GEIR ELLINGSRUD

NOTES FOR MA4210— ALGEBRAIC GEOMETRY I

Contents

These notes ere just informal extensions of the lectures I gave last year. As the course developeds I'll now and then posted new notes on the course's website, but this will certainly happen with irregular intervals. The idea with the notes was to give additional comments and examples which hopefully made reading of the book and the digestion of the lectures easier; and hopefully widened the students mathematical horizon.

It seems that other lectures are interested in the notes, so I try to upgrade them—correct misprints and not to the least give correct proofs of all theorems (important buisiness!!!). This is an ongoing process and the present version is still preliminary. As the students last year survived the notes in the then shaky condition, I am confident that students this year will survive as well; and still better (or more humbly less bad) version are coming!
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Lecture 1

Algebraic sets and the Nullstellensatz

Hot themes in Lecture 1: *The correspondence between ideals and algebraic sets—weak and strong versions of Hilbert's Nullstellensatz—the Rabinowitsch trick—two proofs of the Nullstellensatz, one elementary, and another totally different—radical ideals—intuition, drawings and figures.*

Algebraic geometry has many ramifications, but roughly speaking there are two main branches. One could be called the "geometric" branch where the geometry is the main objective. One studies geometric objects like curves, surfaces, threefolds and varieties of higher dimensions, defined by polynomials (or more generally algebraic functions). The aim is to understand their geometry. Frequently techniques from several other fields are used like from algebraic topology, differential geometry or analysis, and the studies are tightly connected with these other branches of mathematics. This makes it natural to work over the complex field $\mathbb{C}$, even though other fields like function fields are important.

To say that aims of algebraic geometry are totally geometric is half a lie (but a white one). The study of elliptic functions in the beginning of the 19th century, and subsequently of other algebraic functions, was the birth of modern algebraic geometry. The motivation and the origin was found in function theory, but the direction of research quickly took a geometric rout. Riemann surfaces and algebraic curves appeared thogether with their function fields.

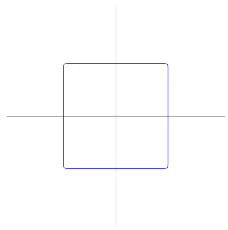


Figure 1.1: The affine Fermat curve $x^{50} + y^{50} = 1$.

The other main branch one could call "arithmetic". Superficially presented, one studies numbers by geometric methods. An ultra famous example is Fermat's last theorem, now Andrew Wiles' theorem, that the equation $x^n + y^n = z^n$ has no integral solutions except the trivial ones. The arithmetic branch also relies on techniques from other fields, like number theory, Galois theory and representation theory. One very commonly applied technique is reduction modulo a prime number p. Hence the importance of including fields of positive characteristic among the base fields. Of course another very natural base field for many of these "arithmetic" studies is the field $\overline{\mathbb{Q}}$ of algebraic numbers.

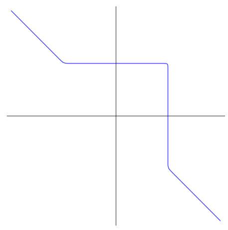


Figure 1.2: The affine Fermat curve $x^{51} + y^{51} = 1$.

Algebraic geometry is to the common benefit a triple marriage of geometry, algebra and arithmetic. All of the spouses claim influence on the development of the field which makes the field quit abstract; but also a most beautiful part of mathematics.

1.1 *Fields and the affine space*

1.1 We shall almost exclusively work over an algebraic closed field which we shall denote by k. In general we do not impose further constraints on k, except for a few results that require the characteristic to be zero. A specific field to have in mind would be the field of complex numbers $\mathbb{C}$, but as indicated above, other important fields are $\overline{\mathbb{Q}}$ and $\overline{\mathbb{F}}_p$.

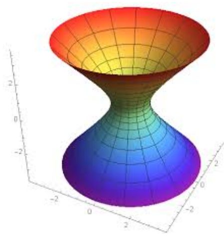


Figure 1.3: A one sheeted-hyperboloid.

1.2 The affine space $\mathbb{A}^n$ is just the space k^n, but the name-change is there to underline that there is more to $\mathbb{A}^n$ than merely being a vector space—and hopefully, this will emerge from the fog during the course. Anyhow, in the beginning think about $\mathbb{A}^n$ as k^n. Often the ground field will be tacitly understood, but when wanting to be precise about it, we shall write $\mathbb{A}^n(k)$. The ground will always be algebraically closed unless the contrary is explicitly stated.

Coordinates are not God-given but certainly man-made. So they are prone to being changed. General coordinate changes in $\mathbb{A}^n$ can be subtle, but translation of the origin and linear changes are unproblematic, and will be done unscrupulously. They are called *affine coordinate changes* and the affine spaces $\mathbb{A}^n$ are named after them.

1.2 *Closed algebraic sets*

The first objects we shall meet are the so called *closed algebraic sets*, and master students in mathematics have already seen a great many examples of such. They are just subsets of the affine space $\mathbb{A}^n$ given by a certain number of polynomial equations. You have probably working with curves in the plane and may be with some surfaces in the space—like conic sections, hyperboloids and paraboloids, for example.

Closed algebraic sets (*lukkede algebraiske mengder*)

1.3 Formally the definition of a closed algebraic set is as follows. If S is a subset of the polynomial ring $k[x_1, \ldots, x_n]$, one defines

$$Z(S) = \{\, x \in \mathbb{A}^n \mid f(x) = 0 \text{ for all } f \in S \,\},$$

and subsets of $\mathbb{A}^n$ obtained in that way are the *closed algebraic sets*. Notice that any linear combination of polynomials from S also vanishes at points of $Z(S)$, even if polynomials are allowed as coefficients. Therefore the *ideal* $\mathfrak{a}$ generated by S has the same zero set as S; that is, $Z(S) = Z(\mathfrak{a})$. We shall almost exclusively work with ideals and tacitly replace a set of polynomials by the ideal it generates.

$\mathfrak{a}, \mathfrak{A}$	$\mathfrak{b}, \mathfrak{B}$	$\mathfrak{c}, \mathfrak{C}$	$\mathfrak{d}, \mathfrak{D}$	$\mathfrak{e}, \mathfrak{E}$
$\mathfrak{f}, \mathfrak{F}$	$\mathfrak{g}, \mathfrak{G}$	$\mathfrak{h}, \mathfrak{H}$	$\mathfrak{i}, \mathfrak{I}$	$\mathfrak{j}, \mathfrak{J}$
$\mathfrak{k}, \mathfrak{K}$	$\mathfrak{l}, \mathfrak{L}$	$\mathfrak{m}, \mathfrak{M}$	$\mathfrak{n}, \mathfrak{N}$	$\mathfrak{o}, \mathfrak{O}$
$\mathfrak{p}, \mathfrak{P}$	$\mathfrak{q}, \mathfrak{Q}$	$\mathfrak{r}, \mathfrak{R}$	$\mathfrak{s}, \mathfrak{S}$	$\mathfrak{t}, \mathfrak{T}$
$\mathfrak{u}, \mathfrak{U}$	$\mathfrak{v}, \mathfrak{V}$	$\mathfrak{w}, \mathfrak{W}$	$\mathfrak{x}, \mathfrak{X}$	$\mathfrak{y}, \mathfrak{Y}$
$\mathfrak{z}, \mathfrak{Z}$				

Mathematicians are always in shortage of symbols and use all kinds of alphabets. The germanic gothic letters are still in use in some context, like to denote ideals in some text.

Any ideal in $k[x_1, \ldots, x_n]$ is finitely generated, this is what Hilbert's basis theorem tells us, so that a closed algebraic subset is described as the set of common zeros of *finitely* many polynomials.

Examples

1.1 The polynomial ring $k[x]$ in one variable is a PID [1], so if $\mathfrak{a}$ is an ideal in $k[x]$, it holds that $\mathfrak{a} = (f(x))$. Because polynomials in one variable merely have finitely many zeros, the closed algebraic subsets of $\mathbb{A}^1$ are just the finite subsets of $\mathbb{A}^1$.

[1] A ring is a PID or a *principal ideal domain* if it is an integral domain where every ideal is principal

1.2 A more spectacular example is the so called *Clebsch diagonal cubic*; a surface in $\mathbb{A}^3(\mathbb{C})$ with equation

$$x^3 + y^3 + z^3 + 1 = (x + y + z + 1)^3.$$

An old plaster model of its reals points; that is, the points in $\mathbb{A}^3(\mathbb{R})$ satisfying the equation, is depicted in the margin.

The Clebsch diagonal cubic

1.3 The traditional conic sections are closed algebraic sets in $\mathbb{A}^2$. A *parabola* is given as the zeros of $y - x^2$ and a *hyperbola* as the zeros of $xy - 1$.

☆

1.4 The more constraints one imposes the smaller the solutions set will be, so if $\mathfrak{b} \subseteq \mathfrak{a}$ are two ideals, one has $Z(\mathfrak{a}) \subseteq Z(\mathfrak{b})$. The *sum* $\mathfrak{a} + \mathfrak{b}$ of two ideals has the intersection $Z(\mathfrak{a}) \cap Z(\mathfrak{b})$ as zero set; remembering that

$$\mathfrak{a} + \mathfrak{b} = \{ f + g \mid f \in \mathfrak{a} \text{ and } g \in \mathfrak{b} \}$$

one easily convinces oneself of this. In the same vein, the *product* $\mathfrak{a} \cdot \mathfrak{b}$ defines the union $Z(\mathfrak{a}) \cup Z(\mathfrak{b})$. With a little thought, this is clear since the products $f \cdot g$ of polynomials $f \in \mathfrak{a}$ and $g \in \mathfrak{b}$ generate $\mathfrak{a} \cdot \mathfrak{b}$. Sending $\mathfrak{a}$ to $Z(\mathfrak{a})$ is a order reversing map from the partially ordered sets of ideals in $k[x_1, \ldots, x_n]$ to the partially ordered set of subsets of $\mathbb{A}^n$.

1.5 It might very well happen that two different ideals define the same algebraic set. The most stupid example being (x) and (x^2); they both define the origin in the affine line $\mathbb{A}^1$. More generally, powers $\mathfrak{a}^n$ of an ideal $\mathfrak{a}$ have the same zeros as $\mathfrak{a}$. Because $\mathfrak{a}^n \subseteq \mathfrak{a}$ it holds that $Z(\mathfrak{a}) \subseteq Z(\mathfrak{a}^n)$, and the other inclusion holds as well since $f^n \in \mathfrak{a}^n$ whenever $f \in \mathfrak{a}$. Recall that the *radical* $\sqrt{\mathfrak{a}}$ of an ideal is the ideal whose members are the polynomials for which a power lies in $\mathfrak{a}$; that is,

$$\sqrt{\mathfrak{a}} = \{ f \mid f^r \in \mathfrak{a} \text{ for some } r \}.$$

The argument above yields that $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$ (in fact, since all ideals in the polynomial ring are finitely generated, a power of the radical is contained in $\mathfrak{a}$). Ideals with the same radical therefore have coinciding zero sets, and we shall soon see that the converse is true as well. This is the content of the famous Hilbert's Nullstellensatz which we are about to formulate and prove, but first we sum up the present discussion in a proposition:

PROPOSITION 1.6 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be two ideals in $k[x_1, \ldots, x_n]$.*

- *If $\mathfrak{a} \subseteq \mathfrak{b}$, then $Z(\mathfrak{b}) \subseteq Z(\mathfrak{a})$;*
- $Z(\mathfrak{a} + \mathfrak{b}) = Z(\mathfrak{a}) \cap Z(\mathfrak{b})$;
- $Z(\mathfrak{a}\mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$;
- $Z(\mathfrak{a}) = Z(\sqrt{\mathfrak{a}})$.

David Hilbert (1862–1943) German mathematician.

By the way, this also shows that $Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$: Because of the inclusion $(\mathfrak{a} \cap \mathfrak{b})^2 \subseteq \mathfrak{a} \cdot \mathfrak{b}$ one has $Z(\mathfrak{a} \cap \mathfrak{b}) \subseteq Z(\mathfrak{a}) \cup Z(\mathfrak{b})$, and the other inclusion follows readily. Notice also that the argument for the second assertion remains valid, *mutatis mutandis*, for any family of ideals $\{\mathfrak{a}_i\}_{i \in I}$; that is, one has

- $Z(\sum_{i \in I} \mathfrak{a}_i) = \bigcap_i Z(\mathfrak{a}_i)$.

1.7 The Nullstellensatz involves the ideal $I(X)$ of polynomials in $k[x_1, \ldots, x_n]$ that vanish along the subset X of $\mathbb{A}^n$, and $I(X)$ acts as a partial converse to $Z(\mathfrak{a})$. To be precise, for any subset $X \subseteq \mathbb{A}^n$ one defines

$$I(X) = \{ f \in k[x_1, \ldots, x_n] \mid f(x) = 0 \text{ for all } x \in X \}.$$

When X is an arbitrary set, there is not much information about X to be extracted form $I(X)$; for instance, if X is any infinite subset of $\mathbb{A}^1$, it holds true that $I(X) = (0)$ (non-zero polynomials have only finitely many zeros). However, if X *a priori* is known to be a closed algebraic subset, it is true that $Z(I(X)) = X$; in other words, one has

- $Z(I(Z(\mathfrak{a}))) = Z(\mathfrak{a})$.

Indeed, it is true for all subsets X of $\mathbb{A}^n$ that $X \subseteq Z(I(X))$ (the functions that vanish in X vanish in X!). Thus $Z(\mathfrak{a}) \subseteq Z(I(Z(\mathfrak{a})))$. The other inclusion follows from $Z(-)$ reversing inclusions and the tautological inclusion $\mathfrak{a} \subseteq I(Z(\mathfrak{a}))$ (a function vanishes where it vanishes!!).

1.3 *The Nullstellensatz*

1.8 Hilbert's Nullstellensatz is about the composition of I and Z the other way around, namely about $I(Z(\mathfrak{a}))$. Polynomials in the radical $\sqrt{\mathfrak{a}}$ vanish along $Z(\mathfrak{a})$ and therefore $\sqrt{\mathfrak{a}} \subseteq I(Z(\mathfrak{a}))$, and the Nullstellensatz tells us that this inclusion is an equality. We formulate the Nullstellensatz here, together with two of its weak avatars, but shall come back with a thorough discussion of the proof(s) a little later.

THEOREM 1.9 (HILBERT'S NULLSTELLENSATZ) *Assume that k is an algebraically closed field, and that $\mathfrak{a}$ is an ideal in $k[x_1, \ldots, x_n]$. Then one has $I(Z(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*

Notice that the ground field must be algebraically closed. Without this assumption the result is not true. The simplest example of an ideal in a polynomial ring with empty zero locus is the ideal (x^2+1) in $\mathbb{R}[x]$.

1.10 Obviously it holds true that $I(\emptyset)$ equals the entire polynomial ring, and if $\mathfrak{a}$ is a proper ideal, it is as obvious that $\sqrt{\mathfrak{a}}$ is not the entire polynomial ring, so in particular, the theorem asserts that $Z(\mathfrak{a}) = \emptyset$ if and only if $\mathfrak{a}$ equals the whole polynomial ring; that is, if and only if $1 \in \mathfrak{a}$. Hence we can conclude that $Z(\mathfrak{a})$ is not empty when $\mathfrak{a}$ is proper. This statement goes under the name of the Weak Nullstellensatz; and is despite the name equivalent to the Nullstellensatz, as we shall see later on.

Theorem 1.11 (Weak Nullstellensatz) *Assume that k is an algebraically closed field. For every proper ideal $\mathfrak{a}$ in $k[x_1,\ldots,x_n]$ there is a point $x \in Z(\mathfrak{a})$.*

1.12 Consider now the ideals $(x_1-a_1,\ldots,x_n-a_n)$ where the a_i's are elements from k. It is easy to see that all these are maximal ideals; indeed, after a linear change of variables it suffices to see that $(x_1,\ldots,x_n)$ is maximal, which is clear since $(x_1,\ldots,x_n)$ obviously is the kernel of the map $k[x_1,\ldots,x_n] \to k$ evaluating a polynomial at the origin.

Amazingly, the converse follows from the Nullstellensatz: Every maximal ideal in the polynomial ring is of this form. If $\mathfrak{m}$ is a maximal ideal, it is certainly a proper ideal, and by the Nullstellensatz there is point $(a_1,\ldots,a_n)$ in $Z(\mathfrak{m})$. Consequently it holds that $(x_1-a_1,\ldots,x_n-a_n) \subseteq \mathfrak{m}$, but since $(x_1-a_1,\ldots,x_n-a_n)$ is also maximal, the two ideals coincide. Hence we have the following equivalent version of the Weak Nullstellensatz:

Theorem 1.13 (Weak Nullstellensatz II) *Let k be an algebraically closed field. Then the maximal ideals in the polynomial ring $k[x_1,\ldots,x_n]$ are those of the form $(x_1-a_1,\ldots,x_n-a_n)$ with $(a_1,\ldots,a_n) \in \mathbb{A}^n$.*

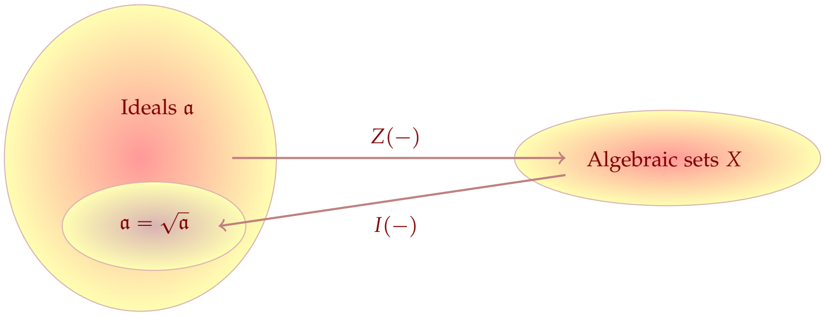
Radical ideals and algebraic subsets

1.14 In view of the Nullstellensatz, it is natural to introduce the notion of a *radical ideal*. It is an ideal equal to its own radical; in other words, it satisfies $\mathfrak{a} = \sqrt{\mathfrak{a}}$. With this concept in place, the two constructions I and Z are mutually inverse mappings from the set of radical ideals to the set of closed algebraic sets.

Radical ideals

Both sets are partially ordered under inclusion, and the two mappings both reverse the partial orders. Moreover, they take "sup's" to "inf's" and *vice versa*.

In a partial ordered set $\inf(a,b)$ is the greatest element less then both a and b, and $\sup(a,b)$ the smallest greater than both. In general they do not exists and do not need to be unique.



The radical of an intersection is the intersection of the radicals, so if $\mathfrak{a}$ and $\mathfrak{b}$ are two radical ideals, their intersection $\mathfrak{a} \cap \mathfrak{b}$ is as well, and it is the "inf" of the two; that is, the greatest radical ideal contained in both.

On the other hand, the sum $\mathfrak{a} + \mathfrak{b}$ of two radical ideals is not in general radical. For instance, the ideals $(y - x^2)$ and (y) are both radical, but $(y - x^2) + (y) = (y - x^2, y) = (y, x^2)$ is not. Hence the "sup" of the two in the set of radical ideals will be $\sqrt{\mathfrak{a} + \mathfrak{b}}$. This means that for *radical ideals* one has the two relations:

- $I(Z(\mathfrak{a}) \cap Z(\mathfrak{b})) = \sqrt{\mathfrak{a} + \mathfrak{b}}$;
- $I(Z(\mathfrak{a}) \cup Z(\mathfrak{b})) = \mathfrak{a} \cap \mathfrak{b}$.

PROBLEM 1.1 Show that $(y - x^2)$ is radical. Let $\alpha \in k$ and let $\mathfrak{a} = (y - x^2, y - \alpha x)$. Show that $\mathfrak{a}$ is a radical ideal when $\alpha \neq 0$, but not when $\alpha = 0$. ★

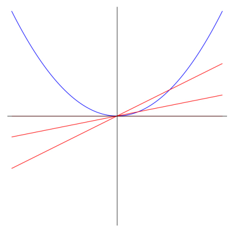


Figure 1.4: The parabola $y = x^2$ and some lines through the origin.

The coordinate ring

Affine coordinate rings

1.15 The ring $A(X) = k[x_1, \ldots, x_n]/I(X)$ is called the *affine coordinate ring* of X. If Y is a closed algebraic sets contained in X, it holds that $I(X) \subseteq I(Y)$, and conversely if $I(Y)$ contains $I(X)$, one has $Y \subseteq X$. Hence there is a one-to-one correspondence between radical ideals in the coordinate ring $A(X)$ and closed algebraic subsets contained in X. If $\mathfrak{a}$ is an ideal in $A(X)$, we denote by $Z(\mathfrak{a})$ the corresponding subvariety of X. And for a point $a = (a_1, \ldots, a_n) \in X$ we let $\mathfrak{m}_a$ denote the image in $A(X)$ of the maximal ideal $(x_1 - a_1, \ldots, x_n - a_n)$ of polynomials vanishing at x.

1.4 *Hilbert's Nullstellensatz—proofs*

In this section we discuss various proofs and various versions of the Nullstellensatz. The Nullstellensatz comes basically in two flavours, the strong Nullstellensatz and the weak one (of which we shall present three variations). Despite their names the different versions are equivalent. The strong version trivially implies the weak, but the reverse implication hinges on a trick frequently called the *Rabinowitsch-trick*.

The Rabinowitsch trick

1.16 We proceed to present the J.L. Rabinowitsch trick proving that the weak version of the Nullstellensatz (Theorem 1.11 on page 12) implies the strong. That is, we need to demonstrate that $I(Z(\mathfrak{a})) \subseteq \sqrt{\mathfrak{a}}$ for any proper ideal $\mathfrak{a}$ in $k[x_1, \ldots, x_n]$.

