to the point p.

The local intersection multiplicities

The *local intersection multiplicity* of the curves X and Y at the point p is defined as the integer $\mu_p(X,Y) = \dim_k \mathcal{O}_{X\cap Y,p}$. Then it holds true that

$$\dim_k \mathcal{O}_{X\cap Y} = \sum_{p\in X\cap Y} \dim_k \mathcal{O}_{X\cap Y,p}.$$

With this in place, we can formulate Bezout's theorem in the plane, but recall that the term curve has the extended meaning explained above.



Two parabolas with triple contact at the origin.

Theorem 6.1 *Let X and Y be two curves in $\mathbb{P}^2$ without a common component. Then*

$$\deg X \cdot \deg Y = \sum_{p\in X\cap Y} \mu_p(X,Y).$$

Transversal intersections

The simplest local behavior two curves X and Y can have in an intersection point p is that the multiplicity $\mu_p(X,Y)$ is one. This happens when and only when the curves intersect *transversally*; *i.e.*, when both have a well defined tangent at p and the two tangents are different. To explain what this means, we assume that p is the origin and use coordinates x and y in a basic open affine neighbourhood about p. Then $f = a_1x + b_1y + H_1(x,y)$ and $g = a_2x + b_2y + H_2(x,y)$ where the terms H_i's are of degree superior to one. The linear forms $L_i = a_ix + b_iy$ define the tangents to the curves, and the curves intersect transversally if L_i are linearly independent; that is, they are non-zero and define two distinct lines.

EXAMPLE 6.2 — TRANSVERSAL INTERSECTIONS Assume that two curves X and Y are given locally round the origin p of $\mathbb{A}^2$ as the zero sets of the polynomials $f = x + H_1$ and $g = y + H_2$ where $\deg H_i \geq 2$. Then $\mu_p(X,Y) = 1$.

Indeed, collecting all terms of H_1 containing x together in a term $xr(x,y)$, one may write

$$f = x(1 + r(x,y)) + h(y)$$

where $r(x,y)$ and $h(y)$ both are of degree one or more. In the local ring $\mathcal{O}_{\mathbb{A}^2,p}$ the element $(1+r)$ is invertible with say $a = (1+r)^{-1}$. Hence $x = -ah(y) \bmod (f,g)$. This gives that

$$g = y(1 + h_2(ah(y), y))$$

mod (f,g), and the important points is that $1 + h_2(ah(y), y)$ is invertible since $H_2(ah(y), y)$ has y^2 as a factor. It follows that $y \in (f,g)\mathcal{O}_{\mathbb{A}^2,p}$ and hence x lies there as well, so that $(f,g)\mathcal{O}_{\mathbb{A}^2,p} = (x,y)$, and the multiplicity is one. ☆

EXAMPLE 6.3 Let $f(x,y) = y - x^2$ and $g(x,y) = y - x^2 - xy$. Then the two curves $X = Z(f)$ and $Y = Z(g)$ have triple contact at the origin; that is, $\mu_p(X,Y) = 3$. One finds

$$(f,g) = (y - x^2, y - x^2 - xy) = (y - x^2, x^3),$$

and hence $\mathcal{O}_{X\cap Y,p} = k[x,y]/(f,g) \simeq k[x]/x^3$ ☆

PROBLEM 6.1 Let $n > m$ be two natural numbers and let $\alpha(x)$ and $\beta(x)$ be two polynomials which do not vanish at $x = 0$. Determine the local intersection multiplicity at the origin of the two curves defined respectively by $y - \alpha(x)x^n$ and $y - \beta(x)x^m$. If $m = n$, show by exhibiting an example that the local multiplicity can take any integral value larger than n. ★ OK

PROBLEM 6.2 Find all intersection points of the two cubic curves defined by the forms $zy^2 - x^3$ and $zy^2 + x^3$ (we assume the characteristic of the ground field to be different from two). Determine all the local intersection multiplicities of the two curves. ★ OK

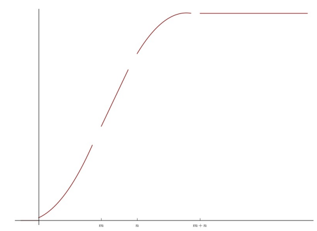
The proof of Bézout in the plane

There are many proofs of Bézout's theorem of various flavours, and the one we shall present, leans on an analysis of the graded k-algebra $A = k[x_0, x_1, x_2]/(F, G)$ and its Hilbert function.