The crux of the trick is to introduce a new auxiliary variable x_{n+1} and for each $g \in I(Z(\mathfrak{a}))$ to consider the ideal $\mathfrak{b}$ in the polynomial ring $k[x_1, \ldots, x_{n+1}]$ given by

$$\mathfrak{b} = \mathfrak{a} \cdot k[x_1, \ldots, x_{n+1}] + (1 - x_{n+1} \cdot g).$$

In geometric terms $Z(\mathfrak{b}) \subseteq \mathbb{A}^{n+1}$ is the intersection of the the subset $Z = Z(1 - x_{n+1} \cdot g)$ and the inverse image $\pi^{-1}Z(\mathfrak{a})$ of $Z(\mathfrak{a})$ under the projection $\pi\colon \mathbb{A}^{n+1} \to \mathbb{A}^n$ that forgets the last and auxiliary coordinate. This intersection is empty, since obviously g does not vanish along Z, but vanishes identically on $\pi^{-1}Z(\mathfrak{a})$.

The weak Nullstellensatz therefore gives that $1 \in \mathfrak{b}$, and hence there are polynomials f_i in $\mathfrak{a}$ and h_i and h in $k[x_1, \ldots, x_{n+1}]$ satisfying a relation like

$$1 = \sum f_i(x_1, \ldots, x_n) h_i(x_1, \ldots, x_{n+1}) + h \cdot (1 - x_{n+1} \cdot g).$$

We substitute $x_{n+1} = 1/g$ and multiply through by a sufficiently[2] high power g^N of g to obtain

$$g^N = \sum f(x_1, \ldots, x_n) H_i(x_1, \ldots, x_n),$$

where $H_i(x_1, \ldots, x_n) = g^N \cdot h_i(x_1, \ldots, x_n, g^{-1})$. Hence $g \in \sqrt{\mathfrak{a}}$.

[2] For instance the highest power of x_{n+1} that occurs in any of the h_i's.

The third version of the Weak Nullstellensatz

1.17 As already mentioned there are several variants of the weak Nullstellensatz. We have already seen two, and here comes number three. This is the one we shall prove and from which we subsequently shall deduce the other versions. It has the virtue of being general, in that it is valid over any field k, and we shall bring it with us into Grothendieck's marvelous world of schemes.

Theorem 1.18 (Weak Nullstellensatz III) *Let k a field and let $\mathfrak{m}$ be a maximal ideal in the polynomial ring $k[x_1, \ldots, x_n]$. Then $k[x_1, \ldots, x_n]/\mathfrak{m}$ is a finite field extension of k.*

Before proceeding to the proof of this version III we show how the Weak Nullstellensatz II (Theorem 1.11 on page 12) can be deduced from version III above.

Proof of II from III: So assume that k is algebraically closed and let $\mathfrak{m}$ be a maximal ideal in $k[x_1, \ldots, x_n]$. The salient point is that the field $k[x_1, \ldots, x_n]/\mathfrak{m}$

is a finite extension of k after version version III above, and since k is algebraically closed by assumption, the two fields coincide. Thus there is an algebra homomorphism $k[x_1, \ldots, x_n] \to k$ having $\mathfrak{m}$ as kernel. Letting a_i be the image of x_i under this map, the ideal $(x_1 - a_1, \ldots, x_n - a_n)$ will be contained in $\mathfrak{m}$, and being maximal, it equals $\mathfrak{m}$. ❏

Proof of version III of the Nullstellensatz

1.19 The by far simplest proof of the Nullstellensatz I know, both technically and conceptually, was found by Daniel Allcock. It relies on no more sophisticated mathematics than the fact the polynomial ring $k[x]$ in one variable is a PID. Allcock establishes the following assertion, which obviously implies version III of the Weak Nullstellensatz:

Lemma 1.20 *If $k \subseteq K$ is a finitely generated extension of fields which is not finite, and $a_1, \ldots, a_r$ are elements in K, then $k[a_1, \ldots, a_r]$ is not equal to K.*

Recall that $k[x]_s$ denotes the localization of $k[x]$ in the multiplicative set $\{1, s, s^2, \ldots\}$. Elements are of the form a/s^r with $a \in k[x]$.

Proof: To begin with we treat the case that K is of transcendence degree one over k. Then there is a subfield $k(x) \subseteq K$ with x transcendental over which K is finite. Let $\{e_i\}$ be a basis for K over $k(x)$ with $e_0 = 1$, and let c_{ijk} be elements in $k(x)$ such that $e_i e_j = \sum_k c_{ijk} e_k$. Let s be a common denominator of the c_{ijk}. Then $A = \bigoplus_i k[x]_s e_i$ is a subalgebra of K which is free as a module over $k[x]_s$. Now, let $a_1, \ldots, a_r$ be elements in K, and express them in the basis $\{e_i\}$; that is, write $a_j = \sum d_{ij} e_i$ with $d_{ij} \in k(x)$. Let t be the common denominator of the d_{ij}'s.

Then $k[a_1, \ldots, a_r]$ is contained in A_t, and therefore can not be equal to K. Indeed, if $u \in k[x]$ is any irreducible element [3] neither being a factor in s nor in t, then u^{-1} will not lie in A_t.

Finally, if the transcendence degree of K is more than one, we let $k' \subseteq K$ be a field containing k over which K is of transcendence degree 1. Then K is never equal to $k'[a_1, \ldots, a_r]$, hence *a fortiori* neither to $k[a_1, \ldots, a_r]$. ❏

[3] Even if k is a finite field, there are infinitely many irreducible polynomials in $k[x]$, see problem 1.12 on page 19.

1.5 *Figures and intuition*

To have some geometric intuition one frequently have real pictures of algebraic sets in mind. Then the ground field must be $\mathbb{C}$ and the algebraic set must be defined by real equations. The object depicted is the subset of the points in $Z(\mathfrak{a})$ whose coordinates are real numbers.

These real pictures can be very instructive (and beautiful) and some times they are unsurpassed to explain what happens. But they can be deceptive and must be taken with a rather large grain of salt—often they do not tell the whole story, and sometimes they do not say any thing at all. For instance, $x^2 + y^2 + 1$ has no real zeros, so $V(x^2 + y^2 + 1)$ has no real points, but of course, complex zeros abound.

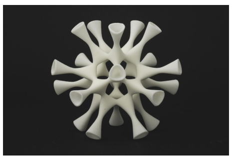


Figure 1.5: The famous surface of degree six constructed by Wolf Barth. It has 65 double points. The picture is of a 3D-print of the surface from http://math-sculpture.com.

1.21 Performing complex coordinate shifts, which is perfectly legitimate when working over $\mathbb{C}$ and does not alter the complex geometric reality, can completely change the real picture. For instance, replacing y by iy in the above example, which is a simple scaling of one of the coordinates; gives the equation $x^2 - y^2 = -1$ whose real points constitute a hyperbola, and scaling both x and y by i gives the circle $x^2 + y^2 = 1$. So the real picture depends heavily on the coordinates one uses.

There is also a shift in dimension. The affine plane $\mathbb{A}^2(\mathbb{C})$ is as real manifold equal to $\mathbb{R}^4$, and a plane in $\mathbb{A}^2(\mathbb{C})$ is a linear subspace of real codimension two; that is, an $\mathbb{R}^2$ in $\mathbb{R}^4$. Complex algebraic sets will be of even (real) dimension and the (real) dimension of their real counterparts will be half that dimension.

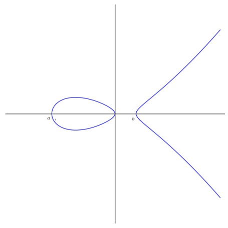


Figure 1.6: The real points of a cubic curve in the so called Weierstrass normal form.

Example 1.4 Consider the curve $y^2 = x(x+a)(x-b)$ in $\mathbb{A}^3(\mathbb{C})$; with a and b both positive. The real points, depicted in Figure 1.6, has two components. One compact, which is homeomorphic to a circle, and one unbounded. The complex points turn out to form a space homeomorphic to a torus $S^1 \times S^1$ (in the topology induced from the standard topology on $\mathbb{C}^2$). Well, to be precise, it is homeomorphic to the torus minus one point.

To underline to what extent the real picture depends on the chosen coordinate system, in figure 1.7 we depicted a cubic curve (virtualy the same as depicted in Figure 1.6) viewed in another coordinate system.

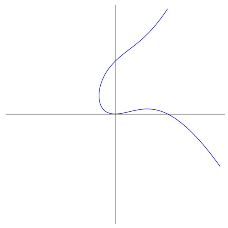


Figure 1.7: The real points of a cubic curve in the so called Tate normal form.

☆

1.6 *A second proof of the Nullstellensatz*

It is worth while to ponder over another proof of the Nullstellensatz which follows a completely different path than the one of Daniel Allcock. We shall present it in a simplified form assuming that $k = \mathbb{C}$ and assuming that $\mathfrak{a}$ is a *prime ideal*.

1.22 The proof is based on the fact that the transcendence degree of the complex numbers $\mathbb{C}$ over the rationals $\mathbb{Q}$ is infinite (in fact, it equals the cardinality $\mathfrak{c}$ of the continuum). It is not hard to see it is infinite; if not, $\mathbb{C}$ would have been countable (it is more challenging to see it equals $\mathfrak{c}$). From this ensues the following lemma:

Lemma 1.23 *Every field K of finite transcendence degree over $\mathbb{Q}$ can be embedded in $\mathbb{C}$.*

Proof: Let $x_1, \ldots, x_r$ be a transcendence basis for K over $\mathbb{Q}$ so that K is algebraic over $\mathbb{Q}(x_1, \ldots, x_r)$. Chose algebraically independent complex numbers $z_1, \ldots, z_r$. Sending x_i to z_i gives an embedding of $\mathbb{Q}(x_1, \ldots, x_r)$ into $\mathbb{C}$ and because $\mathbb{C}$ is algebraically closed, it extends to K. ❑

1.24 Assume now that $\mathfrak{p}$ is a prime ideal in $\mathbb{C}[x_1, \ldots, x_n]$ and choose generators $f_1, \ldots, f_r$ for it. Let k be the field obtained by adjoining all the coefficients of the f_i's to $\mathbb{Q}$; it is clearly of finite transcendence degree over $\mathbb{Q}$. Let $\mathfrak{p}' = \mathfrak{p} \cap k[x_1, \ldots, x_n]$. Then the fraction field of the domain $k[x_1, \ldots, x_n]/\mathfrak{p}'$ is of finite transcendence degree over $\mathbb{Q}$ and therefore it embeds into $\mathbb{C}$. But this means that the images of the x_i's are coordinates for a point where all the f_i's vanish, and consequently $Z(\mathfrak{p})$ is not empty.

PROBLEM 1.2 Contrary to quadratic curves, show that a cubic curve in $\mathbb{A}^2(\mathbb{C})$ defined by an equation with real coefficients always have real points. Generalize to curves with real equations of odd degree. HINT: Intersect with real lines. ★

Examples

1.5 *(Quadratic plane curves or conics)* Curves in $\mathbb{A}^2$ given by irreducible quadratic equations can be classified. Up to an affine change of coordinates there are only two types. Either the equation can be brought on the form $y = x^2$ or on form $xy = 1$.

The quadratic polynomial can be written as $Q(x, y) + L(x, y) + c$ where Q and L are homogeneous polynomials of degree respectively two and one, and where c is a scalar. The quadratic form Q can be factored as the product of two linear form. We change coordinates so that the two factors become x and y; that is, $Q(x, y) = xy$, if they are different, or x if they coincide; that is, $Q(x, y) = x^2$. This brings the original quadratic polynomial on form

$$xy + ax + by + c = (x + a)(y + b) + c - ab$$

if $Q(x, y) = xy$, and

$$x^2 + ax + by + c$$

when $Q(x, y) = y^2$. The last necessary coordinate shifts are then easy to find and left as an exercise.

The following super-trivial lemma is nothing but Taylor expansion to the first order, but is now and then useful:

LEMMA 1.25 *Assume that R is any commutative ring. Let $P(z)$ be a polynomial in $R[z]$. Then $P(z + w) = P(z) + wQ(z, w)$ for some polynomial Q in $R[z, w]$.*

PROOF: Observe that by the binomial theorem one has $(z + w)^i = z^i + wQ_i(z, w)$; the rest of the proof follows from this. ❑

1.6 *(The affine twisted cubic)* In this example we take a closer look at a famous curve called the *twisted cubic*, or rather an *affine* version of it (there is also a *projective* avatar which we come back to later). The word twisted in the name comes from the curve being a space curve not contained in any plane.

The twisted cubic $C \subseteq \mathbb{A}^3$ is the image of the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^3$ given as $\phi(t) = (t, t^2, t^3)$. It is a closed algebraic set; indeed, we shall see that $C = Z(\mathfrak{a})$ where $\mathfrak{a}$ is the ideal

$$\mathfrak{a} = (z - x^3, y - x^2).$$

The inclusion $C \subseteq Z(\mathfrak{a})$ follows readily, and for the other inclusion, we observe that points in $Z(\mathfrak{a})$ are shaped like (x, x^2, x^3) so we can just take $t = x$. Moreover, it holds true that $I(X) = \mathfrak{a}$. To see this, notice that any polynomial f can be represented as

$$f(x, y, z) = f(x, x^2, x^3) + h(x, y, z),$$

where $h \in \mathfrak{a}$. This is just a repeated application of the little lemma (lemma 1.25) above; first with $y = x^2 - (x^2 - y)$ and then with $z = x^3 - (x^3 - z)$. That $f(x, y, z)$ vanishes on C means that $f(x, x^2, x^3)$ vanishes identically and hence $f \in \mathfrak{a}$.

As a by product of this reasoning, we obtain that the ideal $\mathfrak{a}$ is a prime ideal; indeed, it is the kernel of the restriction map

$$k[x, y, z] \to k[t]$$

that sends a polynomial to its restriction to C; in other words, x goes to t, y to t^2 and z to t^3.

☆

Problems

1.3 Let $f \in k[x_1, \ldots, x_n]$. Show that the ideal (f) is radical if and only if no factor of f is multiple.

1.4 Assume that the characteristic of k is zero. Let $f(x)$ be a polynomial in $k[x]$. Show that the relation $\sqrt{(f)} = (f : f')$ holds (where f' is the derivative of f; see exercise 1.13). Give a counterexample if k is of positive characteristic.

For any two ideals $\mathfrak{a}$ and $\mathfrak{b}$ in a ring A recall that one denotes by $(\mathfrak{a} : \mathfrak{b})$ the ideal of those $a \in A$ such that $a \cdot \mathfrak{b} \subseteq \mathfrak{a}$; that is $(\mathfrak{a} : \mathfrak{b}) = \{ a \in A \mid a \cdot \mathfrak{b} \subseteq \mathfrak{a} \}$.

1.5 Let $\mathfrak{p}$ be a prime ideal in $k[x_1, \ldots, x_n]$. Show that $\mathfrak{p}$ is the intersection of all the maximal ideals containing it; that is, $\mathfrak{p} = \bigcap_{\mathfrak{p} \subseteq \mathfrak{m}} \mathfrak{m}$ HINT: Show that $I(Z(\mathfrak{p})) = \bigcap_{\mathfrak{p} \subseteq \mathfrak{m}} \mathfrak{m}$, then use the Nullstellensatz.

1.6 Consider the closed algebraic set in $\mathbb{A}^2$ given by the vanishing of the polynomial $P(x) = y^2 - x(x+1)(x-1)$. Let $\alpha \in \mathbb{C}$ and let $\mathfrak{a} = (x - \alpha, P(x))$. Determine $Z(\mathfrak{a})$ for all α. For which α's is $\mathfrak{a}$ a radical ideal?

1.7 With the same notation as in the previous problem. Let $\mathfrak{b}$ be the ideal $\mathfrak{b} = (y - \alpha, P(x))$. Determine $Z(\mathfrak{b})$ for all α and decide for which α the ideal $\mathfrak{b}$ is radical. HINT: The answer depends on the characteristic of k, characteristic three being special.

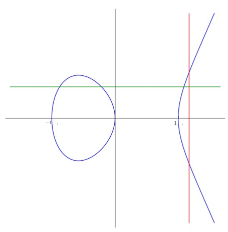


Figure 1.8: A cubic and two lines.

1.8 Let $F_1, \ldots, F_r$ be homogenous polynomials in $k[x_1, \ldots, x_n]$ and let $X = Z(F_1, \ldots, F_r)$ be the closed algebraic subset they define. Show that X is a *cone* with apex at the origin; that is, show that if x is a point in X, the line joining x to the origin lies entirely in X. HINT: Show that $t \cdot x$ lies in X for all $t \in k$.

1.9 Assume that X is a cone in $\mathbb{A}^n$ with apex at the origin, and assume that f is a polynomial that vanishes on X. Show that also all the homogenous components of f vanish along X.

1.10 Let $\mathbb{M}_{n,m}$ be the space of $n \times m$-matrices with coefficients from k. It can be identified with the affine space $\mathbb{A}^{nm}$ with coordinates x_{ij} where $1 \leqslant i \leqslant n$ and $1 \leqslant j \leqslant m$. Let r be a natural number less than both n and m, and let W_r be the set of $n \times m$-matrices of rank at most r. Show that W_r is a closed algebraic subset. Show that all the W_r's are cones over the origin. HINT: Determinants are polynomials.

1.11 Let $C_n \subseteq \mathbb{A}^n$ be the curve with parameter representation $\phi(t) = (t, t^2, \ldots, t^n)$, and let $\mathfrak{a}$ be the ideal $\mathfrak{a} = (x_i - x_1 x_{i-1} \mid 2 \leqslant i \leqslant n)$. Show that C_n is a closed algebraic set, that $I(C_n) = \mathfrak{a}$ and that $\mathfrak{a}$ is a prime ideal. The curves C_n are called *affine normal rational* curves and they are close relatives to the twisted cubic. For $n = 2$ we have a parabola in the plane and for $n = 3$ we get back the twisted cubic.

Affine normal rational curves

1.12 As usual $\mathbb{F}_p$ is the finite field with p elements. The aim of this exercise is to establish that there are infinitely many irreducible polynomials with coefficients in $\mathbb{F}_p$. If you are interested, there is a nice introduction to finite fields in Irelands and Rosen's book[4].

[4]

Let N_d be the number of irreducible, monic polynomials over $\mathbb{F}_p$ of degree d, and let F_d denote their product. Show that $x^{p^n} - x = \prod_{d|n} F_d(x)$ and that $p^n = \sum_{d|n} dN_d$. Conclude that there are infinitely many irreducible polynomials over $\mathbb{F}_p$. (If you know about Möbius inversion, show that $nN_d = \sum_{d|n} \mu(n/d)p^d$.)

1.13 *(The formal derivative.)* Let $f(x) = \sum_i a_i x^i$ be a polynomial. Define the (formal) derivative of f to be $f'(x) = \sum_i i a_i x^{i-1}$. Show that the usual rules are still valid; *i.e.* derivation is a linear operation and Leibnitz's product rule holds true. Show that f' vanishes identically if and only if either f is constant or the characteristic of k is p and $f(x) = g(x^p)$ for some polynomial $g(x)$.

★

Lecture 2

Zariski topolgies

Hot themes in Lecture 2: *The Zariski topology on closed algebraic subsets—irreducible topological spaces—Noetherian topological spaces—-primary decomposition and decomposition of noetherian spaces into irreducibles—hypersurfaces—polynomial maps—quadratic forms—determinantal varieties—Veronese surface.*

The realm of algebraic geometry is much bigger than the corner occupied by the closed algebraic sets. There are many more geometric objects, several of which will be the principal objects of our interest. However, the closed algebraic sets are fundamental and serve as building blocks. Just like a smooth manifold locally looks like an open ball in euclidean space, our spaces will locally look like a closed algebraic set, or in a more restrictive setting, like an affine variety. Before giving the general definition, we need to know what "locally" means, and of course, this will be encoded in a topology. The topologies that are used, are particularly well adapted to algebraic geometry, and they are called *Zariski topologies* after one of the great algebraic geometers Oscar Zariski.

The Zariski topology is of course useful in several other ways as well. For instance, it leads to a general concept of irreducible topological spaces and a decomposition of spaces into irreducible components—a generalization of the primary decomposition of ideals in Noetherian rings.

Oscar Zariski (1899–1986) Russian–American mathematician

2.1 The Zariski topology

In Lecture 1 we established the close relationship between closed algebraic sets and ideals in polynomial rings, and among those relations were the following two:

- $Z(\sum_i \mathfrak{a}_i) = \bigcap_i Z(\mathfrak{a}_i)$;
- $Z(\mathfrak{a} \cap \mathfrak{b}) = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$,

where $\mathfrak{a}$ and $\mathfrak{b}$ are ideals and $\{\mathfrak{a}_i\}_{i\in I}$ any collection of ideals in $k[x_1, \ldots, x_n]$. The first relation shows that the intersection of arbitrarily many closed algebraic sets is a closed algebraic set, the second that the union of two is closed algebraic (hence the union of finitely many). And of course, both the empty set and the entire affine space are closed algebraic sets (zero loci of respectively the whole polynomial ring and the zero ideal). The closed algebraic sets in $\mathbb{A}^n$ therefore fullfil the axioms for being the closed sets of a topology. This topology is called the *Zariski topology*.

The Zariski topology

2.1 Every closed algebraic set X in $\mathbb{A}^n$ carries a Zariski topology as well, namely the topology induced from the Zariski topology on $\mathbb{A}^n$. The closed sets are easily seen to be the closed subsets of $\mathbb{A}^n$ that are contained in X. These are the zero-loci of ideals $\mathfrak{a}$ containing $I(X)$; that is, those shaped like $Z(\mathfrak{a})$ with $I(X) \subseteq \mathfrak{a}$. Such ideals are in one-to-one correspondence with the ideals in the coordinate ring $A(X) = k[x_1, \ldots, x_n]/I(X)$. In other words, the Zariski closed sets in X are the zero-loci of the ideals in $A(X)$. And if we request the $\mathfrak{a}$'s to be radical ideals, the correspondence is one-to-one.

Example 2.1 The closed algebraic sets of $\mathbb{A}^1$ are, apart from the empty set and $\mathbb{A}^1$ itself, just the finite sets. Indeed, the polynomial ring $k[x]$ is a PID so that any ideal is shaped like $(f(x))$, and the zeros of f are finite in number[1]. The Zariski open sets are therefore those with a finite complement. ☆

[1] *Remember that k is algebraically closed*

Example 2.2 The closed sets of the affine plane $\mathbb{A}^2$ are more complicated. Later on we shall show that they are finite unions of either points or subsets shaped like $Z(f(x, y))$ where f is a polynomial in $k[x, y]$. Notice that this is not the product topology on $\mathbb{A}^2$. Indeed, the product topology is generated by the inverse images of closed sets in the factors which in our case are just points (apart from the empty set anf the entire space), and the inverse images are thus sets of the form $Z(x - a)$ or $Z(y - a)$. The closed are finite union of intersections of these, that is, unions of points or lines "parallel to one of the axes". However, a conic like the hyperbola $xy = 1$, for instance, is not among those. ☆

2.2 The open sets are of course the complements of the closed ones, and among the open sets there are some called *distinguished open sets* that play a special role. They are the sets where a single polynomial does not vanish. If f is any polynomial in $k[x_1, \ldots, x_n]$, we define

The distinguished open sets

$$X_f = \{ x \in X \mid f(x) \neq 0 \},$$

which clearly is open in X being the complement of $Z(f) \cap X$. Another common notation for X_f is $D(f)$.

Proposition 2.3 *Let X be a closed algebraic set. The distinguished open sets form a basis*[2] *for the Zariski topology on X.*

[2] Recall that a collection $\{U_i\}$ of open sets is a basis for the topology if every open set in X is the union of members of $\{U_i\}$.

Proof: Fix an open set U. The complement U^c is closed and hence of the form $U^c = Z(\mathfrak{a})$ for some ideal $\mathfrak{a}$ in $A(X)$. If $\{f_i\}$ is a set of generators for $\mathfrak{a}$, it holds true that $Z(\mathfrak{a}) = \bigcap_i Z(f_i)$, and consequently $U = Z(\mathfrak{a})^c = \bigcup_i U_{f_i}$. ❏

2.4 When the ground field is the field of complex numbers $\mathbb{C}$, the affine space $\mathbb{A}^n(\mathbb{C}) = \mathbb{C}^n$ has in addition to the Zariski topology the traditional metric topology, and a closed algebraic set X in $\mathbb{A}^n$ inherits a topology from this. The induced topology on X is called *the complex* or *the strong* topology.

The complex or strong topology

The Zariski topology is very different from the strong topology. Polynomials are (strongly) continuous, so any Zariski-open set is strongly open, but the converse is far from being true. For example, in contrast to the usual topology on $\mathbb{C}$, the Zariski topology on the affine line $\mathbb{A}^1(\mathbb{C})$, as we saw in Example 2.1 above, is the topology of finite complements; a non-empty set is open if and only if the complement is finite.

The Zariski topology has, however, the virtue of being defined whatever the ground field is (as long as it is algebraically closed), and the field can very well be of positive characteristic.

2.2 *Irreducible topological spaces*

2.5 A topological space X is called *irreducible* if it is not the union of two proper closed subsets. That is, if $X = X_1 \cup X_2$ with X_1 and X_2 both being closed, then either $X_1 = X$ or $X_2 = X$.

Irreducible topological spaces (irreduktible topologiske rom)

2.6 If X is a closed algebraic subset of $\mathbb{A}^n$, one may translate the topological property of being irreducible into an algebraic property of the ideal $I(X)$: If $I(X) = \mathfrak{a} \cap \mathfrak{b}$ then either $\mathfrak{a} = I(X)$ or $\mathfrak{b} = I(X)$, and in commutative algebra such ideals are called *irreducible* (guess why!). Prime ideals are examples of irreducible ideals, but there are many more. However, irreducible ideals are *primary ideal*, and this observation is at the base of the theory of primary decomposition in Noetherian rings.

2.7 Taking complements, we arrive at the following characterization of irreducible spaces:

Lemma 2.8 *A topological space X is irreducible if and only if the intersection of any two non-empty open subsets is non-empty.*

Proof: Assume first that X is irreducible and let U_1 and U_2 be two open subset. If $U_1 \cap U_2 = \emptyset$, it would follow, when taking complements, that $X = U_1^c \cup U_2^c$, and X being irreducible, we could infer that $U_i^c = X$ for either $i = 1$ or $i = 2$; whence $U_i = \emptyset$ for one of the i's. To prove the other implication, assume that X is expressed as a union $X = X_1 \cup X_2$ with the X_i's being closed. Then $X_1^c \cap X_2^c = \emptyset$; hence either $X_1^c = \emptyset$ or $X_2^c = \emptyset$, and therefore either $X_1 = X$ or $X_2 = X$. ❑

2.9 There are a few properties irreducible spaces have that follow immediately. Firstly, every open non-empty subset U of an irreducible space X is dense. Indeed, if $x \in X$ and V is any neighbourhood of x, the lemma tells us that $U \cap V \neq \emptyset$, and x belongs to the closure of U.

Secondly, every non-empty open subset U of X is irreducible. This follows trivially since any two non-empty open sets of U are open in X, hence their intersection is *a fortiori* non-empty.

Thirdly, the closure $\overline{Y}$ of an irreducible subset Y of X is irreducible. For if U_1 and U_2 are two non-empty open subsets of $\overline{Y}$, it holds true that $U_i \cap Y \neq \emptyset$, and hence $U_1 \cap U_2 \cap Y \neq \emptyset$ since Y is irreducible, and *a fortiori* the intersection $U_1 \cap U_2$ is non-empty.

Fourthly, continuous images of irreducible spaces are irreducible. If $f\colon X \to Y$ is surjective and continuous and U_i for $i = 1, 2$ are open and non-empty subsets of Y, it follows that $f^{-1}(U_i)$ are open and non-empty (the map f is surjective) for $i = 1, 2$. When X is irreducible, it holds that $f^{-1}(U_1 \cap U_2) = f^{-1}(U1) \cap f^{-1}(U_2) \neq \emptyset$, and so $U_1 \cap U_2$ is not empty.

Summing up for later reference, we state the following lemma:

Lemma 2.10 *Open non-empty sets of an irreducible set are irreducible and dense. Closures and continuous images of irreducible sets are irreducible.*

We should also mention that Zariski topologies are far from being Hausdorff; it is futile to search for disjoint neighbourhoods when all non-empty open subsets meet!

2.11 Closed algebraic sets in the affine space $\mathbb{A}^n$ are of special interest, and we have already alluded to the algebraic equivalent of being irreducible. Here is the formal statement and a proof:

Proposition 2.12 *An algebraic set $X \in \mathbb{A}^n$ is irreducible if and only if the ideal $I(X)$ of polynomials vanishing on X is prime.*

As a particular case we observe that the affine space $\mathbb{A}^n$ itself is irreducible.

Proof: Assume that X is irreducible and let f and g be polynomials such that $fg \in I(X)$, which implies that $X \subseteq Z(f) \cup Z(g)$. Since X is irreducible, it follows that either $Z(g) \cap X$ or $Z(f) \cap X$ equals X. Hence one has either $X \subseteq Z(f)$ or $X \subseteq Z(g)$, which for the ideal $I(X)$ means that either $f \in I(X)$ or $g \in I(X)$.

The other way around, assume that $I(X)$ is prime and that $X = Z(\mathfrak{a}) \cup Z(\mathfrak{b})$ with $\mathfrak{a}$ and $\mathfrak{b}$ being radical ideals. Then it holds that $I(X) = \mathfrak{a} \cap \mathfrak{b}$ and because $I(X)$ is prime, we deduce that $I(X) = \mathfrak{a}$ or $I(X) = \mathfrak{b}$. Hence $X = Z(\mathfrak{a})$ or $X = Z(\mathfrak{b})$. ❑

2.13 Algebraic sets that are the zero-locus of one single polynomial; that is, sets X such that $X = Z(f)$, are called *hypersurfaces*. They are quintessential players in our story. Curves in $\mathbb{A}^2$ and surfaces in $\mathbb{A}^3$ are well known examples of the sort.

Hypersurfaces

In general, hypersurfaces are somehow more manageable than general algebraic sets—even though the equation can be complicated, at least there is just one!

If f is a linear polynomial, $Z(f)$ is called a *hyperplane*—basically a hyperplane is just a linear subspace of dimension $n-1$ in $\mathbb{A}^n$.

Hyperplanes

2.14 The Nullstellensatz tells us that (f) and the radical $\sqrt{(f)}$ have the same zero-locus, so every hypersurface has a polynomial f without multiple factors as defining polynomial. Moreover, we know that a polynomial f generates a prime ideal if and only it is irreducible—this is just the fact that polynomial rings are UFD. Hence a hypersurface is irreducible if and only if it can be defined by an irreducible polynomial.

Proposition 2.15 *Let f be a polynomial in $k[x_1, \ldots, x_n]$. If f is an irreducible polynomial, the hypersurface $Z(f)$ is irreducible. If the hypersurface $Z(f)$ is irreducible, then f is a power of some irreducible polynomial.*

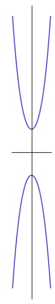


Figure 2.1: A hyperelliptic curve.

Examples

2.3 Polynomials shaped like $f(x) = y^2 - P(x)$ are irreducible unless $P(x)$ is a square; that is $P(x) = Q(x)^2$. Indeed; if $f = A \cdot B$ either both A and B are linear in y or one of them does not depend on y at all. In the former case $A(x,y) = y + a(x)$ and $B(x,y) = y + b(x)$ which gives

$$y^2 - P(x) = (y + a(x)) \cdot (y + b(x)) = y^2 + (a(x) + b(x)) \cdot y + a(x)b(x),$$

and it follows that $a(x) = -b(x)$. In the latter case one finds

$$y^2 - P(x) = (y^2 + a(x)) \cdot b(x)$$

which implies that $b(x) = 1$ and hence $f(x)$ is irreducible. When the polynomial $P(x)$ has merely simple zeros, the curve defined by f is called a *hyperelliptic curve*, and if $P(x)$ in addition is of the third degree, it is said to be an *elliptic curve*. In figures 2.1 and 2.2 in the margin we have depicted (the real points of) two, both with $P(x)$ of the eighth degree. Can you explain the qualitative difference between the two?

Hyperelliptic curves

We already met some elliptic curves in Lecture 1. They are omnipresent in both geometry and number theory and we shall study them closely later on.

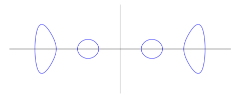


Figure 2.2: Another hyperelliptic curve.

2.4 Our second example is a well known hypersurface, namely the determinant. It is one of many interesting algebraic sets that appear as subsets of matrix-spaces $\mathbb{M}_{n,m} = \mathbb{A}^{nm}$ defined by rank conditions. The example is about the determinant $\det(x_{ij})$ of a "generic" $n \times n$-matrix; that is, one with independent variables as entries. It is a homogenous polynomial of degree n.

We shall see that it is irreducible by a specialization technique. Look at matrices like

$$A = \begin{pmatrix} t & y_1 & 0 & 0 & \ldots & 0 \\ 0 & t & y_2 & 0 & \ldots & 0 \\ 0 & & \ddots & & \ddots & 0 \\ 0 & \ldots & 0 & t & y_{n-2} & 0 \\ 0 & \ldots & 0 & 0 & t & y_{n-1} \\ y_n & \ldots & 0 & 0 & 0 & t \end{pmatrix}$$

with a variable t's along the diagonal, and variables y_i's along the first "supra-diagonal" and y_n in the lower left corner; in other words the specialization consists in putting $x_{ii} = t$, $x_{i,i+1} = y_i$ for $i \leqslant n-1$ and $x_{n1} = y_n$, and the rest of the x_{ij}'s are put to zero.

It is not difficult to show that $\det A = t^n - (-1)^n y_1 \cdot \ldots \cdot y_n$ and that this polynomial is irreducible. A potential factorization $\det(x_{ij}) = F \cdot G$ with F and G both being of degree less than n must persist when giving the variables special values, but as $\det A$ is irreducible and of degree n, we conclude that there can be no such factorization.

Emmy Noether (1882–1935) German mathematician

PROBLEM 2.1 With notation as in the example, show that the determinant $\det A$ is given as $\det A = t^n - (-1)^n y_1 \cdot \ldots \cdot y_n$ and that this is an irreducible polynomial.

Decomposition into irreducibles

From commutative algebra we know that ideals in Noetherian rings have a *primary decomposition*. An ideal $\mathfrak{a}$ in a Noetherina ring can be expressed as an intersection

$$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r$$

where the $\mathfrak{q}_i$'s are primary ideals. Recall that primary ideals have radicals that are a prime, so the ideals $\sqrt{\mathfrak{q}_i}$ are prime. They are called the *primes associated* to $\mathfrak{a}$. Such a decomposition is not always unique. The associated prime ideals are unique as are the components $\mathfrak{q}_i$ corresponding to minimal associated primes, but the so called *embedded components*[3] are not. For instance, one has $(x^2, xy) = (x) \cap (x^2, y)$ but also $(x^2, xy) = (x) \cap (x^2, xy, y^2)$ holds true.

[3] A primary component $\mathfrak{q}_i$ is *embedded* if $\sqrt{\mathfrak{q}_i}$ contains the radical $\sqrt{\mathfrak{q}_j}$ of another component $\mathfrak{q}_j$.

2.16 Properties of ideals in the polynomial ring $k[x_1, \ldots, x_n]$ usually translate into a properties of algebraic sets, and so also for the primary decomposition. In geometric terms it reads as follows. Let $Y = Z(\mathfrak{a})$ for an ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$, and write down the primary decomposition of $\mathfrak{a}$:

$$\mathfrak{a} = \mathfrak{q}_1 \cap \mathfrak{q}_2 \cap \ldots \cap \mathfrak{q}_r.$$

Putting $Y_i = Z(\sqrt{\mathfrak{q}_i})$, we find $Y = Y_1 \cup Y_2 \cup \ldots \cup Y_r$, where each Y_i is an irreducible closed algebraic set in $\mathbb{A}^n$. If the prime $\sqrt{\mathfrak{q}_i}$ is not minimal among the associated primes, say $\sqrt{\mathfrak{q}_j} \subseteq \sqrt{\mathfrak{q}_i}$, it holds that $Y_i \subseteq Y_j$, and the component Y_i contributes nothing to intersection and can be discarded.

2.17 A decomposition $Y = Y_1 \cup \ldots \cup Y_r$ of any topological space is said to be *redundant* if one can discard one or more of the Y_i's without changing the union. That a component Y_j can be discarded is equivalent to Y_j being contained in the union of rest; that is, $Y_j \subseteq \bigcup_{i \neq j} Y_i$. A decomposition that is not redundant, is said to be *irredundant*.

Redundant decompositions

Irredudant decompositions

Proposition 2.18 *Any closed algebraic set $Y \subseteq \mathbb{A}^n$ can be written as an irredundant union*

$$Y = Y_1 \cup \ldots \cup Y_r$$

where the Y_i's are irreducible closed algebraic subsets. The union is unique up to the order of the Y_i's.

2.19 A decomposition result as in Proposition 2.18 above holds for a much broader class of topological spaces than the closed algebraic sets. The class in question is the class of so-called *Noetherian topological spaces*; they comply to the requirement that every descending chain of closed subsets is eventually stable. That is; if $\{X_i\}$ is a collection of closed subsets forming a chain

Noetherian topological spaces

$$\ldots X_{i+1} \subseteq X_i \subseteq \ldots \subseteq X_2 \subseteq X_1,$$

it holds true that for some index r one has $X_i = X_r$ for $i \geqslant r$. It is easy to establish, and left to the zealous students, that any subset of a Noetherian space endowed with the induced topology is Noetherian. Notice also that common usage in algebraic geometry is to call a topological space *quasi-compact* if every open covering can be reduced to a finite covering.

By common usage in mathematics a compact space is Hausdorff. The "Zariski"-like" spaces are far from being Hausdorff, therefore the notion quasi-compact.

Quasi-compact

Lemma 2.20 *Let X be a topological space. The following three conditions are equivalent:*

- *X is Noetherian;*
- *Every open subset of X is quasi-compact;*
- *Every non-empty family of closed subsets of X has a minimal member.*

The last statement leads to the technique called *Noetherian induction*—proving a statement about closed subsets, one can work with a minimal "crook"; *i.e.* a minimal counterexample.

Proof: Assume to begin with that X is Noetherian and let Σ be a family of closed sets without a minimal elements. One then easily constructs a strictly descending chain that is not stationary by recursion. Assume a chain

$$X_r \subset X_{r-1} \subset \ldots \subset X_1$$

of length r has been found; to extend it just append any subset in Σ strictly contained in X_r, which does exist since Σ has no minimal member.

Next, assume that every Σ has a minimal member and let $\{U_i\}$ be an open covering of X. Let Σ be the family of closed sets being finite intersections of complements of members of the covering. It has a minimal element Z. If U_j is any member of the covering, it follows that $Z \cap U_j^c = Z$, hence $U_j \subseteq Z^c$, and by consequence $U = Z^c$.

Finally, suppose that every open U in X is quasi-compact and let $\{X_i\}$ be a descending chain of closed subsets. The open set $U = X \backslash \bigcap_i X_i$ is quasi-compact by assumption and covered by the ascending collection $\{X_i^c\}$, hence it is covered by finitely many of them. The collection $\{X_i^c\}$ being ascending, we can infer that $X_r^c = U$ for some r; that is, $\bigcap_i X_i = X_r$ and consequently it holds that $X_i = X_r$ for $i \geqslant r$. ❏

The decomposition of closed subsets in affine space as a union of irreducibles can be generalized to any Noetherian topological space:

Theorem 2.21 *Every closed subset Y of a Noetherian topological space X has an irredundant decomposition $Y = Y_1 \cup \ldots \cup Y_r$ where each is Y_i is a closed and irreducible subset of X. Furthermore, the decomposition is unique.*

The Y_i's that appear in the theorem are called the *irreducible components* of Y. They are *maximal* among the closed irreducible subsets of Y.

Irreducible components

Proof: We shall work with the family Σ of those closed subsets of X which can not be decomposed into a finite union of irreducible closed subsets; or if you want, the set of counterexamples to the assertion in the theorem—and of course, we shall prove that it is empty!

Assuming the contrary—that Σ is non-empty—we can find a minimal element Y in Σ because X by assumption is Noetherian. The set Y itself can not be irreducible, so $Y = Y_1 \cup Y_2$ where both the Y_i's are proper subsets of Y and therefore do not belong to Σ. Either is thus a finite union of closed irreducible subsets, and consequently the same is true for their union Y. We have a contradiction, and Σ must be empty.

As to uniqueness, assume that we have a counterexample; that is, two irredundant decomposition such that $Y_1 \cup \ldots \cup Y_r = Z_1 \cup \ldots \cup Z_s$ and such that one of the Y_i's, say Y_1, does not equal any of the Z_k's.

Since Y_1 is irreducible and $Y_1 = \bigcup_k (Z_k \cap Y_1)$, it follows that $Y_1 \subseteq Z_k$ for some k. A similarly argument gives $Z_k = \bigcup_i (Z_k \cap Y_i)$ and Z_k being irreducible, it holds that $Z_k \subseteq Y_i$ for some i, and therefore $Y_1 \subseteq Z_k \subseteq Y_i$. By irredundancy we infer that $Y_1 = Y_i$, and hence $Y_1 = Z_k$. Contradiction. ❏

Problem 2.2 Let X be a topological space and $Z \subseteq X$ be an irreducible component of X. Let U be an open subset of X. Assume that $U \cap Z$ is nonempty. Show that $Z \cap U$ is an irreducible component of U. ★

2.22 You should already have noticed the resemblance of the condition to be Noetherian for topological spaces and rings—both are chain conditions—and of course that is where the name Noetherian spaces comes from. When X is a closed algebraic set in $\mathbb{A}^n$, the one-to-one correspondence between the prime ideals in the coordinate ring $A(X)$ and the closed irreducible sets in X, yields that X is a Noetherian space; indeed, Hilbert's basis theorem implies that $A(X)$ is a Noetherian ring, so any ascending chain $\{I(X_i)\}$ of prime ideals corresponding to a descending chain of $\{X_i\}$ of closed irreducibles, is stationary. We have

There are examples of non-noetherian rings with just one maximal ideal, so an ascending chain condition on prime ideals does not imply that the ring is Noetherian. However, by a theorem of I.S. Cohen, a ring is Noetherian if all prime ideals are finitely generated.

Proposition 2.23 *If X is a closed algebraic subset of $\mathbb{A}^n$, then X is a Noetherian space.*

Hypersurfaces once more and some examples

It is often difficult to prove that an algebraic set X is irreducible, or equivalently that the ideal $I(X)$ is a prime ideal. This can be challenging even when X is a hypersurface.