One half of the proof naturally belongs to the realm of what are called coherent sheaves on $\mathbb{P}^2$ and their Euler characteristics, and within that context is obvious. However, we do not have all that advanced machinery to our disposal and have to do with an *ad hoc* calculation (and to be honest, a calculation that would be in some way or the other included in the development of the theory). The point is to link the dimensions $\dim_k A_d$ of the graded pieces of degree d to the local multiplicities. We shall see that, at least when d is sufficiently large, $\dim_k A_d$ do not depend on d, and in fact it holds true that

$$\dim_k A_d = \sum_{p \in X\cap Y} \mu_p(X,Y).$$

The other half of the proof is basically the same in our context as in the context of coherent sheaves. It consists of using what is called a Koszul complex, to show that for $d >> 0$ the dimensions $\dim_k A_d$ are equal to the product $\deg X \cdot \deg Y$; that is, to the left side of the equality in Bézout's theorem.



The graph of the Hilbert function $h_A(d)$.

The Koszul complex

There are a great many Koszul complexes around (in fact one for each finite sequence of elements in a ring), but the one we need, is of the simplest sort. It is a four term[2] exact sequence of graded R-modules[3], and is shaped like

$$0 \longrightarrow R(-n-m) \xrightarrow{\alpha} R(-n) \oplus R(-m) \xrightarrow{\beta} R \longrightarrow R/(F,G) \longrightarrow 0.$$

The maps α and β are defined as $\alpha(a) = (aF, -aG)$ and $\beta(a,b) = aG + bF$, and the last map is the quotient map. One easily checks that $\beta \circ \alpha = 0$, and indeed, that $\ker \beta = \alpha$. This last equality hinges on the assumption that F and G be without common components. Indeed, if $\beta(a,b) = 0$; that is, if $aF = -bG$, it follows that $a = cG$ and $b = -cF$ for some polynomial c since the polynomial ring is a UFD). The sequence obviously being exact at all other places than at the middle one, is therefore exact.

[2] Strictly speaking, only the three leftmost terms belong to the complex. The quotient $R/(F,G)$ is the zeroth homology of the complex.

[3] Hence the maps are homogeneous of degree zero.

Lemma 6.1 *The Hilbert function $h_A(d)$ of A satisfies $h_A(d) = nm$ for $d \geq n+m$.*

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French mathematician

PROOF: It is a general fact that the Hilbert function is additive over exact sequences of graded modules[4], and according to this principle we find for $d > n+m$ that

$$h_A(d) = \binom{d+2}{2} - \binom{d-n+2}{2} - \binom{d-m+2}{2} + \binom{d-m-n+2}{2} = nm,$$

where the last equality results from a trivial computation with the binomial coefficients. ❑

[4] The maps must of course be homogeneous of degree zero. Additivity means that for each degree d the alternating sum of the dimensions of the graded pieces of degree d equals zero

PROBLEM 6.3 Perform the trivial computation in the proof. ★

The local connection

Now, we have come to the point where to establish the link between the graded pieces of A and the algebra $\mathcal{O}_{X \cap Y}$. Recall that when setting the scene, we chose coordinates so that the entire intersection $X \cap Y$ is contained in the basic open set $D = D_+(x_2)$. So it is very natural to localize at x_2 and to consider the localized algebra A_{x_2}. Since x_2 is homogeneous, A_{x_2} is a graded algebra whose homogeneous elements are of the form $H \cdot x_2^{-r}$ with H (the residue class of) a homogeneous polynomial. The degree of the homogeneous element $H \cdot x_2^{-r}$ is of course equal to $\deg H - r$.

The first crucial fact is the following description of the graded pieces of A_{x_2}; they are all isomorphic to $\mathcal{O}_{X\cap Y}$:

Lemma 6.2 *The degree zero part of A_{x_2} equals $\mathcal{O}_{X\cap Y}$, and the decomposition of A_{x_2} into homogeneous pieces takes the form:*

$$A_{x_2} = \bigoplus_{i\in\mathbb{Z}} k[x_0x_2^{-1}, x_1x_2^{-1}]/(f,g) \cdot x_2^i = \bigoplus_{i\in\mathbb{Z}} \mathcal{O}_{X\cap Y} \cdot x_2^i.$$

PROOF: The first thing to observe, is that in the ring A_{x_2} where x_2 is invertible the equality, $(F,G) = (f,g)$ holds; indeed, $x_2^n f = F$ and $x_2^m g = G$. The second is that both f and g are homogeneous of degree zero.

Given that $(A_{x_2})_0 = \mathcal{O}_{X\cap Y}$ clearly $\mathcal{O}_{X\cap Y} \cdot x_2^d \subseteq (A_{x_2})_d$, and because x_2 is invertible, the multiplication map is injective.