Generally, to find the primary decomposition of an ideal a difficult. In addition to the problems of finding the minimal primes and the corresponding primary components, which frequently can be attacked by geometric methods, one has the notorious problem of embedded components. They are annoyingly well hidden from geometry.

If $X = Z(f)$ is a hypersurface in $\mathbb{A}^n$, there will be no embedded components since the polynomial ring is a UFD. Indeed, one easily sees that $(f) \cap (g) = (fg)$ for polynomial without common factors. Hence one infers by induction that

$$(f) = (f_1^{a_1}) \cap \ldots \cap (f_r^{ar}),$$

where $f = f_1^{a_1} \cdot \ldots \cdot f_r^{a_r}$ is the factorization of f into irreducibles, and observes there are no inclusions among the prime ideals (f_i).

There is a vast generalization of this. A very nice class of rings are formed by the so called Cohen–Macaulay rings. If the coordinate ring $A(X)$ is Cohen–Macaulay, the ideal $I(X)$ has no embedded components. This is a part of the Macaulay's unmixed theorem—which even says that all the components of $I(X)$ have the same dimension.

Examples

2.5 *(Homogeneous polynomials)* Recall that a polynomial f is *homogenous* if all the monomials that appear (with a non-zero coefficient) in f are of the same total degree. Recollecting terms of the same total degree, one sees that any polynomial can be written as a sum $f = \sum_i f_i$ where the f_i's are homogeneous of degree i; and since homogeneous polynomials of different total degrees are linearly independent, such a decomposition is unique.

Homogeneous polynomials

If a homogeneous polynomial f factors as a product $f = a \cdot b$, the polynomials a and b will also *homogeneous*. (Sometimes this can make life easier if you want *e.g.* to factor f or to show that f is irreducible.) Indeed, if $a = \sum_{0 \leqslant i \leqslant d} a_i$ and $b = \sum_{0 \leqslant j \leqslant e} b_j$ with a_i's and b_j's homogeneous of degree i and j respectively and with $a_d \neq 0$ and $b_e \neq 0$, one finds

$$f = ab = \sum_{i+j<d+e} a_i b_j + a_d b_e$$

Since the decomposition of f in homogeneous parts is unique, it follows that $f = a_e b_d$.

2.6 The polynomial $f(x) = x_1^2 + x_2^2 + \ldots + x_n^2$ is irreducible when $n \geqslant 3$ and the characteristic of k is not equal to two. To check this, we may assume $n = 3$. Suppose there is a factorization like

$$x_1^2 + x_2^2 + x_3^2 = (a_1x_1 + a_2x_2 + a_3x_3)(b_1x_1 + b_2x_2 + b_3x_3).$$

Observing that $a_1 \cdot b_1 = 1$ and replacing a_i by a_1/a_1 and b_i by b_i/b_1, one may assume that $a_1 = b_1 = 1$ and the equation takes the shape

$$x_1^2 + x_2^2 + x_3^2 = (x_1 + a_2x_2 + a_3x_3)(x_1 + b_2x_2 + b_3x_3).$$

Putting $x_3 = 0$, and using the factorization $x_1^2 + x_2^2 = (x_1 + ix_2)((x_1 - ix_2)$, one easily brings the equation on the form

$$x_1^2 + x_2^2 + x_3^2 = (x_1 + ix_2 + a_3x_3)(x_1 - ix_2 + b_3x_3),$$

from which one obtains $a_3 = b_3$ and $a_3 = -b_3$. Since the characteristic is not equal to two, this is a contradiction. If k is of characteristic two, however, $x_1^2 + x_2^2 + x_3^2$ is not irreducible; it holds true that $x_1^2 + x_2^2 + x_3^2 = (x_1 + x_2 + x_3)^2$.

2.7 *(Monomial ideals)* An ideal $\mathfrak{a}$ is said to be *monomial* if it is generated by monomials. Such an ideal has the property that if a polynomial f belongs to $\mathfrak{a}$, all the monomials appearing in f belong to $\mathfrak{a}$ as well. To verify this one writes f as a sum $f = \sum M_i$ of monomial terms[4] and let $\{N_j\}$ be monomial generators for $\mathfrak{a}$. Then one infers that

Monomial ideals

[4] A *monomial term* is of the form $\alpha \cdot M$ where α is a scalar and M a monomial.

$$f = \sum_i M_i = \sum_j P_j N_j = \sum_{j,k} A_{kj} N_j$$

where P_j are polynomials whose expansions in monomial terms are $P_j = \sum_k A_{kj}$. Since different monomials are linearly independent (by definition of polynomials), every term M_i is a linear combination of the monomial terms $A_{kj}N_j$ corresponding to the same monomial, and hence lies in the ideal $\mathfrak{a}$.

2.8 Monomial ideals are much easier to work with than general ideals. As an easy example, consider the union of the three coordinate axes in $\mathbb{A}^3$. It is given as the zero locus of the ideal $\mathfrak{a} = (xy, xz, yz)$, and one has

$$(xy, xz, yz) = (x, y) \cap (x, z) \cap (z, w)$$

Indeed, one inclusion is trivial; for the other it suffices to show that a monomial in $(x, y) \cap (x, z) \cap (y, z)$ belongs to (xy, xz, yz). But $x^n y^m z^l$ lies in $(x, y) \cap (x, z) \cap (y, z)$ precisely when at least two of the three integers n, m and l are non-zero, which as well is the requirement to lie in $\mathfrak{a}$.

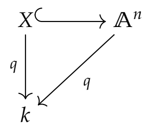
☆

2.3 *Polynomial maps between algebraic sets*

The final topic we approach in this chapter are the so called *polynomial maps* between closed algebraic sets. This anticipates the introduction of "morphisms" between general varieties in the next section, but it is still worthwhile to mention it here. Polynomial maps between algebraic sets are conceptually much simpler than morphisms, and in concrete cases one works with polynomials. In the end, the two concepts of polynomial maps and morphisms between closed algebraic sets turn out to coincide.

The coordinate ring

Polynomial functions



2.24 Let $X \subseteq \mathbb{A}^n$ be a closed algebraic set. A *polynomial function* (later on they will also be called *regular functions*) on X is just the restriction to X of a polynomial on $\mathbb{A}^n$; that is, it is a polynomial $q \in k[x_1, \ldots, x_n]$ regarded as a function on X. Two polynomials p and q restrict to the same function precisely when the difference $p - q$ vanishes on X; that is to say, the difference $p - q$ belongs to the ideal $I(X)$. We infer that the polynomial functions on X correspond exactly the elements in the coordinate ring $A(X) = k[x_1, \ldots, x_n]/I(X)$.

Polynomial maps

Polynomial maps

2.25 Now, given another closed algebraic subset $Y \subseteq \mathbb{A}^m$ and a map $\phi \colon X \to Y$. Composing ϕ with the inclusion of Y in $\mathbb{A}^m$, we may consider ϕ as a map from X to $\mathbb{A}^m$ that takes values in Y; and as such, it has m component functions $q_1, \ldots, q_m$. We say that ϕ is a *polynomial map* if these components are polynomial functions on X. The set of polynomial maps from X to Y will be denoted by $\mathrm{Hom}_{\mathsf{Aff}}(X, Y)$.

EXAMPLE 2.9 We have already seen several examples. For instance, the parametrization of a rational normal curve C_n is a polynomial map from $\mathbb{A}^1$ to C_n whose component functions are the powers t^i. ☆

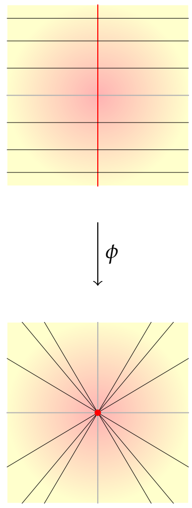
PROPOSITION 2.26 *Polynomial maps are Zariski continuous.*

PROOF: Assume that $\phi \colon X \to Y$ is the polynomial map. Given a polynomial function on Y. Clearly $\phi \circ q$ is a polynomial function on X, and hence $\phi^{-1}(Z(q)) = Z(\phi \circ q)$ is closed in X, and consequently, for any closed set $Z(q_1, \ldots, q_m)$, we find that the inverse image $\phi^{-1}Z(q_1, \ldots, q_r) = \bigcap_i \phi^{-1}Z(q_i)$ is closed. ❑

Examples

We conclude this lecture with a few examples to illustrate different phenomena surrounding polynomial maps . One example to warn that images of polynomial maps can be complicated, followed by two examples to illustrate that important and interesting subsets naturally originating in linear algebra can be irreducible algebraic sets.

EXAMPLE 2.10 Images of polynomial maps can be complicated. In general they are neither closed nor open. For example, consider the map $\phi\colon \mathbb{A}^2 \to \mathbb{A}^2$ given as $\phi(u, v) = (u, uv)$. Pick a point (x, y) in $\mathbb{A}^2$. If $x \neq 0$, it holds that $\phi(x, x^{-1}y) = (x, y)$ so points lying off the y-axis are in the image. Among points with $x = 0$ however, only the origin belongs to the image. Hence the image is equal to the union $\mathbb{A}^2 \backslash Z(x) \cup \{(0, 0)\}$. This set is neither closed (it contains an open set and is therefore dense) nor open (the complement equals $Z(x) \backslash \{(0, 0)\}$ which is dense in the closed set $Z(x)$, hence not closed).



The map ϕ collapses the v-axis to the origin, and, consequently, lines parallel to the u-axis are mappes to lines through the origin; the intersection point with the v-axis is mapped to the origin. Pushing these lines out towards infinity, their images approach the v-axis. So in some sense, the "lacking line" that should have covered the v axis, is the "line at infinity". ☆

PROBLEM 2.3 Let ϕ be the map in Example 2.10. Show that ϕ maps lines parallel to the u-axis (that is, those with equation $v = c$) to lines through the origin. Show that lines through the origin (those having equation $v = cu$) are mapped to parabolas. ★

PROBLEM 2.4 Describe the image of the map $\phi\colon \mathbb{A}^3 \to \mathbb{A}^3$ given as $\phi(u, v, w) = (u, uv, uvw)$. ★

EXAMPLE 2.11 *(Determinantal varieties)* Determinantal varieties are as the name indicates, closed algebraic sets defined by determinants. They are much studied and play a prominent role in mathematics. In this example we study one particular instance of the species.

The space $\mathbb{M}_{2,3} = \mathrm{Hom}_k(k^3, k^2)$ of 2×3-matrices may be consider to be the space $\mathbb{A}^6$ with the coordinates indexed like the entries of a matrix; that is, the points are like

$$\begin{pmatrix} x_{11} & x_{12} & x_{13} \\ x_{21} & x_{22} & x_{23} \end{pmatrix}. \tag{2.1}$$

We are interested in the subspace W of matrices of rank at most one. For a 2×3-matrix to be of rank at most one, is equivalent to the vanishing of the maximal minors (in the present case there are three[5] maximal minors and all three are quadrics). This shows that W is a closed algebraic set.

[5] The three are $x_{11}x_{22} - x_{12}x_{21}$, $x_{11}x_{23} - x_{13}x_{21}$, $x_{12}x_{23} - x_{13}x_{22}$

To see it is irreducible we express W as the image of an affine space under a polynomial map. Indeed, any matrix of rank at most one, can be factored as

$$a = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} (b_1, b_2, b_3).$$

So W is the image of $\mathbb{A}^5 = \mathbb{A}^2 \times \mathbb{A}^3$ under the map—which clearly is polynomial—that sends the tuple $(a_1, a_2, b_1, b_2, b_3)$ to the matrix $(a_i b_j)$. ☆

PROBLEM 2.5 Show that rank one maps can be factored as in the example above. HINT: The linear map corresponding to the matrix has image of dimension one and can be factored as $k^3 \xrightarrow{\alpha} k \xrightarrow{\iota} k^2$ ★

PROBLEM 2.6 Show that the ideal generated by the three minors of the matrix in (2.1) is a prime ideal in $k[x_{ij}]$. ★

EXAMPLE 2.12 *(Quadratic forms)* Recall that a *quadratic form* is a homogeneous polynomial of degree two. That is, one that is shaped like $P(x) = \sum_{i,j} a_{ij} x_i x_j$ where both i and j run from 1 to n. In that sum $a_{ij} x_i x_j$ and $a_{ji} x_j x_i$ appear as separate terms, but as a matter of notation, one organizes the sum so that $a_{ij} = a_{ji}$. Coalescing the terms $a_{ij} x_i x_j$ and $a_{ji} x_j x_i$, the coefficient in front of $x_i x_j$ becomes equal to $2a_{ij}$. For instance, when $n = 2$, a quadratic form is shaped like

Quadratic forms

$$P(x) = a_{11} x_1^2 + 2a_{12} x_1 x_2 + a_{22}^2 x_2^2.$$

The *coefficient matrix* of the form is the symmetric matrix $A = (a_{ij})$. Then one can express $P(x)$ as the matrix product

$$P(x) = xAx^t$$

where $x = (x_1, \ldots, x_n)$.

The set of such forms—or of such matrices—constitute a linear space which we shall denote by S_n. It has a basis x_i^2 and $2x_i x_j$ so in our language S_n is isomorphic to an affine space $\mathbb{A}^N$ whose dimension N equals the number of distinct monomials $x_i x_j$; that is $N = n(n+1)/2$. The coordinates with respect to this basis, are denoted by a_{ij}.

We are interested in the subspaces $W_r \subseteq \mathbb{A}^N$ where the rank of A is as most r. They form a descending chain; that is $W_{r-1} \subseteq W_r$; and clearly $W_n = \mathbb{A}^N$ and $W_0 = \{0\}$.

The W_r's are all closed algebraic subsets, and the aim of this example is to show they are irreducible:

PROPOSITION 2.27 *The subsets W_r are closed irreducible algebraic subset of $\mathbb{A}^N$.*

PROOF: That the W_r's are closed, hinges on the fact that a matrix is of rank at most r if and only if all its $(r+1) \times (r+1)$-minors vanish.

To see that the W_r's are irreducible, we shall use a common technique. Every matrix in W_r can be expressed in terms of a "standard matrix" in a a continuous manner.

By the classical Gram-Schmidt process, any symmetric matrix can be diagonalized. There is a relation

$$BAB^t = D$$

where B is an invertible matrix and where D is a diagonal matrix of a special form. If the rank of A is r, the first r diagonal elements of D are 1's and the rest are 0's. Introducing $C = B^{-1}$, we obtain the relation

$$A = CDC^t.$$

Allowing C to be any $n \times n$-matrix, not merely an invertible one, one obtains in this way all matrices A of rank at most r. Rendering the above considerations into geometry, we introduce a parametrization of the locus W_r of quadrics of rank at most r. It is not a one-to-one map, several parameter values correspond to the same point, but it is a polynomial map and serves our purpose, to prove that W_r is irreducible. We define a map

Jacob Steiner (1996–1863) Swiss mathematician

$$\Phi\colon \mathbb{M}_{n,n} \to \mathbb{A}^N$$

byt letting it send an $n \times n$-matrix C to CDC^t. The map Φ is a polynomial map because the entries of a product of two matrices are expressed by polynomials in the entries of the factors, and by the Gram-Schmidt process described above, its image equals W_r. Hence W_r is irreducible. ❑

Giuseppe Veronese (1854–1917) Italian mathematician

To get a better understanding of how a form of rank r is shaped, one introduces new coordinates $\{y_i\}$ adapted to a form specific with matrix $A = BDB^t$ by the relations $yB = x$, which is legitimate since B and therefore B^t is invertible. Then $xAx^t = yB^tAB^ty^t = yDy^t$. So, in view of the shape of D, expressed in the new coordinates the quadratic form $P(x)$ has the shape:

$$P(y) = y_1^2 + \ldots + y_r^2.$$

Of special interest are the sets W_1 of rank one quadrics. By what we just saw, these quadrics are all squares of a linear form in the variables x_i's (remember that y_1 is a linear form in the original coordinates, the x_i's); that is, one has an expression

$$P(x) = (\sum_i u_i x_i)^2 = \sum_i u_i^2 x_i^2 + \sum_{i<j} 2u_i u_j x_i x_j.$$

This gives us another parametrization of W_1, namely the one sending a linear form to its square. The linear forms constitute a vector space of dimension n (one coefficient for each variable), so "the square" is map

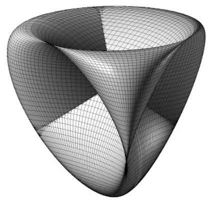


Figure 2.3: The Roman surface—a projection of a real Veronese surface

$$\nu\colon \mathbb{A}^n \to \mathbb{A}^N \qquad (2.2)$$

sending $(u_1, \ldots, u_n)$ to the point whose coordinates are all different products $u_i u_j$ with $i \leqslant j$ (remember we use the basis for the space of quadrics made up of the squares x_i^2 and the cross terms $2x_i x_j$, in some order). When $n = 3$ we get a mapping of $\mathbb{A}^3$ into $\mathbb{A}^6$ whose image is called the *cone over the Veronese surface*.

The *Veronese surface* is a famous *projective* surface living in the projective space $\mathbb{P}^5$ (we shall study these spaces closely in subsequent lectures). The real points of the Veronese surface be can be realized as projection into $\mathbb{R}^3$, at least if one allows the surface to have self intersections. The surface depicted in the margin is parametrized by three out of the six quadratic terms in the parametrization (2.2) above, and is the image of the unit sphere in $\mathbb{R}^3$ under the map $(x, y, z) \mapsto (xy, xz, yz)$. This specific real surface is often called the *Steiner surface* after the Swiss mathematician Jacob Steiner who was the first to describe it, but also goes under the name of the *Roman surface* since Steiner was in Rome when he discovered it. ☆

PROBLEM 2.7 Show that the map v above in (2.2) is not injective, but satisfies $\nu(-u) = \nu(u)$. Show that if $\nu(u) = \nu(u')$, then either $u = u'$ or $u = -u'$. ★

Problems

2.8 Show that an irreducible space is Hausdorff if and only if it is reduced to single point.

2.9 Endow the natural numbers $\mathbb{N}$ with the topology whose closed sets apart from $\mathbb{N}$ itself are the finite sets. Show that $\mathbb{N}$ with this topology is irreducible. What is the dimension?

2.10 Show that any countable subset of $\mathbb{A}^1$ is Zariski-dense.

2.11 Let X be an infinite set and $Z_1, \ldots Z_r \subseteq X$ be proper infinite subsets of X such any two of them intersect in at most a finite set. Let $\mathcal{T}$ be the set of subsets of X that are either finite, the union of some of the Z_i's and a finite set, the empty set or the entire set X. Show that $\mathcal{T}$ is the set of closed sets for a topology on X. When is it irreducible?

2.12 Let $X \subseteq \mathbb{A}^4$ be the union of the four coordinate axes. Determine the ideal $I(X)$ by giving generators. Describe the Zariski topology on X.

2.13 Let $\mathfrak{a}$ be the ideal $\mathfrak{a} = (xz, xw, zy, wy)$ in the polynomial ring $k[x, y, z, w]$. Describe the algebraic set $W = Z(\mathfrak{a})$ in $\mathbb{A}^4$ geometrically, and show that the primary decomposition of $\mathfrak{a}$ is

$$\mathfrak{a} = (x, y) \cap (z, w).$$

2.14 Continuing the previous exercise, let $\mathfrak{b}$ be the ideal $\mathfrak{b} = (w - \alpha y)$ with α a non-zero element in k, and let $X = Z(\mathfrak{b})$. Describe geometrically the intersection $W \cap X$. Show that the image $\mathfrak{c}$ of the ideal $\mathfrak{a} + \mathfrak{b}$ in $k[x, y, z]$ under the map that sends w to αy is given as

$$\mathfrak{c} = (xz, xy, zy, y^2),$$

and determine a primary decomposition of $\mathfrak{c}$. What happens if $\alpha = 0$?

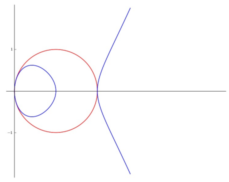


Figure 2.4: The curves in problem 2.16.

2.15 Let two quadratic polynomials f and g in $k[x, y, z, w]$ be given as $f = xz - wy$ and $g = xw - zy$. Describe geometrically the algebraic subset $Z(f, g)$ and find a primary decomposition of the ideal (f, g).

2.16 Let $f = y^2 - x(x-1)(x-2)$ and $g = y^2 + (x-1)^2 - 1$. Show that $Z(f, g) = \{(0, 0), (2, 0)\}$. Determine the primary decomposition of (f, g).

2.17 Let $\mathfrak{a}$ be the ideal $(wy - x^2, wz - xy)$ in $k[x, y, z, w]$. Show that the primary decomposition of $\mathfrak{a}$ is

$$\mathfrak{a} = (w, x) \cap (wz - xy, wy - x^2, y^2 - zx).$$

2.18 Let $\mathfrak{a} = (wz - xy, wy - x^2, y^2 - zx)$. Show that $Z(\mathfrak{a})$ is irreducible and determine $I(X)$.

2.19 Show that any reduced[6] algebra of finite type over k is the coordinate ring of a closed algebraic set.

[6] Reduced means that there are no non-zero nilpotent elements.

2.20 Show that any integral domain finitely generated over k is the coordinate ring of an irreducible closed algebraic set.

2.21 Show that the multiplication map $\mathbb{M}_{n.m} \times \mathbb{M}_{m,k} \to M_{n,k}$ is a polynomial map.

2.22 Let W_r be the subset of $\mathrm{Hom}_k(k^n, k^m) = \mathbb{M}_{n,m} = \mathbb{A}^{nm}$ of maps of rank at most r. Show that W_r is irreducible.

★

Lecture 3

Sheaves and varities

Hot themes in Lecture 3: *Sheaves of rings— regular function—rational functions—affine varieties—general varieties—morphisms—morphisms into affine varieties—the Hausdorff axiom—products of varities.*

Jean Leray (1906–1998)
French mathematician

A central feature of modern geometry is that a space of some geometric type comes equipped with a distinguished set of functions. For instance, topological spaces carry continuous functions and smooth manifolds carry C^{∞}-functions.

There is a common way of *axiomatically* introducing the different types of "functions" on a topological space, namely by the so called *sheaves of rings*.

There are many variants of sheaves involving all kinds of structures other than ring structures. They are omnipresent in modern algebraic geometry—and there is a vast theory about them. However, we confine ourselves to "sheaves of functions" in this introductory course. Our sole reason to introduce sheaves is that we need them to give a uniform and clear definition of varieties, which are after all our main objects of study. So we cut the story to a bare minimum (those pursuing studies of algebraic geometry will certainly have the opportunity to be well acquainted with sheaves of all sorts, and hopefully it will be a help already having seen some sheaves when meeting the full crowd).

Figure 3.1: A sheaf

Sheaves were invented by the french mathematician Jean Leray during his imprisonment as a prisoner of war during WWII. The (original) french name[1] is *faisceau*.

[1] In Norwegian one says "knippe" which is close to the meaning of the French word faisceau. When sheaves were introduced in Norway, a discussion arose among the mathematicians about the terminology, some proposed "feså"!!

3.1 *Sheaves of rings*

The introduction of sheaves is a two step processe. It turns out to be very natural and fruitful to divide the sheaf-axioms into two parts. We begin with defining *presheaves*. This is broader class than the sheaves which must abide to further conditions.

Presheaves

As a preparation to the intrduction on one introduces so-called presheaves.

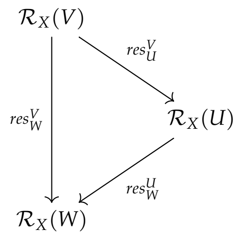
3.1 Let X be a topological space. A *presheaf* of rings has two constituents. Firstly, one associates to any open subset $U \subseteq X$ a ring $\mathcal{R}_X(U)$, and secondly, to any pair $U \subseteq V$ of open subsets a ring homomorphism

Presheaves

$$res^V_U \colon \mathcal{R}_X(V) \to \mathcal{R}_X(U),$$

subjected to the following two conditions:

- $res^V_V = id_{\mathcal{R}_X(V)}$,
- $res^U_W \circ res^V_U = res^V_W$,



sections of a sheaf

where the last condition must be satisfied for any three open sets $W \subseteq U \subseteq V$.

The elements of $\mathcal{R}_X(U)$ are frequently called *sections* of $\mathcal{R}_X$ over U, although they often have particular names in specific contexts. It is also common usage to denote $\mathcal{R}_X(U)$ by $\Gamma(U, \mathcal{R}_X)$ or by $H^0(U, \mathcal{R}_X)$ (indicating that there are mysterious gadgets $H^i(U, \mathcal{R}_{\mathcal{X}})$ around), but we shall stick to $\mathcal{R}_X(U)$. The homomorphisms res^V_U are called *restriction maps*. An alternative notation for the restriction maps is the traditional $f|_U$.

The two conditions above reflect familiar properties of functions (to fix the ideas, think of continuous functions on X). The first reflects the utterly trivial fact that restriction from V to V does not change a function, and the second the fact that restricting from V to W can be done by restricting via any intermediate open set U.

3.2 Students initiated in the vernacular of category theory will recognize a presheaf of rings as a contravariant functor from the category of open subsets of X with inclusions as maps to the category of rings. That absorbed, one easily imagines what a presheaf with values in any given category is; for instance, the commonly met presheaves of abelian groups. Giving such a sheaf, amounts to giving an abelian group $\mathcal{A}(U)$ for every open U of X and restrictions maps $\mathcal{A}(V) \to \mathcal{A}(U)$ that are group homomorphisms, and of course the two axioms must be fulfilled.

Sheaves

3.3 Many classes of functions comply to a principle that can be subsumed in the phrase "Functions of the class are determined locally". There are two aspects of this principle. Given an open U in X and a covering $\{U_i\}$ of U of open sets. Firstly, two functions on U that agree on each U_i, are equal on U. Secondly, a collection of functions, one on each U_i, agreeing on the intersections $U_i \cap U_j$, can be patched together to a give a global function on U. The defining properties of function-classes obeying to this principle must be of a local nature; like being continuous or differentiable. But being bounded, for instance, is not a local property, and the set of bounded functions do not in general form sheaves.

Sheaf of rings

This leads to two new axioms. A presheaf $\mathcal{R}_X$ is called a *sheaf of rings* when the following conditions are fulfilled: For every open $U \subseteq X$ and every covering $\{U_i\}_{i \in I}$ of U by open subsets it holds true that:

- Whenever $f, g \in \mathcal{R}_X(U)$ are two sections satisfying $res^U_{U_i} f = res^U_{U_i} g$ for every $i \in I$, it follows that $f = g$.
- Assume there are given sections $f_i \in \mathcal{R}_X(U_i)$, one for each $i \in I$, satisfying

$$res^{U_i}_{U_j \cap U_i} f_i = res^{U_j}_{U_i \cap U_j} f_j$$

for each pair of indices i, j. Then there exists an $f \in \mathcal{R}_X(U)$ such that $res^U_{U_i} = f_i$.

EXAMPLE 3.1 The simplest examples of sheaves of rings are the sheaves of continuous real or complex valued functions on a topological space X. ☆

The sheaf $\mathcal{C}_X$ of continuous $\mathbb{A}^1$-valued functions.

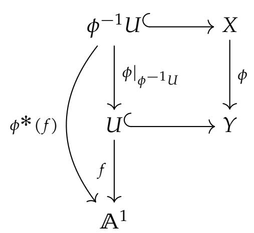
EXAMPLE 3.2 In our algebraic world it is also natural to consider the *sheaf* $\mathcal{C}_X$ of continuous functions with values in $\mathbb{A}^1$ on a topological space X. Where, of course, $\mathbb{A}^1$ is equipped with the Zariski topology, so that a function into $\mathbb{A}^1$ is continuous if and only if the fibres are closed. Since $\mathbb{A}^1 = k$ we can consider $\mathbb{A}^1$ being a field, and ring operations in $\mathcal{C}_X$ can be defined pointwise. So that the space of sections $\mathcal{C}_X(U)$—that this, the set of continuous maps $U \to \mathbb{A}^1$—is a k-algebra equipped with pointwise addition and multiplication.

The sheaf $\mathcal{C}_X$ is an ancilliary introduced to make the development of the theory; the functions that really interests us are the polynomial functions. ☆

3.4 Assume that $\phi\colon X \to Y$ is a continuous map between two topological spaces. The two spaces carry their sheaves of continuous functions into $\mathbb{A}^1$, respectively $\mathcal{C}_X$ and $\mathcal{C}_Y$. Composition with ϕ gives us some kind of "map" ϕ^* between the two sheaves $\mathcal{C}_Y$ and $\mathcal{C}_X$; or rather a well organized collection of maps, one would called it.

In precise terms, for any open $U \subseteq Y$ and any $f \in \mathcal{C}_Y(U)$ one forms the composition $\phi^*(f) = f \circ \phi|_{\phi^{-1}U}$, which is a section in $\mathcal{C}_X(\phi^{-1}U)$. Clearly ϕ^* is a k-algebra homomorphism (operations are defined pointwise), and it is compatible with the restriction maps: For any open $V \subseteq U$ it holds true that

$$\phi^*(f)|_{\phi^{-1}V} = \phi^*(f|_V).$$



In particular, one has a k-algebra homomorphism $\phi^*\colon \mathcal{C}_Y(Y) \to \mathcal{C}_X(X)$ between the k-algebras of global continuous functions to $\mathbb{A}^1$.

This "upper star operation" is *functorial* in the sense that if ϕ and ψ are composable continuous maps, it holds true that $(\psi \circ \phi)^* = \phi^* \circ \psi^*$. Notice the change of order of the two involved maps—the "upper star operation" is *contravariant*, as one says.

3.5 Given a topological space X with a sheaf of rings $\mathcal{R}$ on it. It should be intuitively clear what is meant by a *subsheaf or rings* $\mathcal{R}' \subseteq \mathcal{R}$; namely, for every open subset U of X one is given a subring $\mathcal{R}'(U) \subseteq \mathcal{R}(U)$ that satisfies two conditions. First of all, the different subrings $\mathcal{R}'(U)$ must be *compatible with the restrictions*; that is, for every pair of open subsets $U \subseteq V$, the restriction maps res^V_U takes $\mathcal{R}'(V)$ into $\mathcal{R}'(U)$. This makes $\mathcal{R}'$ a presheaf, and the second condition requires $\mathcal{R}'$ to be a *sheaf*. The first sheaf axiom for $\mathcal{R}'$ is inherited from $\mathcal{R}$, but the second imposes a genuine condition on $\mathcal{R}'$. Patching data in $\mathcal{R}'$ gives rise to a section in $\mathcal{R}$, and for $\mathcal{R}'$ to be a sheaf, the resulting section must lie in $\mathcal{R}'$.

Subsheaves or rings

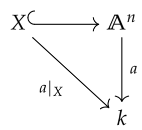
3.6 At a certain point we shall be interested in subsheaves of rings of the sheaves $\mathcal{C}_X$ of continuous $\mathbb{A}^1$-functions on topological spaces and isomorphisms between such. So let X and Y be topological spaces with a *homeomorphism* $\phi\colon X \to Y$ given. Then the composition map ϕ^* maps $\mathcal{C}_Y$ isomorphically into $\mathcal{C}_X$. If $\mathcal{R}_Y$ and $\mathcal{R}_X$ are subsheaves of rings of respectively $\mathcal{C}_X$ and $\mathcal{C}_Y$, there is a very natural criterion for when ϕ^* induces and isomorphism between $\mathcal{R}_X$ and $\mathcal{R}_Y$—if it is true locally, it holds globally:

Lemma 3.7 *If there is a basis $\{U_i\}_{i\in I}$ for the topology of X such that ϕ^* takes $\mathcal{R}_X(U_i)$ into $\mathcal{R}_Y(\phi^{-1}U_i)$, then ϕ^* maps $\mathcal{R}_X$ isomorphically into $\mathcal{R}_Y$.*

Proof: It suffices to see that ϕ^* maps $\mathcal{R}_X(U)$ into $\mathcal{R}_Y(\phi^{-1}U)$ for every open subset $U \subseteq X$.

So, let $U \subseteq X$ be open, and take a section s in $\mathcal{R}_X(U) \subseteq \mathcal{C}_X(U)$. It is sent to a section $\phi^*(s) \in C_Y(\phi^{-1}U)$. Now, there is subset J of the index set I so that $\{U_j\}_{j\in J}$ is an open covering of U, and by assumption, $\phi^*(s|_{U_J})$ lies in $\mathcal{R}_Y(\phi^{-1}U_j)$. Since $\phi^*(s|_{U_J}) = \phi^*(s)|_{\phi^{-1}U_j}$, these sections coincide on the intersections $\phi^{-1}U_j \cap \phi^{-1}U_{j'}$, and consequently they patch together to a section in $\mathcal{R}_\mathcal{Y}(\phi^{-1}U)$. ❑

3.2 *Functions on irreducible algebraic sets*



In this section we shall work with an irreducible closed algebraic set $X \subseteq \mathbb{A}^n$. It has a coordinate ring $A(X) = k[x_1, \ldots, x_n]/I(X)$, which is an integral domain. The coordinate ring can easily be identified with the ring of *polynomial functions* on X which we met in Lecture 2 (Paragraph 2.24 on page 31); that is, the functions on X that are restrictions of polynomials in $k[x_1, \ldots, x_n]$.

3.8 For points $p \in X$ we denote by $\mathfrak{m}_p$ the ideal in $A(X)$ of polynomial functions vanishing at p. It is a maximal ideal, and the Nullstellensatz tells us it is generated by the elements $x_i - p_i$ for $1 \leqslant i \leqslant m$ where the p_i's are the coordinates of the point p.

Rational and regular functions on irreducible algebraic sets

3.9 We shall denote the fraction field of $A(X)$ by $K(X)$. It is called the *rational function field*, or for short *the function field* of X, and the elements of $K(X)$ are called *rational functions*. The name stems from the case of the affine line $\mathbb{A}^1$ whose coordinate ring is the polynomial ring $k[x]$, and whose function field therefore equals $k(x)$; the field of rational functions in one variable—a class of functions familiar from earlier days. Similarly, the function field of $\mathbb{A}^n$ is the field $k(x_1, \ldots, x_n)$ of rational functions in n variables whose elements are quotients of polynomials in the x_i's.

The field of rational functions (kroppen av rasjonale funkjsoner)

Rational functions (rasjonale funksjoner)

3.10 Properly speaking rational functions are not functions on X; they are only defined on open subsets of X. However, a statement as that requires a precise definition of what is meant by a function being defined at a point[2]. So let $p \in X$ be a point. One says that a rational function $f \in K(X)$ is *defined* at x, or is *regular* at p, if f can be represented as a fraction $f = a/b$ of two elements in $A(X)$ where the denominator b does not vanish at p; that is, it holds true that $b \notin \mathfrak{m}_p$. The subring of $K(X)$ consisting of functions regular at p is just the localization $A(X)_{\mathfrak{m}_p}$ of $A(X)$ at the maximal ideal $\mathfrak{m}_p$. This ring is commonly denoted by $\mathcal{O}_{X,p}$ and called *the local ring* at p.

[2] Remember those endless problems in calculus courses with L'Hôpital's rule?

Regular functions (regulære funksjoner)

The local ring at a point (den lokale ringen i et punkt)

EXAMPLE 3.3 An element $f = a/b$ with $a, b \in A(X)$ is certainly defined at all points in the distinguished open set X_b where the denominator b does not vanish. Be aware, however, that it can be defined on a bigger set; the stupid example being $a = b$. For a less stupid example see Example 3.4 below. ☆

3.11 To a rational function $f \in K(X)$ one associates the ideal $\mathfrak{a}_f$ of *denominators* for f. It is defined by $\mathfrak{a}_f = \{ b \in A(X) \mid bf \in A(X) \}$; so that $\mathfrak{a}_f$ consists of the denominators that appear when expressing f as a fraction in different ways. The role of $\mathfrak{a}_f$ is made clear by the following lemma:

The ideal of denominators (nevneridealer)

LEMMA 3.12 *The maximal open set where the rational function f is defined, is the complement of $Z(\mathfrak{a}_f)$. A function f on X is regular if and only if it belongs to $A(X)$.*

The last statement in the lemma says that the regular functions on X are precisely the polynomial functions.

PROOF: Let $p \in X$ be a point. If $\mathfrak{a}_f \nsubseteq \mathfrak{m}_p$, there is an element $b \in \mathfrak{a}_f$ not vanishing at p with $f = a/b$ for some a, hence f is regular at p. If f is regular at p, one can write $f = a/b$ with $b \notin \mathfrak{m}_p$, hence $\mathfrak{a}_f \nsubseteq \mathfrak{m}_p$. For the second statement, the Nullstellensatz tells us that $Z(\mathfrak{a}_f)$ is empty if and only if $1 \in \mathfrak{a}_f$, which is equivalent to f lying in $A(X)$. ❑

LEMMA 3.13 *Let $b \in A(X)$. The regular functions on X_b equals $A(X)_b$.*

The ring $A(X)_b$ is the localization of $A(X)$ in the element b; *i.e.* in the multiplicative system $S = \{ b^i \mid i \in \mathbb{N} \}$.

PROOF: Clearly functions of the form a/b^r are regular on the distinguished open set X_b where b does not vanish. For the other way around: Assume that f is regular on X_b. Because $X_b^c = Z(b)$, this means that $Z(\mathfrak{a}_f) \subseteq Z(b)$. We infer by the Nullstellensatz that $b \in \sqrt{\mathfrak{a}_f}$, so $f = ab^r$ for some r and some $a \in A(X)$, which is precisely to say that $f \in A(X)_b$. ❑

EXAMPLE 3.4 Consider the algebraic set $X = Z(xw - yz)$ in $\mathbb{A}^4$. It is irreducible, and one has the equality $f = x/y = z/w$ in the fraction field $K(X)$. The rational function f is defined on the open set $X_y \cup X_w$ which is strictly larger than both X_y and X_w, and the maximal open set where f is defined, is not a distinguished open set.

Indeed, assume it were, say it was equal to X_b for some $b \in A(X)$ (then there would be an inclusion $X_y \cup X_w \subseteq X_b$). By lemma 3.12 above, it would follow that $\mathfrak{a}_f = (b)$ and hence $(y, w) \subseteq (b)$. Now letting $A = k[x, y, z, w]/(xw - yz)$, one would have $A/(y, w) = k[x, z]$. Hence the prime ideal (z, w) would be of height two, contradicting Krull's Hauptidealsatz[3] which says that the principal ideal (b) is of height at most one. ☆

[3] We haven't spoken about Krull's Hauptidealsatz yet, but we'll do in due course

EXAMPLE 3.5 The coordinate ring $A(X)$ from the previous example is not a UFD—in fact, it is in some sense the arche-type of a k-algebra that is not a UFD—and this is the reason behind f not being defined on a distinguished open subset. One has

PROPOSITION 3.14 *Let $X \subseteq \mathbb{A}^n$ be a irreducible closed algebraic set, and assume that the coordinate ring is a* UFD. *Then the maximal open subset where a rational function f is defined, is of the form X_b.*

PROOF: Let $f \in K(X)$ be a rational function and assume let $b', b \in \mathfrak{a}_f$ be two elements. That is, it holds true that $f = a/b = a'/b'$ so that $ab' = a'b$, and we may well cancel common factors and assume that a and b (respectively a' and b') are without common factors (remember, $A(X)$ is a UFD). Now, we can write $b = cg$ and $b' = c'g$ with c and c' without common factors. It follows that $ac' = a'c$ and hence c is a factor in a and c' one in a'. We infer that c and c' are units, and b and b' are both equal to g up to a unit. Hence $\mathfrak{a}_f = (g)$. ❑

☆

EXAMPLE 3.6 When $n \geqslant 2$, any regular function on $\mathbb{A}^n \backslash \{0\}$ extends to $\mathbb{A}^n$ and is thus a polynomial function. Indeed, the coordinate ring of $\mathbb{A}^n$ is the polynomial algebra $k[x_1, \ldots, x_n]$ which is UFD, hence the maximal set where a regular function is defined is of the form $\mathbb{A}^n_f$, but when $n \geqslant 2$, $\mathbb{A}^n \backslash \{0\}$ is not of this form (the ideal $(x_1, \ldots, x_n)$ is not a principal ideal). ☆

The sheaf of regular functions on affine varieties

The sheaf $\mathcal{O}_X$ of regular functions

Time has come to define *the sheaf $\mathcal{O}_X$ of regular functions* on X—remember, the assumption that X is an irreducible closed algebraic set in $\mathbb{A}^n$ is still in force. The ring $\mathcal{O}_X(U)$ associated to an open subset $U \subseteq X$ is simply defined by

$$\mathcal{O}_X(U) = \{ f \in K(X) \mid f \text{ is regular on } U \} = \bigcap_{x \in U} \mathcal{O}_{X,x},$$

and the restriction maps are just, well, the usual restrictions. The all the rings $\mathcal{O}_X(U)$ are subrings of the function field $K(X)$, and when $U \subseteq V$ are two open subsets, the restriction from V to U is just the inclusion $\bigcap_{x\in V} \mathcal{O}_{X,x} \subseteq \bigcap_{x\in U} \mathcal{O}_{X,x}$. This gives us a *presheaf* of rings—the two presheaf axioms are trivially verified— and shortly it will turn out to be a sheaf.

When working with the sheaf $\mathcal{O}_X$, one should have in mind that all sections of $\mathcal{O}_X$ are elements of $K(X)$ and all restriction maps are identities. For a given open subset U, the presheaf merely picks out which rational functions in $K(X)$ are regular in U. This simplifies matters considerably and makes the following proposition almost trivial:

PROPOSITION 3.15 *Let X be an irreducible closed algebraic set. The presheaf $\mathcal{O}_X$ is a sheaf.*

PROOF: There are two axioms to verify. The first one is trivial: If $U \subseteq V$ are two opens, the restriction map, just being the inclusion $\bigcap_{x\in V} \mathcal{O}_{X,x} \subseteq \bigcap_{x\in U} \mathcal{O}_{X,x}$, is injective.

As to the second requirement, assume first that f and g are regular on opens subsets U and V respectively, and that $f|_{U\cap V} = g|_{U\cap V}$. Then $f = g$ as elements in $K(X)$. Next, let $\{U_i\}$ be a covering of U and assume given sections f_i of $\mathcal{O}_X$ over U_i coinciding on the pairwise intersections. Since all the intersections $U_j \cap U_i$ are non-empty, the f_i's all correspond to the same element $f \in K(X)$, and since the U_i's cover U, that element is regular in U. ❑

3.16 Notice that Lemma 3.12 on page 41 when interpreted in the context of sheaves, says that the global sections of the structure sheaf $\mathcal{O}_X$ is the coordinate ring $A(X)$; in other words, one has $\mathcal{O}_X(X) = A(X)$. In particular, when $X = \mathbb{A}^m$, one has $\mathcal{O}_{\mathbb{A}^m}(\mathbb{A}^m) = k[x_1, \ldots, x_n]$.