Any homogeneous element z in A_{x_2} is of the form $z = ax_2^s$ where a is homogeneous of degree zero and $s \in \mathbb{Z}$; indeed, z equals $H(x_0,x_1,x_2)x_2^{-r}$ for some homogeneous polynomial H and some integer r, and therefore $z = H(x_0x_2^{-1}, x_1x_2^{-1}, 1)x_2^{d-r}$ where d denotes the degree of H. Hence $A_{x_2} = \bigoplus_{d\in\mathbb{Z}}(A_{x_2})_0 \cdot x_2^d$.

What remains, is to identify the degree zero piece $(A_{x_2})_0$. To that end, consider the quotient map $R \to R/(F,G)$. When localized in x_2 it yields the quotient map

$$R_{x_2} \to R_{x_2}/(f,g)R_{x_2} = A_{x_2}.$$

Considering the degree zero part of this map, and observing that the degree zero part of the ideal (f,g) in R_{x_2} is the ideal $(f,g)(R_{x_2})_0$ in $(R_{x_2})_0$—since f and g both are of degree zero—we are done; indeed, the degree zero part of R_{x_2} equals $(R_{x_2})_0 = k[x_0x_2^{-1}, x_1x_2^{-1}]$. ❑

The second, and last, crucial element in the proof is the the following:

Lemma 6.3 *For d sufficiently large, the localization map $A \to A_{x_2}$ induces an isomorphism $A_d \to (A_{x_2})_d$ between the graded pieces of degree d.*

PROOF: There are two things to prove; that the map is injective and that it is surjective.

First of all, the kernel of the localization map has support at the origin because the locus $Z(x_2)$ and the support of A only has the origin in common: Since $Z_+(F,G,x_2) = \emptyset$, the Projective Nullstellensatz implies that (F,G,x_2) is $\mathfrak{m}_+$-primary (recall that $\mathfrak{m}_+ = (x_0,x_1,x_2)$). Any element in the kernel of the localization map is killed by some power of x_2 and being an element in A, it is killed by F and G. Hence the kernel is killed by some power $\mathfrak{m}_+^N$, and is therefore of finite dimension as a vector space over k. Being of finite dimension over k, the kernel can merely have finitely many graded pieces different from zero, and hence the localization maps induce injections $A_d \to (A_{x_2})_d$ in large degrees.

Turning to the surjectivity, we observe that any homogeneous element $ax_2^r \in A_{x_2}$ with a of degree zero, can be expressed as a product $ax_2^r = Hx_2^{r-d}$ where H is the residue class mod (F, G) of a homogeneous polynomial of degree d, and consequently, when $r > d$, the element ax_2^r lies in the image of the localization map. So, take any basis $a_1, \ldots, a_r$ for $(A_{x_2})_0$ and write the members as products $a_j = H_j x_2^{-d_j}$ where H_j is the residue class of a homogeneous polynomial of degree d_j. If now $d > \max d_j$, all the products $a_j x_2^d$ lie in the image by our observation above; and since multiplication by x_2 is an isomorphism $(A_{x_2})_0 \to (A_{x_2})_d$, this shows that the localization map $A_d \to (A_{x_2})_d$ is onto, and we are sauf and saint. ❑

$$\begin{array}{ccc} A_d & \longrightarrow & (A_{x_2})_d \\ {\scriptstyle x_2^d}\uparrow & & \simeq\uparrow{\scriptstyle x_2^d} \\ A_0 & \longrightarrow & (A_{x_2})_0 \end{array}$$

Summing up, the two lemmas combined yields the result we want:

Proposition 6.1 *For $d >> 0$, the localization map $A \to A_{x_2}$ induces an isomorphism between the graded piece A_d of A and $\mathcal{O}_{X\cap Y} \cdot x_2^d$. In particular, the following equality holds true*

$$\dim_k A_d = \dim_k \mathcal{O}_{X\cap Y}.$$

PROOF OF BÉZOUT'S THEOREM: Finally, to finish of the proof of Bézout's theorem the lemmas we have established and the definitions we have given yields the following sequence of equalities:

$$mn = h_A(d) = \dim A_d = \dim_k \mathcal{O}_{X\cap Y} = \sum_p \dim_k \mathcal{O}_{X\cap Y, p} = \sum_p \mu_p(X, Y)$$