3.3 *The definition of a variety*

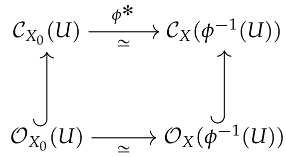
In this section we introduce the main objects of study in this course, namely the varieties. We begin by telling what an affine variety is, and subsequently the affine varieties will serve as building blocks for general varieties. The general definition may appear rather theoretical, but soon, when we come to projective varities, there will be many examples illustrating its necessity and how it functions in practice.

As alluded to in the introduction, varieties will be topological spaces endowed with sheaves of rings of regular functions.

Affine varities

The definition of an *affine variety* which we are about to give, can appear unnecessarily complicated. Of course, the model affine variety is an irreducible closed algebraic set X endowed with the sheaf $\mathcal{O}_X$ of regular functions, but the theory requires a slightly wider and more technical definition. We must accept gadgets that in a certain sense are "isomorphic" to one of the models. An *affine variety* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ a subsheaf of rings of the sheaf $\mathcal{C}_X$ of continuous functions on X with values in $\mathbb{A}^1$. The pair is subjected to the following condition: There is an irreducible algebraic set X_0 and a homeomorphism $\phi\colon X \to X_0$, so that the map $\phi^*\colon \mathcal{C}_{X_0} \to \mathcal{C}_X$ induces an isomorphism between $\mathcal{O}_X$ and $\mathcal{O}_{X_0}$. This means that for all open subsets $U \subseteq X_0$ the map ϕ^* takes $\mathcal{O}_{X_0}(U)$ isomorphically into $\mathcal{O}_X(\phi^{-1}(U))$.

Affine varities



3.17 The distinguished open sets X_f of an algebraic set X we met during the second Lecture illustrate well the reason for this somehow cumbersome definition of an affine variety. *Per se*—as open subsets of X—they are not closed algebraic sets, but endowed with the restriction $\mathcal{O}_X|_{X_f}$ of the sheaf of regular functions as sheaf of rings, they turn out to be affine varieties:

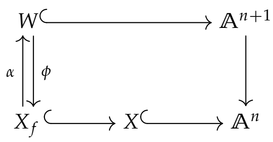
Proposition 3.18 *Let X be an irreducible closed algebraic set and let $f \in A(X)$. Then the pair $(X_f, \mathcal{O}_X|_{X_f})$ is an affine variety.*

Proof: We need to exhibit a closed algebraic set W and a homeomorphism $\phi\colon W \to X_f$ inducing an isomorphism between the sheaf of rings.

To this end, assume that $X = Z(\mathfrak{a})$ for an ideal $\mathfrak{a} \subseteq k[x_1, \ldots, x_n]$. The requested W will be the closed algebraic subset $W \subseteq \mathbb{A}^n \times \mathbb{A}^1 = \mathbb{A}^{n+1}$ that is the zero locus of the following ideal[4]:

$$\mathfrak{b} = \mathfrak{a}k[x_1, \ldots, x_{n+1}] + (1 - f \cdot x_{n+1}).$$

The subset W is contained in inverse image $X \times \mathbb{A}^1$ of X under the projection onto $\mathbb{A}^n$, and consists of those points there where $x_{n+1} = 1/f(x_1, \ldots, x_n)$. We let ϕ denote the restriction of the projection to W; it is bijective onto X_f with the the map α sending $(x_1, \ldots, x_n)$ to $(x_1, \ldots, x_n, 1/f(x_1, \ldots, x_n))$ as inverse.



The salient point is that ϕ^* and α^* are mutually inverse homomorphism between $A(W)$ and $A(X_f)$. As α and ϕ are mutually inverse, the only ting to verify is that $A(W)$ and $A(X_f)$ are mapped into each other.

A regular function g on W is a polynomial in the coordinates $x_1, \ldots, x_{n+1}$, and substituting $1/f(x_1, \ldots, x_n)$ for x_{n+1}, gives a regular function on X_f since $A(X_f) = A(X)_f$ (this is lemma 3.13 on page 41). So α^* takes $A(W)$ into $A(X_f)$.

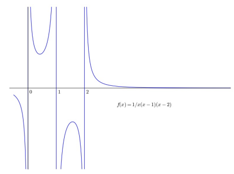


Figure 3.2: The set W is the graph of the function $1/f$ with $f(x) = x(x-1)(x-2)$.

Similarly, if g is regular on X_f, it is expressible in the form a/f^r where a is a polynomial in $x_1, \ldots, x_n$, and therefore $g \circ \phi$ is regular on W; indeed, it holds true that

$$a(\phi(x_1, \ldots, x_{n+1}))/f(\phi(x_1, \ldots, x_{n+1}))^r = a(x_1, \ldots, x_n)x_{n+1}^r.$$

To finish the proof, we have to show that ϕ^* takes the sheaf of rings $\mathcal{O}_{X_f}$ into the sheaf of regular functions $\mathcal{O}_W$, but because of lemma 3.7 on page 40, it suffices to show that for any distinguished open set $X_g \subseteq X_f$ and any regular function h on X_g, the composite $g \circ \phi$ is regular on $\phi^{-1}X_g = W_{\phi^* g}$;

[4] We already came across this ideal when performing the Rabinowitsch trick, but contrary to then, in the present situation f does not belong to $\mathfrak{a}$

but this is now obvious since ϕ^* is an isomorphism and $A(X_g) = A(X_f)_g$ and $A(X_{\phi^* g}) = A(W)_{\phi^*(g)}$.

❑

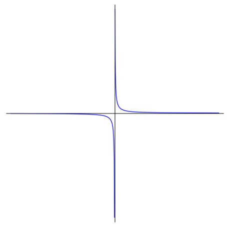


Figure 3.3: Projections onto the x-axis, makes the hyperbola $xy = 1$ is isomorphic to $\mathbb{A}^1 \backslash \{0\}$.

The set W in the proof is nothing but the *graph* of the function $1/f$ embedded in $\mathbb{A}^{n+1}$. Two simple examples of the situation are depicted in the margin (Figure 3.2 and 3.3) in both cases $X = \mathbb{A}^1$. In the first figure the function f is given as $f(x) = x(x-1)(x-2)$, and in the second f is the coordinate x. In the latter case $X_f = \mathbb{A}^1 \backslash \{0\}$, and W is the hyperbola $xy = 1$.

General prevarieties

To begin with we define a version of geometric gadgets called *prevarieties*, which at least for us is provisional. One of the axioms that varieties must fulfil—the so called Hausdorff axiom—is momentary lacking since its formulation requires the concept of a morphism. Therefore the outline is first to introduce *prevarieties*, then *morphisms* between such and finally define what is meant by a *variety*.

Prevarieties

Prevarieties are defined as follows: A *prevariety* is a topological space X endowed with a subsheaf of rings $\mathcal{O}_X$ of the sheaf $\mathcal{C}_X$ such that

- X is an irreducible topological space;
- There is an open covering $\{X_i\}$ of X such that each $(X_i, \mathcal{O}_X|_{X_i})$ is an affine variety.

The structure sheaf

Regular functions

The sheaf $\mathcal{O}_X$ is called *the structure sheaf* of X, and the sections of $\mathcal{O}_X$ over an open subset U are called *regular functions* on U.

The first axiom can be weakened to requiring that X be *connected*, since connectedness in the presence of the second axiom implies that X irreducible.

Problem 3.1 Show that a connected space having an open covering of irreducible open sets is irreducible. ★

Example 3.7 Assume $U \subseteq X$ is an open subset of a prevariety X. We may endow U with the restriction of the structure sheaf $\mathcal{O}_X$ to U; that is, we put $\mathcal{O}_U = \mathcal{O}_X|_U$. Then $(U, \mathcal{O}_U)$ will be a prevariety. In fact, this follows from the slightly more general statement:

Proposition 3.19 *A prevariety X has a basis for the topology consisting of open affine subsets.*

Proof: Let $\{X_i\}$ be an open affine covering of X as in the second axiom. If $U \subseteq X$ is an open subsets of X, the sets $U_i = U \cap X_i$ form an open covering of U. The U_i's will not necessarily be affine, but we know that the distinguished open sets in X_i form a basis for its topology, and by proposition 3.18 on page 44 above they are affine varieties. Hence we can cover each of the U_i's, and thereby U, by affine opens. ❑

☆

3.4 *Morphisms between prevarieties*

On the fly, we also define what is meant by a *morphism* between prevarieties (the usage will soon degenerate into the more practical term "maps of prevarieties"). Morphisms are always maps that conserve structures; in our present context this means they are continuous maps that conserve the sheaves of regular functions.

Morphisms of varieties

Assume that X and Y are two prevarieties. A continuous map from $\phi\colon X \to Y$ is called a *morphism* if for all open subsets $U \subseteq Y$ and all regular functions f on U, the function $f \circ \phi|_{\phi^{-1}(U)}$ is regular on $\phi^{-1}(U)$. This is equivalent to requiring that the map ϕ^* from $\mathcal{C}_Y$ to $\mathcal{C}_X$ sends the structure sheaf $\mathcal{O}_Y$ of Y into the structure sheaf $\mathcal{O}_X$ of X.

Morphisms

If X and Y are varieties, an *isomorphism* from X to Y is a morphism $\phi\colon X \to Y$ which has an inverse morphism; that is, there is a morphism $\psi\colon Y \to X$ such that $\psi \circ \phi = \mathrm{id}_X$ and $\phi \circ \psi = \mathrm{id}_Y$.

Isomorphisms

Being a morphism is a *local property* of a continuous map $\phi\colon X \to Y$ between two prevarieties; that is, one can check it being a morphism on appropriate open coverings. One has:

Lemma 3.20 *Let X and Y be two prevarieties and let $\phi\colon X \to Y$ be a continuous map between them. Suppose one can find open coverings $\{U_i\}_{i \in I}$ and $\{V_i\}_{i \in I}$ of respectively X and Y such that ϕ maps U_i into V_i's, and such that $\phi|_{U_i}$ is a morphism between U_i and V_i, then ϕ is a morphism.*

Proof: Let f be a regular function on some open $V \subseteq Y$ and let $U \subseteq X$ be an open subset with $\phi(U) \subseteq V$. To see that $f \circ \phi|_U$ is regular in U, it suffices by the patching property of sheaves, to show that its restriction to each $U_i \cap U$ is regular. But $U_i \cap U$ maps into V_i, and by hypothesis $f \circ \phi|_{U_i}$ is regular, and because restrictions of regular functions are regular, it follows that $f \circ \phi|_{U_i \cap U}$ is regular. ❑

Problem 3.2 Show that the composition of two composable morphisms is a morphism. Show that morphisms to $\mathbb{A}^1$ are just regular functions. ★

Maps into affine space

3.21 Given a prevariety X and a set $f_1, \ldots, f_m$ of regular functions on X. Letting the f_i's serve as component functions one builds a mapping $\phi\colon X \to$

$\mathbb{A}^m$ by putting $\phi(x) = (f_1(x), \ldots, f_m(x))$. It is obviously continuous, and as would be expected, ϕ is a *morphism*.

Indeed, since being a morphism is a local property (Lemma3.20 above), it suffices to check the defining property on the distinguished open subsets of $\mathbb{A}^m$. So let $U = \mathbb{A}^m_b$ be one. Regular functions on U are (Lemma 3.13 on page 41) of the form $g = a/b^r$ where a is a polynomial and r a non-negative integer. For points x in the inverse image $\phi^{-1}(U)$, it holds true that $b(f_1(x), \ldots, f_m(x)) \neq 0$, hence

$$g \circ \phi(x) = a(f_1(x), \ldots, f_m(x))/b(f_1(x), \ldots, f_m(x))^r$$

is regular in $\phi^{-1}(U)$.

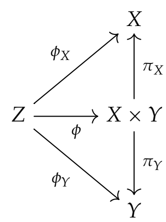
On the other hand, if $\phi\colon X \to \mathbb{A}^m$ is a morphism, the component functions f_i of ϕ being the compositions $f_i = x_i \circ \phi$ of the morphism ϕ with the coordinate functions, are morphisms. Hence we have proven the following proposition whose content is the quite natural property that morphisms from a prevariety X to the affine m-space $\mathbb{A}^m$ are determined by giving regular component functions:

Proposition 3.22 *Assume that X is a prevariety. Sending ϕ to ϕ^* sets up a one-to-one correspondence between morphisms $\phi\colon X \to \mathbb{A}^m$ and k-algebra homomorphisms $\phi^*\colon k[x_1, \ldots, x_m] \to \mathcal{O}_X(X)$.*

3.23 This property may be interpreted in yet another way. There is standard way of defining what is meant by a product in a general category, model on the property of the Cartesian product of two sets that maps into it are given by the two component maps. In our context, it goes as follows: If X and Y are two prevarieties, a *product* of X and Y is *triple* consisting of a prevariety $X \times Y$ and two morphisms π_X and π_Y from $X \times Y$ to respectively X and Y, with the property that given two morphism $\phi_X\colon Z \to X$ and $\phi_Y\colon Z \to Y$, there is a unique morphism $\phi\colon Z \to X \times Y$ such that $\phi_X = \pi_X \circ \phi$ and $\phi_Y = \pi_Y \circ \phi$. In other words, giving ϕ is the same as giving the component morphism.

The product of two prevarieties (produktet av prevariteter)

What we checked above is, when $m = 2$, equivalent to saying that $\mathbb{A}^2$ is the product of $\mathbb{A}^1$ in the category of prevarieties. The definition of a product of two prevarieties generalizes *mutatis mutandis* to a product of a finite number of prevarieties; and what we proved above is that $\mathbb{A}^m$ is the m-fold product of $\mathbb{A}^1$.



3.24 The Proposition 3.22 has an immediate generalization. We may replace the affine n-space with any affine variety, and then we obtain the following :

Theorem 3.25 (Morphisms into affine varieties) *Assume that X is a prevariety and Y an affine variety. The the assignment $\phi \mapsto \phi^*$ sets up a one-to-one correspondence between morphisms $\phi\colon X \to Y$ and k-algebra homomorphisms $\phi^*\colon A(Y) \to \mathcal{O}_X(X)$.*

PROOF: Suppose that $Y \subseteq \mathbb{A}^n$. Giving a morphism $\phi\colon X \to Y$ is the same as giving a morphism $\phi\colon X \to \mathbb{A}^n$ that factors through Y. By proposition 3.22 above, giving a $\phi\colon X \to \mathbb{A}^n$ is the same as giving the algebra homomorphisms $\phi^*\colon k[x_1,\ldots,x_n] \to \mathcal{O}_X(X)$, and ϕ takes values in Y if and only if $f(\phi(x)) =$ for all $f \in I(Y)$; that is, the composition map ϕ^* vanishes on the ideal $I(Y)$. Hence ϕ takes values in Y if and only if ϕ^* factors through the quotient $A(Y) = k[x_1,\ldots,x_n]/I(X)$. ❑

3.26 Specializing the prevariety X to be affine as well, we get the following corollary, the main theorem for morphism of affine varieties:

THEOREM 3.27 (MORPHISM OF AFFINE VARIETIES) *Assume that X and Y are two affine varieties. Then $\phi \mapsto \phi^*$ is a one-to-one correspondence between morphisms $\phi\colon X \to Y$ and k-algebra homomorphisms $\phi^*\colon A(Y) \to A(X)$.*

An immediate corollary is the following:

THEOREM 3.28 *Let X and Y be two affine varieties and $\phi\colon X \to Y$ a morphism. Then ϕ is an isomorphism if and only if ϕ^* is an isomorphism. In particular, X and Y are isomorphic if and only if $A(X)$ and $A(Y)$ are isomorphic as k-algebras.*

EXAMPLE 3.8 The variety $\mathbb{A}^n \backslash \{0\}$ is not affine if $n \geqslant 2$. Indeed, by Example 3.6 on page 42 the inclusion $\iota\colon \mathbb{A}^n \backslash \{0\} \to \mathbb{A}^n$ induces an isomorphism i^* between the spaces of global sections of the two structure sheaves. If $\mathbb{A}^n \backslash \{0\}$ were affine, the inclusion would therefore have been an isomorphism after Theorem 3.28 above, but this is of course not the case. ☆

3.29 In a categorical language Proposition 3.25 above says in view of exercise 2.20 on page 36 in Notes 2, that the category of affine varieties is equivalent to the category of finitely generated k-algebras that are integral domains. So the study of the affine varieties is in some sense equivalent to the study of k-algebras; but luckily, the world is more versatile than that! There is the vast host of projective varieties—beautiful, challenging and intricate and sometimes even untouchable!

3.5 *The Hausdorff axiom*

The Hausdorff axiom is the third axiom required of varieties. Zariski topologies are, as we have seen, far from being Hausdorff, but some properties[5] of Hausdorff spaces can be salvaged by this third axiom, so in some sense it is a substitute for the topologies being Hausdorff.

[5] That is, properties expressed in terms of morphisms not in terms continuous maps.

Varieties (*varieteter*)

3.30 A prevariety $(X, \mathcal{O}_X)$ is called a *variety* if the following condition is fulfilled

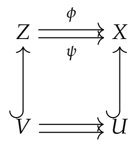
- ❑ Given any two morphisms $\phi, \psi\colon U \to X$ where U is a prevariety, the set of points in U where ϕ and ψ coincide is closed; that is, the subset $\{\, x \in U \mid \phi(x) = \psi(x) \,\}$ is closed.

Of course, one may as well require the set of points where ϕ and ψ assumes distinct values to be open.

3.31 The first we check is that affine varieties deserve having the term varietry in their name:

Proposition 3.32 *Any affine variety is a variety.*

Proof: To begin with, observe that if f and g are two regular functions on a prevariety U, the set where they coincide is closed. Indeed, the diagonal in $\mathbb{A}^1 \times \mathbb{A}^1 = \mathbb{A}^2$ being the zero locus of $x_1 - x_2$ is closed and the map $U \to \mathbb{A}^2$ given as $x \mapsto (f(x), g(x))$ is continuos. Since preimages of closed sets by continuous maps are closed, it follows that $\{\, x \mid f(x) = g(x) \,\}$ is closed.



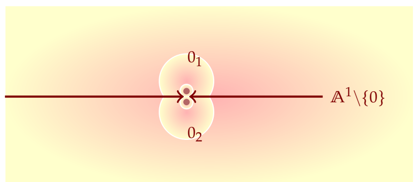
Now, let $X \subseteq \mathbb{A}^m$ be affine, and assume that ϕ and ψ map U into X. If the coordinate functions on $\mathbb{A}^n$ are $y_1, \ldots, y_m$, the compositions $y_i \circ \psi$ and $y_{i\circ}\phi$ are regular functions on U. The set where ϕ and ψ coincide is the intersection of the subsets where each pair $y_i \circ \psi$ and $y_i \circ \phi$ coincide. By the initial observations each of these subsets is closed, hence their intersection is closed. ❑

Lemma 3.33 *Assume that X is a prevariety such any two different points are contained in an open affine subset. Then X is a variety.*

Proof: Let Z be a prevariety and ϕ and ψ two maps from Z to X. Let $x \in Z$ be a point such that $\phi(x) \neq \psi(x)$. Then by assumption there is an open affine set U in X containing both $\phi(x)$ and $\psi(x)$, and $V = \phi^{-1}U \cap \psi^{-1}U$ is an open set in Z where x lie. Now U is a variety by proposition 3.32 above, hence the set $W \subseteq V$ where the two maps ψ and ϕ coincide is closed; but this means that $V \backslash W$ is an open set in Z containing x and being entirely contained in the complement of $\{\, z \in Z \mid \phi(z) = \psi(z) \,\}$. It follows that the complement of $\{\, z \in Z \mid \phi(z) = \psi(z) \,\}$ is open since x was an arbitrary member. ❑

Example 3.9 *(A bad guy)* This is an example of a prevariety X for which the Hausdorff axiom is not satisfied. These "*non separated prevarieties*", as they often are called, exist on the fringe of the algebro-geometric world, you very seldom meet them—although now and then they materialize from the darkness and serve a useful purpose. Anyhow, this is the only place such a creature will appear in this course, and the only reason to include it is to convince you that the Hausdorff axiom is needed.

The intuitive way to think about X is as an affine line with "two origins. It does not carry enough functions that the two origins can be separated—if a function is regular in one, it is regular in both and takes the same value there.



The underlying topological space is the set $X = (\mathbb{A}^1 \backslash \{0\}) \cup \{0_1, 0_2\}$ endowed with the topology of finite complements. It has two copies of the affine line $\mathbb{A}^1$ lying within it; either with one of the twin origins as origin; that is $A_1 = (\mathbb{A}^1 \backslash \{0\}) \cup \{0_1\}$ and $A_2 = (\mathbb{A}^1 \backslash \{0\}) \cup \{0_2\}$. Both these sets are open sets and their intersection A is given as $A = A^1 \cap A^2 = \mathbb{A}^1 \backslash \{0\}$. Obviously, the

Hausdorff axiom is not satisfied, because the two inclusions of $\mathbb{A}^1$ in X are equal on $\mathbb{A}^1 \backslash \{0\}$ which is not closed in X.

To tell what regular functions X carries, let $U \subseteq X$ be any open subset. There are two cases:

- The complement $X \backslash U$ contains both the twin origins: Then the ring $\mathcal{O}_X(U)$ of regular functions in U is the set of rational functions $a(x)/b(x)$ in one variable with $b(x) \neq 0$ for $x \in U$—so the sheaf $\mathcal{O}_X|A$ equals $\mathcal{O}_{\mathbb{A}^1}|_A$.