❑

General Bézout

Notice that in the above argument, we never used in an essential way that the scene is set in $\mathbb{P}^2$. The arguments go through *mutatis mutandis* for the intersection of n hypersurfaces $X_i = Z_+(F_i)$ in $\mathbb{P}^n$ as long as the intersection is finite. One chooses coordinates such that $X_1 \cap \ldots \cap X_n$ is entirely contained in $D_+(x_n)$, and one lets f_i be the polynomial F_i dehomogenized with respect to x_n. The algebra

$$\mathcal{O}_{X_1\cap\ldots\cap X_n} = k[x_0x_n^{-1}, \ldots, x_{n-1}x_n^{-1}]/(f_1, \ldots, f_n)$$

is then finite dimensional as a vector space over k. Moreover, letting $A = k[x_0, \ldots, x_n]/I$ where I is the ideal generated by the F_i's; that is $I = (F_1, \ldots, F_n)$, one obtains the following

Proposition 6.2 *For $d >> 0$, the localization map $A \to A_{x_n}$ induces an isomorphism between the graded piece A_d of A and $\mathcal{O}_{X_1 \cap \ldots \cap X_n} \cdot x_n^d$. In particular*

$$\dim_k A_d = \dim_k \mathcal{O}_{X_1 \cap \ldots \cap X_n}.$$

There is a Koszul complex build on the n forms F_i of length $n+1$. It requires more work to construct than the simple one we used above, and shall not do that. But established, via a computation with binomial coefficients, it shows that $\dim A_d = \deg X_1 \cdot \ldots \cdot X_n$.

Introducing local intersection multiplicities by splitting $\mathcal{O}_{X_1 \cap \ldots \cap X_n}$ into a product of local factors

$$\mathcal{O}_{X_1 \cap \ldots \cap X_n} = \prod_{p \in X_1 \cap \ldots \cap X_n} \mathcal{O}_{X_1 \cap \ldots \cap X_n, p}$$

and setting $\mu_p X_1, \ldots, X_n = \dim_k \mathcal{O}_{X_1 \cap \ldots \cap X_n, p}$, one arrives at the general Bézout theorem

Theorem 6.2 *Assume that X_i for $1 \leq i \leq n$ is a hypersurface in $\mathbb{P}^n$ and assume that their intersection is finite. Then*

$$\prod_i \deg X_i = \sum_{p \in X_1 \cap \ldots \cap X_n} \mu_p(X_1, \ldots, x_n).$$

The Koszul complex in dimension three

EXAMPLE 6.4 — THE KOSZUL COMPLEX ON THREE ELEMENTS To get an idea of the shape of the general Koszul complex, we take a look at the one on three elements. Given three forms F_1, F_2 and F_3 and let $d_i = \deg F_i$. The Koszul complex build on these forms is the following complex:

$$0 \longrightarrow R(-d_{123}) \xrightarrow{\beta} R(-d_{23}) \oplus R(-d_{13}) \oplus R(-d_{13}) \xrightarrow{M} R(-d_1) \oplus R(-d_2) \oplus R(-d_3) \xrightarrow{\alpha} R$$

where $d_{ij} = d_i + d_j$ and $d_{123} = d_1 + d_2 + d_3$. The middle map is given by the antisymmetric matrix

$$M = \begin{pmatrix} 0 & -F_3 & F_2 \\ F_3 & 0 & -F_1 \\ -F_2 & F_1 & 0 \end{pmatrix}$$

and two others, α and β, respectively by the matrices (F_1, F_2, F_3) and $(F_1, F_2, F_3)^t$. It is fairly straightforward to check that we have a complex; that is, $M \circ \beta = 0$ and $\alpha \circ M = 0$. The condition for the complex being exact is that F_1 and F_2 do not have common component and that F_3 does not belong to (F_1, F_2). In case the forms are in four variables, this is equivalent to the intersection in $\mathbb{P}^3$ of the three corresponding hypersurfaces being finite. ☆

Problem 6.4 Show that $(R/(F_1, F_2, F_3))_d$ is of dimension $d_1 d_2 d_3$ when $d \geq d_1 + d_2 + d_3$. ★

Example 6.5 As a warm up, we take a quick look at the localization of the polynomial ring $R = k[x_0, x_1, x_2]$ in the variable x_2. This ring is a graded algebra whose homogeneous elements are shaped like $F \cdot x_2^{-r}$ with F a homogeneous polynomial; the degree of $F \cdot x_2^{-r}$ equals $\deg F - r$.

One obviously has $R_{x_2} = k[x_0 x_2^{-1}, x_1 x_2^{-1}, x_2, x_2^{-1}]$, and since the two first elements, $x_0 x_2^{-1}$ and $x_1 x_2^{-1}$ are of degree zero, the decomposition of R_{x_2} into homogeneous pieces takes the form

$$R_{x_2} = \bigoplus_{i \in \mathbb{Z}} k[x_0 x_2^{-1}, x_1 x_2^{-1}] x_2^i. \quad (1)$$

Dekomp

Be aware that the index i also takes all negative values—there are rationale functions of negative degree. A fraction $F(x_0, x_1, x_2) x_2^{-b}$ in R_{x_2}, *i.e.*, of a homogenous form F of degree a and the power x_2^{-b}, corresponds to the element $F(x_0 x_2^{-1}, x_1 x_2^{-1}, 1) x_2^{a-b}$ in the right hand sum in the equation (1) above. ☆

Problem 6.5 Show that the degree zero part of R_{x_2} equals $k[x_0 x_2^{-1}, x_1 x_2^{-1}]$. ★

Example 6.6 This describes an observation that can be useful when computing local multiplicities. Assume that X, X' and Y are curves (in the extended meaning, *i.e.*, effective divisors) passing through the point p and assume that neither X nor X' has a common component with Y that passes through p. If f and f' are the equations of X and X', we denote by $X + X'$ the curve whose local equation is ff'.

Then it holds true that

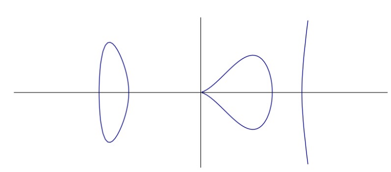
$$\mu_p(X + X', Y) = \mu_p(X, Y) + \mu_p(X', Y). \quad (2)$$

AddLokMult

One has the exact sequence of algebras

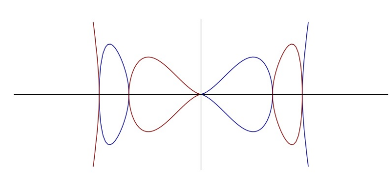
$$0 \longrightarrow \mathcal{O}_{\mathbb{P}^2,p}/(f', h) \xrightarrow{\alpha} \mathcal{O}_{\mathbb{P}^2,p}/(ff', h) \longrightarrow \mathcal{O}_{\mathbb{A}^2,p}/(f, h) \longrightarrow 0.$$

The map α is multiplication by f and is injective; indeed, if $af = bff' + ch$ it follows that $c = c'f$ and hence $a = bf' + c'h$; that is $a = 0$ in $\mathcal{O}_{\mathbb{P}^2,p}/(f', h)$. The exactness of the sequence at the two other places follows easily, and hence taking dimensions over k we obtain the equality (2) ☆



The affine pieces in $D_+(z)$ of one the two curves in problem 6.6

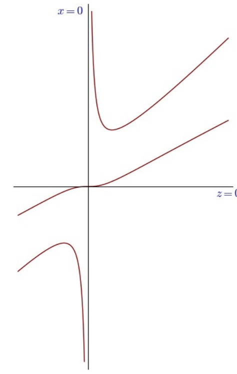
Problem 6.6 Let X and Y be two curves in $\mathbb{P}^2$ being the zero loci of the polynomials $z^5y^2 - x^3(z^2 - x^2)(2z^2 - x^2)$ and $z^5y^2 - x^3(z^2 - x^2)(2z^2 - x^2)$. Determine all intersection points and the local multiplicities in all the intersection points of X and Y ★



The affine pieces in $D_+(z)$ of the two curves in problem 6.6

ToKurver

PROBLEM 6.7 Let C be the curve given as $zy^2 - x(x-z)(x-2z)$. Determine the intersection points and the local multiplicities that X has withe the line $z = 0$. Same task, but with the line $x - z = 0$. ★



Appendix: Some graded algebra

Graded modules

Recall that a graded k-algebra is a k-algebra S with a decomposition $S = \bigoplus_d S_d$ into a direct sum of k-vector spaces. The summands are called the *homogeneous parts* of S, and the elements of S_d are said to be homogeneous of degree d. The decomposition is subjected to the requirement

$$S_d \cdot S_{d'} \subseteq S_{d+d'},$$

which can be considered a compatibility relation between the grading and the multiplicative structure of S. The part S_0 of elements of degree zero acts on each of the parts S_d making them S_0-modules. The field k is contained in S_0.