- One or both the twin origins belongs to U: Then $\mathcal{O}_X(U)$ will be the set of $a(x)/b(x)$ of rational functions in one variable such that $b(x) \neq 0$ for $x \in U \cap A$ and additionally $b(0) \neq 0$.

So the point, is that when U is an open subset containing both 0_1 and 0_2 the subsets U, $U \backslash \{0_1\}$ and $U \backslash \{0_2\}$ all carry the *same* regular functions.

We leave it as an exercise to students interested in the dark corners at the fringes of the algebro-geometrical universe to fill in details and check that the axioms for a prevariety are fulfilled. ☆

3.6 *Products of varieties*

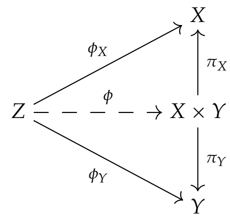
An invaluable tool when working with varieties, is the unrestricted possibility to form the product of two prevarieties. In this section we shall describe the construction of such a product.

The definition of the product is by means of a "universal property" —or phrased in a more pretentious manner, the product with its two projections solves "a universal problem". In this way the definition may be formulated in any category, but wether objects have a product or not depends of course on the category—here are by no means any thing close to general existence theorems. This will be topical if one starts studying *schemes*. They have products, but the underlying set of a product can be very different from the product of the underlying sets of the factors. Prevarieties, however, behave more pleasantly, and their products will be the cartesian products of the underlying sets of the involved prevarieties endowed with a Zariski topology and a sheaf of rings.

3.34 The proof that products exist is in a natural way a three step process. After having defined what we mean by a product, we construct the product of affine varieties and give some of their properties. Based on this, the product will be constructed for general prevarieties, and finally we show that the product of to varieties, which *a priori* is just a prevariety, in fact is a variety.

The universal property of a product

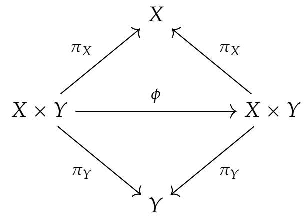
3.35 The product of two prevarieties X and Y is a prevariety $X \times Y$ together with two morphisms $\pi_X\colon X \times Y \to X$ and $\pi_Y\colon X \times Y \to Y$ called *the projections*, and the three must comply to the following condition:



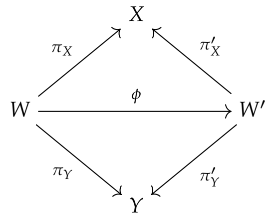
- For any prevariety Z and any pair of morphisms $\phi_X\colon Z \to X$ and $\phi_Y\colon Z \to Y$, there is a morphism $\phi\colon Z \to X \times Y$ such that $\phi_X = \pi_X \circ \phi$ and $\phi_Y = \pi_Y \circ \phi$. Moreover, ϕ is uniquely defined by these conditions.

In a more laid-back language, the product has the constituting property that giving a morphism into it is the same thing as giving the two component morphisms.

As usual with objects defined by universal properties, the product is uniquely defined up to a unique morphism. For the benefit of the students we offer a proof of this, and remark that the argument does not refer to any property specific for varities and is valid in any category.



PROPOSITION 3.36 *The product is unique up to a unique isomorphism.*



PROOF: Observe first that a morphism $\phi\colon X \times X \to X \times Y$ such that $\pi_X \circ \phi = \pi_X$ and $\pi_Y \circ \phi = \pi_Y$ must, by the uniqueness part of the definition, be equal to $\mathrm{id}_{X\times Y}$. So assume that W and W' with projections π_X, π_Y and π'_X, π'_Y are two products of X and Y—*i.e.* they both have the universal property. By the existence part there is a unique morphism $\phi\colon W \to W'$ such that $\pi'_x \circ \phi = \pi_X$ and $\pi'_Y \circ \phi = \pi_Y$ and a unique morphism $\psi\colon W' \to W$ with $\pi_X \circ \psi = \pi'_X$ and $\pi_Y \circ \psi = \pi'_Y$.

From the observation at the top of the proof it ensues that $\psi \circ \phi = \mathrm{id}_{W'}$ and $\phi \circ \psi = \mathrm{id}_W$. Indeed, by symmetry it suffices to check the first. One has $\pi_X \circ \psi \circ \phi = \pi'_X \circ \phi = \pi_X$ and ditto $\pi_Y \circ \psi \circ \phi = \pi_Y$, and we can conclude that $\psi \circ \phi = \mathrm{id}_{W'}$. ❏

3.37 The main theorem we shall prove in this section is formulated as follows, and as already explained, the second part will be establish first. The proof occupies the rest this chapter.

THEOREM 3.38 (EXISTENCE OF PRODUCTS) *Any two prevarieties X and Y has a product $X \times Y$. It is a prevariety whose underlying set is the Cartesian product of X and Y, and together with the two projections π_X and π_Y it satisfies the universal property.*

- *If X and Y are varieties, the product $X \times Y$ is a variety.*
- *When X and Y are affine varieties, the product $X \times Y$ will be affine, and it holds true that the coordinate ring is given as $A(X \times Y) = A(X) \otimes_k A(Y)$.*

The product of affine varieties

We start out by proving that products exist in the category of affine varities.

3.39 Assume that $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$ are two affine varieties and choose coordinates $x_1, \ldots, x_n$ on $\mathbb{A}^n$ and $y_1, \ldots, y_m$ on $\mathbb{A}^m$. The product of X and Y will be constructed as a closed algebraic subset of $\mathbb{A}^{n+m}$, and to this end, we let $\mathfrak{a}$ be the ideal in the polynomial ring $k[x_1, \ldots, x_n, y_1, \ldots, y_m]$ generated by $I(X)$ and $I(Y)$. That is, we put

$$\mathfrak{a} = (f_1(x), \ldots, f_r(x), g_1(y), \ldots, g_s(y)),$$

where the $f_i(x)$'s are generators for $I(X)$ and the $g_j(y)$'s for $I(Y)$. Moreover, we put $W = Z(\mathfrak{a})$. Notice that by general theory of tensor products of algebras, it holds true that

$$k[x_1, \ldots, x_n, y_1, \ldots, y_m]/\mathfrak{a} \simeq A(X) \otimes_k A(Y),$$

which is very near the statement about coordinate rings in the theorem; it remains to be seen that $\mathfrak{a}$ is a radical ideal.

For the moment W is just a closed algebraic subset, but in the final step of the construction, which is the hardest part, it will turn out to be an affine variety; *i.e.* it will be irreducible.

The first, and easy, step of the construction is to show that the subset W of $\mathbb{A}^{n+m}$ satisfies the universal property among closed algebraic sets and polynomial maps.

The two projection p_X and p_Y are induced from the natural linear projections π_n and π_m mapping $\mathbb{A}^{n+m}$ onto $\mathbb{A}^n$ and $\mathbb{A}^m$ respectively. They clearly sends points in W to respectively $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$.

Lemma 3.40 *The subset W has the cartesian product $X \times Y$ as underlying set, and together with the two maps $p_X = \pi_n|_W$ and $p_Y = \pi_m|_W$ satisfies the universal property of a product in the category of closed algebraic sets and polynomial maps.*

Proof: If the point $(x_1, \ldots, x_r, y_1, \ldots, y_s)$ of $\mathbb{A}^{n+m}$ lies in W, it holds true that $f_i(x_1, \ldots, x_r) = 0$ for all i and $g_j(y_1, \ldots, y_s) = 0$ for all j by the definition of the ideal $\mathfrak{a}$. Hence $(x_1, \ldots, x_r) \in X$ and $(y_1, \ldots, y_s) \in Y$, and we can conclude that W coincides with $X \times Y$.

Given two polynomial maps ϕ_X and ϕ_Y from a closed algebraic set Z into respectively X and Y. Since $X \subseteq \mathbb{A}^n$ and $Y \subseteq \mathbb{A}^m$, the two maps take values in $\mathbb{A}^n$ and $\mathbb{A}^m$, and consequently $z \mapsto (\phi_X(z), \phi_Y(z))$ is a polynomial map $Z \to \mathbb{A}^{n+m}$ with values in W. One easily convinces oneself that it solves the universal problem. ❑

The next step is to establish that W is irreducible and is the product in the category of prevarieties. The first lemma, about W being irreducible, holds for a large class of topologies on the product $X \times Y$; the salient hypothesis being that all the sets $\{x\} \times Y$ and $X \times \{y\}$ are closed.

Lemma 3.41 *Assume that X and Y are two irreducible topological spaces. Assume $X \times Y$ is equipped with a topology such that all sets of the form $\{x\} \times Y$ and $X \times \{y\}$ are closed. Then $X \times Y$ is irreducible as well.*

PROOF: Assume the product $X \times Y$ can be expressed as a union $X \times Y = Z_1 \cup Z_2$ of two closed subsets Z_1 and Z_2. Let $X_i = \{ x \in X \mid \{x\} \times Y \subseteq Z_i \}$. It holds true that $X_i = \bigcap_{y \in Y} X \times \{y\} \cap Z_i$, and consequently the X_i's are both closed sets.

For every $x \in X$ the set $\{x\} \times Y$ is contained in either Z_1 or Z_2 since Y is irreducible, and it ensues that $X = X_1 \cup X_2$. Now, X is assumed to irreducible, so either it holds that $X_1 = X$, and therefore that $Z_1 = X \times Y$, or $X_2 = X$, and $Z_2 = X \times Y$. ❏

3.42 We now know that W is irreducible and hence may infer from Lemma 3.40 above that W is the product of X and Y in the category of affine varieties. A straightforward gluing argument applied to maps into W, extends this to the category of all prevarieties, and shows that W, indeed, is the product of X and Y in the category of prevarieties;

LEMMA 3.43 *The set W together with the projections $\pi_X = \pi_n|_W$ and $\pi_Y = \pi_m|_W$ is the product of X and Y in the category of prevarieties.*

PROOF: As already observed, the closed algebraic set W is irreducible and it merely remains to establish the universal property.

Given two morphisms ϕ_X and ϕ_Y from a prevariety Z into respectively X and Y. Cover Z by open affine sets Z_i and consider the restrictions $\phi_X|_{Z_i}$ and $\phi_Y|_{Z_i}$. Since W satisfies the universal property among affine varieties, they give rise to morphisms $\phi_i \colon Z_i \to W$ such that $\pi_X \circ \phi_i = \phi_X|_{Z_i}$ and $\pi_Y \circ \phi_i = \phi_Y|_{Z_i}$.

On the intersections $Z_{ij} = Z_i \cap Z_j$ the morphism ϕ_i and ϕ_j must agree; indeed, both are solutions to the universal problem posed by the morphism $\phi_X|_{Z_{ij}}$ and $\phi_Y|_{Zij}$, and this solution being unique, it holds that $\phi_i|_{Z_{ij}} = \phi_j|_{Z_{ij}}$.

The different ϕ_i's therefore patch together, and we obtain a morphism ϕ with the requested property that $\pi_X \circ \phi = \phi_X$ and $\pi_Y \circ \phi = \phi_Y$. ❏

3.44 The last thing to establish about the affine products is that the coordinate rings are as announced:

LEMMA 3.45 *In the present setting $A(X \times Y) = A(X) \otimes_k A(Y)$.*

PROOF: Any element f in the tensor product $A(X) \otimes_k A(Y)$ can be represented as $f = \sum g_i \otimes h_i$ where $g_i \in A(X)$ and $h_i \in A(Y)$, and we may assume that the h_i's are linearly independent over k.

Assume that f is nilpotent and fix a point x_0 in X. Considered as a function of y, the element $f(x_0, y) \in A(Y)$ will be nilpotent, and hence $f(x_0, y) = \sum_i g_i(x_0) h_i(y) = 0$. Since the h_i's are linearly independent, it follows that $g_i(x_0) = 0$ for all i. Now, the point x_0 was an arbitrary point in X so that $g_i = 0$ as function on X, and we are done. ❏

PROBLEM 3.3 Lemma 3.45 is a special case of the following result from commutative algebra. If A and B are two reduced k-algebras finitely generated over the algebraically closed field k, then $A \otimes_k B$ is reduced. Show this. HINT: Adapt the proof of lemma 3.45. ★

Products of prevarieties

Let X and Y be two prevarieties. We shall work with affine open coverings $\{X_i\}$ and $\{Y_i\}$ of respectively X and Y.

The definition of the product as a prevariety requires the specification of an underlying set, a topology on that set and a sheaf of rings on the that topological space, and this must be constructed in a manner that the resulting space has an open covering by affine varieties.

The underlying set of the product will be nothing but the cartesian product $X \times Y$. To introduce the topology, we observe that $X \times Y$ is the union of the sets $U_{ij} = X_i \times Y_j$, and requiring these to form an open covering, we obtain a topology. Indeed, one declares a subset U to be open when $U \cap U_{ij}$ is open in U_{ij} for each pair of indices i and j.

PROBLEM 3.4 Show that this gives a topology on $X \times Y$. Show that the induced topology on the sets U_{ij} coincides with the original topology, and that the projections onto X and Y are continuous. ★

It remains to define the structure sheaf on $X \times Y$. This is also pretty straightforward. We simply say that a function f which is continuous on an open set U (*i.e.* a section of $\mathcal{C}_{X \times Y}$) is regular at a point $p \in U$ if the restriction $f|_{U_{ij}}$ is regular at p for one (hence for all, see lemma 3.46 below) of the affine subsets U_{ij} that contain p. Next, we let $\mathcal{O}_{X \times Y}$ be the subsheaf of $\mathcal{C}_{X \times Y}$ whose sections $\mathcal{O}_{X \times Y}(U)$ over an open U consists of the functions regular at all points p in U. Since $\mathcal{C}_{X \times Y}$ is a sheaf, and since being regular is local property, one gets for free that $\mathcal{O}_{X \times Y}$ is a sheaf of rings.

The following lemma ensures that $\mathcal{O}_{X \times Y}|_{U_{ij}} = \mathcal{O}_{X_i \times Y_j}$, so that U_{ij} is an affine cover of $X \times Y$.

LEMMA 3.46 *The setting is as above. If $p \in U_{ij} \cap U_{kl}$, then $f|_{U_{ij}}$ is regular at p if and only if $f|_{U_{kl}}$ is regular at p.*

PROOF: Since the functions $f|_{Uij}$ and $f|_{U_{kl}}$ are both continuous, and since being regular is a local property, we can finish the proof by observing that $(X_i \cap X_l) \times (Y_j \cap Y_k)$ is an open neighbourhood of p both in U_{ij} and in U_{kl} on which $f|_{U_{ij}}$ and $f|_{U_{kl}}$ coincide. ❑

The final point in establishing that $X \times Y$ is a product, is to verify that the two projections are morphisms, and that the universal property is satisfied. The underlying continuous map ϕ associated to two maps ϕ_X and ϕ_Y, is the obvious one, namely the one defined by $\phi(z) = (\phi_X(z), \phi_Y(z))$, and it is a matter of simple verifications to check that ϕ so defined is a morphism. And as usual, we leave the work to the zealous students.

PROBLEM 3.5 Show that the projections p_X and p_Y are morphisms as is the map ϕ described in the text above. ★

Consequences

The *diagonal* Δ_X of a space X is the subset $\Delta_X = \{ (x,x) \mid x \in X \}$ of $X \times X$. More generally if $\phi \colon X \rightarrow\rightarrow Y$ is a morphism the *graph* of ϕ is the subset $\Gamma_\phi = \{ (x,y) \mid \phi(x) = y \}$ of the product $X \times Y$. Putting $\phi = \mathrm{id}_X$ we see that $\Delta_X = \Gamma_{\mathrm{id}_X}$.

The diagonal (diagonalen)

The graph of morphisms (grafen til morphismer)

3.47 In topology a space is Hausdorff if and only if the diagonal is closed in the product topology. This hinges on the observation that two open neighbourhoods U and V of points x and y respectively, are disjoint precisely when $U \times V \subseteq X \times Y$ does not meet the diagonal. In algebraic geometry a corresponding statement holds. A prevariety X satisfies the Hausdorff axiom if and only if the diagonal is closed in $X \times X$, but this time in the Zariski topology.

PROPOSITION 3.48 *A prevariety X is a variety if and only if the diagonal $\Delta_X \subseteq X \times X$ is closed.*

PROOF: The diagonal being the equalizer of the two projections, it will be closed when X is a variety. Assume then that Δ_X is closed and let $\phi, \psi \colon Z \to X$ be two morphisms. Their equalizer $\{ z \mid \phi(z) = \phi(z) \}$ is the inverse image of Δ_X by the morphism $Z \to X \times Y$ whose components are ϕ and ψ. Hence it is closed. ❑

As an immediate corollary one has

COROLLARY 3.49 *When X and Y are varieties and $\phi \colon X \to Y$ is a morphism, the graph Γ_ϕ is closed in the product $X \times Y$.*

3.50 The second application illustrates a general principle often referred to as "reduction to the diagonal". In its simplest form—formulated for sets—it is the observation that the intersection $U \cap V$ of two subsets U and V of a set X, is naturally bijective to the intersection $U \times C \cap \Delta_X$. This relation persists when U and V are open subsets of a prevariety X, but of course with the annotation "bijective" replaced with "isomorphic". A consequence is that the intersection of two open affines in a variety is affine:

PROPOSITION 3.51 *Assume that U and V are open affine subsets of the variety X, then the intersection $U \cap V$ is affine*

PROOF: The intersection $U \cap V$ is isomorphic to the intersection $U \times V \cap \Delta_X$, but the product $U \times V$ is affine and since Δ_X is closed, x being a variety, it ensues that $U \times V \cap \Delta_X$ is closed in $U \times V$. Hence affine. ❑

PROBLEM 3.6 Show that $U \cap V$ is isomorphic to $U \times V \cap \Delta_X$ whenever U and V are open subsets of a prevariety X. ★

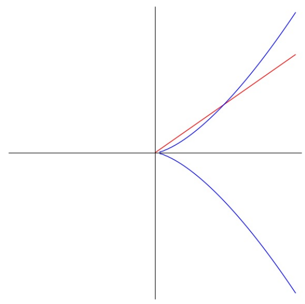
An epilogue

As an epilogue, we remind you that a variety has two ingredients: a topological space X and the structure sheaf $\mathcal{O}_X$. Among the two the structure sheaf is the main player, the Zariski topology having a more supportive role. For instance, if X is an irreducible and Noetherian space whose only closed irreducible sets are the points, the closed sets, apart form the entire space, are precisely the finite subsets. This means that all such spaces are homeomorphic as longs as their cardinality is the same. So for instance, the affine lines $\mathbb{A}^1$ over different countable[6] fields are homeomorphic, and they are even homeomorphic to the bad guys we just constructed.

[6] The algebraic closure of finite fields $\mathbb{F}_q$ and of the rationals $\mathbb{Q}$ are all countable.

Later on, after having introduced the concept of dimension, we shall see that any irreducible space of dimension one fall in this category, so they are all homeomorphic. But there is an extremely rich fauna of such varieties!

In higher dimensions the Zariski topologies play a more decisive role, but still they do not distinguish varieties very well.

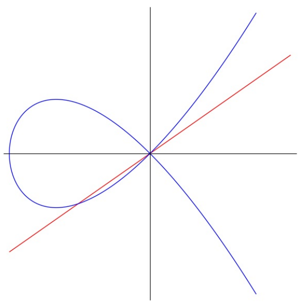


The rational cusp $y^2 = x^3$.

Problems

3.7 *(Rational cusp.)* Consider the curve C in $\mathbb{A}^2$ whose equation is $y^2 - x^3$. Show that C can be parametrized by the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ defined as $\phi(t) = (t^2, t^3)$. Describe the map $\phi^*\colon A(C) \to A(\mathbb{A}^1)$. Show that ϕ is bijective but not an isomorphism. Show that the function field of C equals $k(t)$.

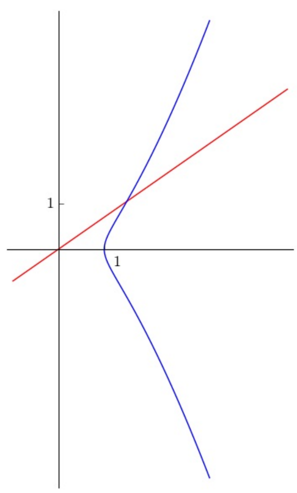
3.8 *(Rational node.)* In this exercise we let C be the curve in $\mathbb{A}^2$ whose equation is $y^2 - x^2(x+1)$. Define a map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ by $\phi(t) = (t^2 - 1, t(t^2 - 1))$. Show that $\phi(\mathbb{A}^1) = C$, and describe the map $\phi^*\colon A(C) \to A(\mathbb{A}^1)$. Show that ϕ is not an isomorphism, but induces an isomorphism $\mathbb{A}^1 \backslash \{\pm 1\} \to C \backslash \{0\}$. Show that the function field of C equals $k(t)$.



The rational node $y^2 = x^2(x+1)$.

3.9 Let C be one of the curves from the two previous exercises. Show that, except for finitely many, every line through the origin intersects C in exactly one other point. What are the exceptional lines in the two cases? Use this to give a geometric interpretation of the parametrizations in the previous exercises.

3.10 *(An acnode.)* Consider the curve D given by $y^2 = x^2(x-1)$ in $\mathbb{A}^2$. Make a sketch of the real points of D (see the figure in the margin); notice that the origin is isolated among the real points—such a point is called an *acnode*. Show that $(t^2 + 1, t(t^2 + 1))$ is a parametrization of D. Exhibit a complex linear change of coordinates in $\mathbb{A}^2$ that brings D on the form in problem 3.8 above.



An acnode at the origin $y^2 = x^2(x+1)$.