EXAMPLE 6.7 The archetype of a graded ring is the polynomial ring $R = k[x_0, \ldots, x_n]$ with the homogenous part of degree consisting of the homogenous forms of degree d. ☆

EXAMPLE 6.8 If one localizes R in x_n, the resulting algebra R_{x_n} is graded. The homogeneous elements of R_{x_n} are the ones of the form $z = H(x_0, \ldots, x_n)x_n^{-r}$ for some homogeneous polynomial H and some non-zero integer r. The degree of the element z equals $\deg H - r$. When $\deg z = 0$, it holds true that $z = H(x_0x_n^{-1}, \ldots, x_{n-1}x_n^{-1})$; that is the dehomogenization of H. This implies that the degree zero piece of R_{x_n} is given as the polynomial ring $(R_{x_2})_0 = k[x_0x_n^{-1}, \ldots, x_{n-1}x_n^{-1}]$. Hence the decomposition of R_{x_n} into homogeneous pieces is shaped like

$$R_{x_2} = \bigoplus_{i \in \mathbb{Z}} k[x_0x_n^{-1}, \ldots, x_{n-1}x_n^{-1}] \cdot x_n^i.$$

☆

Graded S-modules

A graded S-module is an S module M with a decomposition $M = \bigoplus_d M_d$ into a direct sum of k-vector space such that

$$S_{d'} \cdot M_d \subseteq M_{d+d'}$$

Notice that all the summands M_d are modules over the degree zero piece S_0.

EXAMPLE 6.9 Every homogenous ideal $\mathfrak{a}$ in R is a graded R_n-module. It satisfies the equality $\mathfrak{a} = \bigoplus_d \mathfrak{a} \cap R_d$ so that the homogeneous part $\mathfrak{a}_d$ of degree d is given as the intersection $\mathfrak{a}_d = \mathfrak{a} \cap R_d$.

The quotient $R/\mathfrak{a}$ is a graded module over R_n as well as a graded k-algebra. It holds true that $R/\mathfrak{a} = \bigoplus_d R_d/\mathfrak{a}_d$. ☆

THE INTRODUCTION of a new concept in mathematics is almost always followed by the introduction of corresponding "morphism's'; that is, "maps" preserving the new structure. In the present case a "morphism" between two graded S-modules M and M' is an S-homomorphism $\phi\colon M \to M'$ that respects the grading; that is $\phi(M_d) \subseteq M'_d$. One says that ϕ is a homogeneous homomorphism of degree zero, or *homomorphism of graded modules*. Two graded modules are *isomorphic* if there is a homomorphism of graded modules $\phi\colon M \to M'$ having an inverse.

Homomorphism of graded modules

Isomorphic graded modules

One easily checks that the kernel and the cokernel of a homomorphism of degree zero are graded in a natural way. Students initiated in the categorical language would say that the graded modules form an *abelian category*.

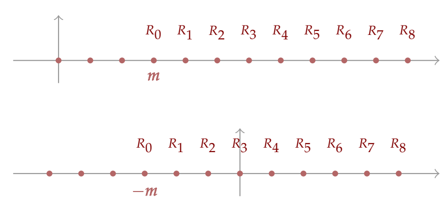
PROBLEM 6.8 Show that if $\phi\colon M \to M'$ is invertible and homogeneous of degree zero, the inverse is automatically is homogeneous of degree zero. ★

THERE IS A COLLECTION OF SHIFT OPERATORS acting on the category of graded S-modules. For each graded module M and each integer $m \in \mathbb{Z}$ there is fresh graded module $M(m)$ associated to a graded module M. The shift operators do not alter the module structure of M, not even the set of homogeneous elements is affected, but they give new degrees to the homogeneous elements. The new degrees are defined by setting

$$M(m)_d = M_{m+d}.$$

In other words, one declares the degree of elements in M_m to be $d - m$.

EXAMPLE 6.10 For instance, when $m > 0$, the shifted polynomial ring $R(-m)$ has no elements of degree d when $d < f$, indeed, $R(m)_d = R_{d-m}$, and the ground field k sits as the graded piece of degree m. Whereas the twisted algebra $R(m)$ has non-zero homogeneous elements of degree down to $-m$ with the ground field sitting as the piece of degree $-m$. ☆



The graded modules $R(-m)$ and $R(m)$

EXAMPLE 6.11 One simple reason for introducing the shift operators, is to keep track of the degrees of generators. For instance, consider the principle ideal $\mathfrak{a} = (F)$ in the polynomial ring R generated by a homogeneous form F of degree m. As every principal ideal in R is, $\mathfrak{a}$ is isomorphic to R as an R-module—multiplication by F gives an isomorphism. However, this not an isomorphism of *graded* modules since it alters the degrees; a homogeneous element a is mapped to the product aF which is of degree $\deg a + m$. But multiplication by F induces a graded isomorphism between $R(-m)$ and $\mathfrak{a}$, since for elements $a \in R(-m)_d$ it holds that $\deg a = d - m$ and consequently $\deg aF = d$.

The classical short exact sequence is therefore an exact sequence of graded modules:

$$0 \longrightarrow R(-m) \xrightarrow{\mu} R \longrightarrow R/F \longrightarrow 0,$$

where thew map μ is multiplication by F. ☆

ALL GRADED MODULES we shall meet in this course are finitely generated over the polynomial ring R. Their generators may be taken to be homogeneous, but they can of course be of different degrees. If the degrees of generators are $d_1, \ldots, d_r$, then M is quotient of a module shaped like a finite direct sum $\bigoplus_{1\leq i\leq r} R(-d_i)$; the factor $R(-d_i)$ is sent to the generator of degree d_i. The twists make the quotient map homogenous of degree zero.

Lemma 6.4 *If M is a graded module finitely generated over the polynomial ring R, then all the graded pieces M_d are finite dimensional vector spaces over k.*

PROOF: This is more or less obvious. It is true for R itself, hence for all twists $R(m)$, hence for direct sums $\bigoplus_i R(-d_i)$. And if M is a quotient of $\bigoplus_i R(-d_i)_d$, the graded piece M_d of M of degree d is a quotient of the graded piece $\bigoplus_i R(-d_i)_d$. ❑

Hilbert functions and Hilbert polynomials

There are some numerical invariants attached to a graded module M finitely generated over a polynomial ring R, which makes working with graded modules much easier. These functions, or their alter egos, are ubiquitous in algebraic geometry and they play an extremely important role. One is the so called *Hilbert function* $h_M(d)$ of M defined as $h_M(d) = \dim_k M_d$. It turns out that $h_M(d)$ behaves like a polynomial for d sufficiently large; that is, there is a unique polynomial $P_M(d)$ coinciding with $h_M(d)$ when $d >> 0$. This is the *Hilbert polynomial* of M.

Hilbert functions

Hilbert polynomials

A FUNDAMENTAL PROPERTY of the Hilbert functions that makes it possible to calculate at least of them, is that, just like the vector space dimension, they are additive over short exact sequences. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of graded module, it holds true that $h_M = h_{M'} + h_{M''}$. Indeed, for each degree d the graded pieces of degree d fit into an exact sequence

$$0 \longrightarrow M'_d \longrightarrow M_d \longrightarrow M''_d \longrightarrow 0$$

of vector spaces, and the assertion follows since vector space dimension is additive.

FOR FUNCTIONS $h\colon \mathbb{Z} \to \mathbb{Z}$; *i.e.*, functions taking integral values on the integers, one introduces a *difference operator* Δ. It is some sort of *discrete derivative* and it is defined as

The difference operator or the discrete derivative

$$\Delta h(d) = h(d) - h(d-1).$$

Just like a derivative, if $\Delta h(d) = 0$ for all d, then h is constant. And so two functions h and h' having the same discrete derivative are equal up to a constant.

Multiplication by an element $x \in R$ of degree one which is not a zero-divisor in the graded module M, induces an exact sequence

$$0 \longrightarrow M(-1) \longrightarrow M \longrightarrow M/xM \longrightarrow 0,$$

from which we infer the equality

$$h_{M/xM}(d) = h_M(d) - h_M(d-1) = \Delta h_M(d). \qquad (3)$$

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A POLYNOMIAL $P(t)$ with rational coefficient is called a *numerical polynomial* if it assumes integral values for integral arguments; that is, if $P(t) \in \mathbb{Z}$ whenever $t \in \mathbb{Z}$.

Numerical polynomials

EXAMPLE 6.12 The binomial coefficients are archetypes of numerical polynomials. Recall that they are defined for any t by the identity

$$\binom{t+n}{n} = (t+n)(t+n-1) \cdot \ldots \cdot (t+1)/n!,$$

and it is well known they are numerical polynomials. A straightforward calculation shows that

$$\Delta \binom{t+n}{n} = \binom{t+n-1}{n-1}. \qquad (4)$$

☆

BinomLikhetDelta

EXAMPLE 6.13 The Hilbert function of the polynomial[5] ring ${}^nR$ vanishes for negative arguments and is given as the binomial coefficient

[5] Recall that ${}^nR = k[x_0, \ldots, x_n]$

$$h_{{}^nR}(d) = \binom{n+d}{n}$$

when $d \geq 0$. Indeed, multiplication by x_n induces the exact sequence

$$0 \longrightarrow {}^{n}R(-1) \longrightarrow {}^{n}R \longrightarrow {}^{n-1}R \longrightarrow 0$$

of graded modules, and hence $\Delta h_{^nR} = h_{^{n-1}R}$. By induction on n and the identity (4) above the assertion follows. Because ${}^0R = k[x_0]$ it obviously holds that $h_{^0R}(d) = 1$ for $d \geq 0$ and $h_{^0R} = 0$ when $d < 0$, so that the induction can start. ☆

EXAMPLE 6.14 For a graded R-modules M of finite support, the Hilbert polynomial P_M vanishes identically. Indeed, the module M is finite dimensional as a vector space over k, and there is only room for finitely many non-zero graded pieces. But of course, the Hilbert *function* of M is not identically zero. ☆

WE SHALL MOSTLY be concerned with the leading term of numerical polynomials; they are of a special form as described in the following lemma:

Lemma 6.5 *Assume that $P(t)$ is a numerical polynomial of degree m. Then*

$$P(t) = c_m/m!t^m + \dots$$

where c_m is an integer[6]*. The discrete derivative $\Delta P(t)$ is of degree $m-1$ and its leading coefficient equals $c_m/(m-1)!$*

[6] As is customary, the dots stand for terms of lower degree than m.

PROOF: We proceed by induction on m. The lemma holds for $m = 0$ because a numerical polynomial of degree zero is an integral constant. For $m > 0$ we write $P(t) = a_m t^m + Q(t)$ with Q of degree at most $m-1$. Appealing to the binomial theorem, one finds

$$\Delta P(t) = a_m t^m - a_m(t-1)^m + \Delta Q(t) =$$
$$= a_m t^m - a_m t^m + m a_m t^{m-1} + \Delta Q(t) = m a_m t^{m-1} + \Delta Q(t),$$

by induction $\Delta Q(t)$ is of degree at most $m-2$, the leading coefficient of $\Delta P(t)$ is $m a_m$, and again by induction, it is shaped like $m a_m = c_{m-1}/(m-1)!$ where c_{m-1} is an integer. The lemma follows. ❑

Theorem 6.3 *Let $\mathfrak{a}$ be a homogenous ideal in R. Then $P_{R/\mathfrak{a}}$ is of degree* $\dim R/\mathfrak{a}$.

PROOF: We proceed by induction on $\dim R/\mathfrak{a}$. Let $\mathfrak{p}_i$ be the associated prime ideals to $\mathfrak{a}$. Then there is an element $x \in \mathfrak{m}_+$ of degree one not contained in any of the minimal primes of $\mathfrak{a}$, and $\dim R/\mathfrak{a} + (x) = \dim R/\mathfrak{a} - 1$. Hence there is an exact sequence

$$0 \longrightarrow K \longrightarrow S(-1) \xrightarrow{x} S \longrightarrow S/xS \longrightarrow 0$$

where K is of finite support. By example xxx above, the Hilbert polynomial of K vanishes identically, and hence $\Delta P_S = P_{S/xS}$. By induction we are through. ❑

EXAMPLE 6.15 Let $F \in R = k[x_0, x_1, x_2]$ be a homogeneous polynomial of degree m. Then there is a short exact sequence of graded modules

$$0 \longrightarrow R(-m) \xrightarrow{\alpha} R \longrightarrow R/F \longrightarrow 0$$

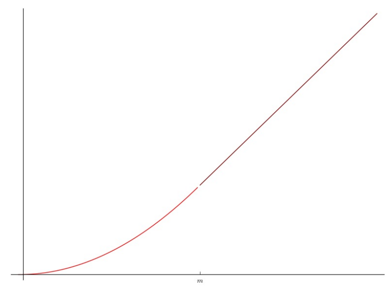
where the map α is multiplication by F. Additivity of the Hilbert functions yields, when $d \geq m$, that

$$h_{R/F}(d) = h_R(d) - h_R(d-m) = md + m^2 - 3m/2,$$

while

$$h_{R/F}(d) = h_R(d) = d^2/2 + 3d/2 + 1$$

when $0 \leq d < m$ since then $h_{R(d-m)} = 0$. For $d < 0$ it obviously holds true that $h_{R/F}(d) = 0$. So the Hilbert function is constant and equal to zero for negative values of the argument, it grows quadratically for d between 0 and m and settles with a linear grows for $d \geq m$. The Hilbert polynomial is linear and has leading term md. The geometric interpretation of the algebra $R/(F)R$ is as the homogeneous coordinate ring $S(X)$ of of the curve $X = Z_+(F) \subseteq \mathbb{P}^2$. Notice that the degree of the Hilbert polynomial $P_{S(X)}(d)$ equals the dimension of X (both are one) and that the leading coefficient equals the degree of F. ☆



The graph of $h_{R/(F)}$.