3.11 Let R be a UFD. Show that any prime ideal of height one (that is a prime ideal properly containing no other prime ideals than the zero ideal) is principal.

3.12 Let X be a variety and let $Y \subseteq X$ be a closed irreducible subset. For any open $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the subset of regular functions on U that vanish on $Y \cap U$. Show that $\mathcal{I}_Y(U)$ is an *ideal* in $\mathcal{O}_X(U)$. Show that if $V \subseteq U$ are two open sets, then res^U_V takes $\mathcal{I}_Y(U)$ into $\mathcal{I}_Y(V)$. Show that $\mathcal{I}_Y$ is a sheaf (of abelian groups, in fact of rings if one ignores the unit element).

3.13 Let X be a prevariety and assume that $Y \subseteq X$ is a closed irreducible subset. Show that Y can be given the structure of a prevariety in a unique way so that the inclusion $Y \to X$ is a morphism.

3.14 Let $\mathcal{B}$ be the presheaf of bounded continuous real valued functions on $\mathbb{R}$. Show that $\mathcal{B}$ is not a sheaf. HINT: It does not satisfy the second sheaf axiom.

3.15 Let X be a topological space and let $\mathbf{A}$ be a ring equipped with the discrete topology. For any open set $U \subseteq X$ let $\mathbf{A}(U)$ be the set of continuous functions $U \to A$. Show that $\mathbf{A}(U)$ is a sheaf.

3.16 *(For fringy people.)* Let X be any closed algebraic set and let $Y \subseteq X$ be a proper closed subset. Construct a prevariety $X_{\amalg}$ containing unseparable twin copies of Y and two different open subsets both isomorphic to X that intersect along $X \backslash Y$.

3.17 Show that the algebraic closure of a countable field is countable.

3.18 Show that $\mathbb{A}^2$ is not the same as the product topology on $\mathbb{A}^1 \times \mathbb{A}^1$

3.19 Mimic the construction of "the bad guy" with $\mathbb{A}^2$ and the origin to get an "even worse guy" X (cfr. exercise 3.16). Exihibit two affine open subsets of X whose intersection equals $\mathbb{A}^2 \backslash \{0\}$. Conclude that the hypothesis that X be a varity in proposition 3.51 on page 55 can not be skipped.

★

Lecture 4

Projective varieties

Hot themes in Lecture 4: *Projective spaces—homogeneous coordinates closed projective sets—homogenous ideals and closed projective sets—projective Nullstellensatz—distinguished open sets—Zariski topology and regular functions—projective varieties—global regular functions on projective varieties—morphism from quasi projective varieties—linear projections—Veronese embeddings—Segre embeddings.*

Projective geometry arose in the wake of the discovery of the perspective by italian renaissance painters like for instance Filippo Brunelleschi. In a perspective drawing one considers bundles of light rays emanating from or meeting at a point (the observers eye) or meeting at an apparent point at infinity, the so-called vanishing point, when rays are parallel. Figures are perceived the same if one is the projection of the other.

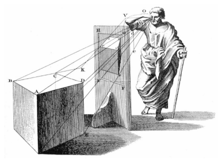


Figure 4.1: The discovery of perspective in art.

In the beginning projective geometry was purely a synthetic geometry (no coordinates, no functions, merely points and lines). The properties of different figures that where studied were the properties invariant under projection from a point. Subsequently, an analytic theory developed and eventually became the basis for the projective geometry as we know it in algebraic geometry today.

The synthetic theory still persists, especially since some finite projective planes are important combinatorial structures[1]. The simplest being the *Fano plane* with seven points and seven lines!

[1] The axiomatics of the synthetic projective plane geometry is exceedingly simple. There are to sets of objects, points and lines, and there are two axioms: Through any pair of points there goes a unique line, and any two lines intersect in a unique point.

The projective spaces and the projective varieties are in some sense the algebro-geometric counterparts to compact spaces, with which they share many nice properties.

4.1 Non-compact spaces are on the other hand typically difficult to handle; if you discard a bunch of points in an arbitrary manner from a compact space (for instance, a sphere) it is not much you can say about the result unless you know the way the discarded points were chosen, and moreover, functions can tend to infinity near the missing points. Compact spaces and projective varieties are in some sense complete, they do not suffer from the deficiencies of these "punctured" spaces—hence their importance and popularity!

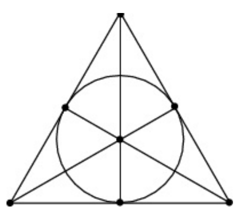


Figure 4.2: The Fano plane; the projective plane over the field with two elements $\mathbb{P}^2(\mathbb{F}_2)$.

4.1 *The projective spaces* $\mathbb{P}^n$

The projective n-spaces $\mathbb{P}^n$ (*de projektive rommene*)

4.2 Let n be a non-negative integer. The underlying set of *the projective n-space* $\mathbb{P}^n$ over k is the set of lines passing through the origin in $\mathbb{A}^{n+1}$; or in other words, the set of one-dimensional vector subspaces. Since any point, apart from the origin, lies on a unique line in $\mathbb{A}^{n+1}$ passing by 0, there is map

$$\pi\colon \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n,$$

sending a point to the line on which it lies. It will be convenient to denote by $[x]$ the line joining x to the origin; that is, $[x] = \pi(x)$.

One may as well consider $\mathbb{P}^n$ as the set of equivalence classes in $\mathbb{A}^{n+1}\backslash\{0\}$ under the equivalence relation for which two points x and y are equivalent when $y = tx$ for some $t \in k^*$.

EXAMPLE 4.1 There is merely one line through the origin in $\mathbb{A}^1$, so $\mathbb{P}^0$ is just one point. ☆

4.3 We shall begin with getting more acquainted with the projective spaces, and subsequently equip $\mathbb{P}^n$ with a variety structure. This amounts to endowing it with a topology (which naturally will be called the *Zariski topology*) and telling what functions on $\mathbb{P}^n$ are regular; that is, defining the sheaf $\mathcal{O}_{\mathbb{P}^n}$ of regular functions. Finally, we shall introduce the larger class of *projective varieties*. They will be the closed irreducible subsets of $\mathbb{P}^n$ given the topology induced from the Zariski topology on $\mathbb{P}^n$ and equipped with a sheaf of rings of regular functions.

Homogeneous coordinates

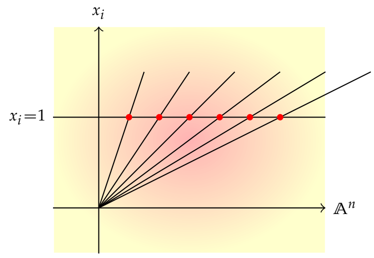
Homogeneous coordinates (*homogene koordinater*)

4.4 Coordinates are of course very useful and desirable tools, but on $\mathbb{P}^n$ there are no *global* coordinates. However, there is a good substitute. If $[x] \in \mathbb{P}^n$ corresponds to the line through the point $x = (x_0 : \cdots : x_n)$, we say that $(x_0 : \cdots : x_n)$ are *homogeneous coordinates* of the point $[x]$—notice the use of colons to distinguish them from the usual coordinates in $\mathbb{A}^{n+1}$. The homogeneous coordinates of $[x]$ depend on the choice of the point x in the line $[x]$ and are not unique; they are only defined up to a scalar multiple, so that $(x_0 : \cdots : x_n) = (tx_0 : \cdots : tx_n)$ for all elements $t \in k^*$. Be aware that $(0\colon \ldots : 0)$ is forbidden; it does not correspond to any line through the origin and thence are not coordinates of any point in $\mathbb{P}^n$.

4.5 Visualizing projective spaces can be quite challenging, but the following is one way of thinking about them. This description of $\mathbb{P}^n$ will also be important in the subsequent theoretical development and is an invaluable tool when working with projective spaces.

Fix one of the coordinates, say x_i, and let $D_+(x_i)$ denote the set of lines $[x] = (x_0 \ldots : x_n)$ for which $x_i \neq 0$. These sets are called the *distinguished* *open subsets* of $\mathbb{P}^n$ like their affine cousins. The subvariety A_i of $\mathbb{A}^{n+1}$ where $x_i = 1$ will also be useful. Every line $[x]$ with $x_i \neq 0$ intersects the subvariety A_i in precisely one point, namely the point $(x_0 x_i^{-1}, \ldots, x_n x_i^{-1})$. Thus there is a natural one-to-one correspondence α_i between the subsets $D_+(x_i)$ of $\mathbb{P}^n$ and A_i of $\mathbb{A}^{n+1}$. Now, obviously the subvariety A_i is isomorphic to affine n-space $\mathbb{A}^n$ (the projection that forgets the i-th coordinate is an isomorphism); hence $D_+(x_i)$ is in a natural bijective correspondence (later on we shall see it is an isomorphism) with $\mathbb{A}^n$. To avoid unnecessary confusion, let us denote the coordinates on A_i by t_j where j runs from 0 to n but stays different from i. Then the bijection α_i from $D_+(x_i)$ to A_i is given by the assignment $t_j = x_j x_i^{-1}$; it has the restriction $\pi|_{A_i}$ of the canonical projection as inverse.

The distinguished open subsets $D_+(x_i)$ (åpne basismengder)



4.6 The complement of the basic open subset $D_+(x_i)$ consists of the lines lying in the subvariety of $\mathbb{A}^{n+1}$ where $x_i = 0$; that is, the subvariety $Z(x_i)$. This is an affine n-space with coordinates[2] $(x_0, \ldots, \hat{x}_i, \ldots, x_n)$, and so the complement $\mathbb{P}^n \backslash D_+(x_i)$ is equal to the projective space $\mathbb{P}^{n-1}$ of lines in that affine space. It is called the *hyperplane at infinity*.

[2] A hat indicates that a variable is missing.

Hyperplane at infinity (hyperplanet i det uendelige)

Be aware that the "hyperplane at infinity" is a *relative* notion; it depends on the choice of the coordinates. In fact, given any linear functional $\lambda(x)$ in the x_i's, one may chose coordinates so that the hyperplane $\lambda(x) = 0$ is the hyperplane at infinity.

Examples

4.2 *(The projective line)* When $n = 1$ we have the *projective line* $\mathbb{P}^1$. It consists of a "big" subset isomorphic to $\mathbb{A}^1$ to which one has added a point at infinity. Every point can be made the point at infinity by an appropriate coordinate change.

The projective line over the complex numbers, endowed with the strong topology, is the good old *Riemann sphere* we became acquainted with during courses in complex function theory. Indeed, let $(x_0 : x_1)$ be the homogeneous coordinates on $\mathbb{P}^1$. In the set $D_+(x_0)$—which is isomorphic to $\mathbb{A}^1$; that is, to $\mathbb{C}$—one uses $z = x_1/x_0$ as coordinate, where as in $\mathbb{D}_+(x_1)$ the coordinate is $z^{-1} = x_0/x_1$; and we recognize the patching on $D_+(x_0) \cap D_+(x_1)$ used to construct the Riemann sphere.

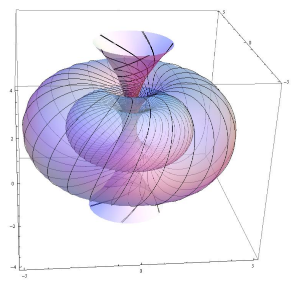
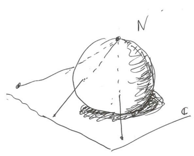


Figure 4.3: The Hopf fibration

The projection map $\pi\colon \mathbb{C}^2 \backslash \{0\} \to \mathbb{CP}^1$ is interesting. Restricting it to the unit sphere S^3 in $\mathbb{C}^2$ one obtains a map $S^3 \to S^2$, which is very famous and goes under the name of the *Hopf fibration*. It is easy to see that its fibres are circles, so that π is a fibration of the three sphere S^3 over S^2 in circles.



The projective line over the reals $\mathbb{R}$ is just a circle, but notice there is only one point at infinity. One uses lines and not rays through the origin, and so there is no distinction between ∞ and $-\infty$.

4.3 *(The projective plane)* The variety $\mathbb{P}^2$ is called *the projective plane*. It has a "big" open sutset $\mathbb{A}^2 = D_+(x_i)$ with a projective line at infinity "wrapped" around it.

The projective plane contains many subsets that are in a natural one-to-one correspondence with the projective line $\mathbb{P}^1$. The set of one dimensional subspaces contained in a fixed two dimensional vector subspace of $\mathbb{A}^3$ is such a $\mathbb{P}^1$, and of course any two dimensional subspace will do. These subsets are called *lines* in $\mathbb{P}^2$.

By linear algebra two different two dimensional vector subspaces of $\mathbb{A}^3$ intersect along a line through the origin, hence the fundamental observation that the two corresponding lines in $\mathbb{P}^2$ intersect in a *unique* point. Two lines do not necassarily meet in the "finite part"; that is, in the affine 2-space $\mathbb{A}^2$ where $x_i \neq 0$. This occures if and only they have a common intersection with the line at infinity, and trhen one says that the two lines meet at infinity. And naturally, when they do not meet in the finite part, they are experienced to be parallel; hence "parallel" lines meet at a common point at infinity.

The projective plane over the reals, is a subtle creature. After having picked one of the coordinates x_i, we find a big cell of shape $D_+(x_i)$ in $\mathbb{P}^2$, which is bijective to $\mathbb{R}^2$, enclosed by the line at infinity; a circle bordering the affine world like the Midgard Serpent. Again, be aware that in constructing $\mathbb{P}^2$ one uses lines through the origin and not rays emanating from the origin. This causes $\mathbb{P}^2$ to be non-orientable—tubular neighbourhoods of the lines are in fact Möbius bands.

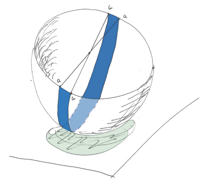


Figure 4.4: A Möbius band in the real projective plane.

☆

The Zariski topology and projective varieties

One may use the projection map $\pi\colon \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ to equip $\mathbb{P}^n$ with a topology: We declare a subset in $X \subseteq \mathbb{P}^n$ to be closed if and only if the inverse image $\pi^{-1}(X)$ is closed in $\mathbb{A}^{n+1}\backslash\{0\}$. Since the operation of forming inverse images behaves well with respect to intersections and unions (*i.e.* commutes with them), these sets are easily seen to fulfill the axioms for the closed sets of a topology. Naturally, this topology is called the *Zariski topology* on $\mathbb{P}^n$. It is the quotient topology with respect to the equivalence relation on $\mathbb{A}^{n+1}$ giving $\mathbb{P}^n$.

The Zariski topology (*Zariski topologien*)

4.7 Polynomials on $\mathbb{A}^{n+1}$ do not descend to functions on $\mathbb{P}^n$ unless they are constant—non-constant polynomials are not constant along lines through the origin. However, if F is a *homogeneous* polynomial it holds true that $F(tx) = t^d F(x)$ where d is the degree of F, so if F vanishes at a point x, it vanishes along the entire line joining x to the origin. Hence it is lawful to say that F is *zero* at a point $[x] \in \mathbb{P}^n$; and it is meaningful to talk about the zero locus in $\mathbb{P}^n$ of a set of homogeneous polynomials. A homogeneous ideal $\mathfrak{a}$ in $k[x_0, \ldots, x_n]$ is generated by homogeneous polynomials, and we can hence speak about the *zero locus* $Z_+(\mathfrak{a})$ in $\mathbb{P}^n$ as the common set of zeros of the generators.

The zero locus of a homogeneous ideal

In the same spirit as one defined the basic open sets $D_+(x_i)$, one defines the *distinguished open subset* $D_+(F) = \{ [x] \in \mathbb{P}^n \mid F(x) \neq 0 \}$ for any homogeneous polynomial F. All these sets are open in $\mathbb{P}^n$, their complements being the closed sets $Z_+(F)$.

Distinguished open subsets

Problem 4.1 Let $\mathfrak{a} \subseteq k[x_0, \ldots, x_n]$ be an ideal. Show that $\mathfrak{a}$ is homogeneous if and only if it satisfies either of the following two equivalent properties:

a) A polynomial $f(x)$ belongs to $\mathfrak{a}$ precisely when $f(tx)$ lies there for all scalars $t \in k$.

b) The zero set $Z(\mathfrak{a})$ in $\mathbb{A}^{n+1}$ is a cone; that is, if $x \in Z(\mathfrak{a})$, then the whole line through x and the origin is contained in $Z(\mathfrak{a})$. ★

4.8 A homogeneous ideal $\mathfrak{a}$ also has a zero set $Z(\mathfrak{a})$ in the affine space $\mathbb{A}^{n+1}$, and since homogeneous polynomials vanish along lines through $\{0\}$, it clearly holds that $\pi^{-1}Z_+(\mathfrak{a}) = Z(\mathfrak{a}) \cap \mathbb{A}^{n+1} \setminus \{0\}$. Thus the closed sets of the Zariski topology on $\mathbb{P}^n$ are the exactly subsets of type $Z_+(\mathfrak{a})$ with $\mathfrak{a}$ being a homogeneous ideal in $k[x_0, \ldots, x_n]$. Such a subset X is called a *closed projective subset* and the topology induced from the Zariski topology on $\mathbb{P}^n$ is called the Zariski topology on X. If additionally X is an *irreducible* space, it is said to be a *projective variety*. And an open subset of a projective variety is said to be a *quasi projective variety*.

Closed projective subset (lukkede projektive mengder)
Projective varieties (projektive variteter)
Quasi projective varieties (kvasiprojektive variteter)

The *affine cone* $C(X)$ over a projective variety X is defined as the closed subset $C(X) = \pi^{-1}X \cup \{0\}$. It is a cone in the sense that it contains the line joining any one its points to the origin; or phrased differently, if $x \in C(X)$ then $tx \in C(X)$ for all scalars $t \in k$. The inverse image $\pi^{-1}X$ will now and then be called the *punctured cone* over X and denoted by $C_0(X)$; so that $C_0(X) = C(X) \cap (\mathbb{A}^{n+1} \setminus \{0\})$.

The affine cone over a projective variety (affine kjegler)
The punctured cone (den punkterte kjeglen)

In this story there is one ticklish point. The maximal ideal $\mathfrak{m}_+ = (x_0, \ldots, x_n)$ vanishes only at the origin in $\mathbb{A}^{n+1}$, and so it defines the *empty set* in $\mathbb{P}^n$; indeed, for no point in $\mathbb{P}^n$ do all the coordinates x_i vanish. Hence $\mathfrak{m}_+$ goes under the name of *the irrelevant ideal*.

The irrelevant ideal (de irrelevante idealet)

Problem 4.2 Show that if $\mathfrak{a}$ and $\mathfrak{b}$ are two homogeneous ideals, then $\mathfrak{a} \cdot \mathfrak{b}$ and $\mathfrak{a} + \mathfrak{b}$ are homogeneous, and it holds true that $Z_+(\mathfrak{a} \cdot \mathfrak{b}) = Z_+(\mathfrak{a}) \cup Z_+(\mathfrak{b})$ and $Z_+(\mathfrak{a} + \mathfrak{b}) = Z_+(\mathfrak{a}) \cap Z_+(\mathfrak{b})$. ★

The big open subsets

4.9 Since π obviously is continuous, the Zariski topology makes the projective spaces irreducible. It also clear that the "big" affine subsets $D_+(x_i)$ where $x_i \neq 0$ are open subsets, their complements—the hyperplanes at infinity—being the closed sets $Z_+(x_i)$. In Paragraph 4.5 the subsets A_i of $\mathbb{A}^{n+1}$ where the i-th coordinate equals one were introduced, and we demonstrated that the restriction $\pi|_{A_i}$ was a bijection between A_i and $D_+(x_i)$; now we go one step further:

Proposition 4.10 *The restriction $\pi|_{A_i}$ of π is a homeomorphism bewteen A_i and $D_+(x_i)$.*

The proof of this needs the process of *homogenization* of a polynomial which, when the variable x_i is fixed, is a systematic way of producing a homogeneous polynomial f^h from a polynomial f. If d is the degree of f, one puts

Homogenization of polynomials (homogenisering)

$$f^h(x_0, \ldots, x_n) = x_i^d f(x_0 x_i^{-1}, \ldots, x_n x_i^{-1}). \tag{4.1}$$

For example, if $f = x_1 x_2^3 + x_3 x_0 + x_0$, one finds that relative to the variable x_0 one has

$$f^h(x_0, x_1, x_2, x_3) = x_0^4\big(x_1 x_0^{-1}(x_2 x_0^{-1})^3 + x_3 x_0^{-1} + 1\big) = x_1 x_2^3 + x_3 x_0^3 + x_0^4.$$

The net effect of the homogenization process is that all the monomial terms are filled up with the chosen variable so that they have the same degree. To verify that f^h in (4.1) is homogeneous of degree d, let t be any scalar. Each fraction $x_j x_i^{-1}$ is invariant when the variables are scaled, and the front factor x_i^d changes by the multiple t^d. The important relation $f|_{A_i} = f^h|_{A_i}$, which is easy to establish (just put $x_i = 1$) holds true as well.

Proof of Proposition 4.10: Now, we come back to the proof of the proposition. The restriction $\pi|_{A_i}$ is, as already observed, continuous, so our task is to show that the inverse is continuous, or what amount to same, that $\pi|_{A_i}$ is a closed map.

Since any closed subset of A_i is the intersection of sets of the form $Z = Z(f) \cap A_i$, and π being bijective takes intersections to intersections, it suffices to demonstrate that $\pi(Z(f) \cap A_i)$ is closed in $D_+(x_i)$ for any polynomial f on A_i. But this is precisely what the homogenization f^h is constructed for. Indeed, the subset $Z(f^h)$ of $\mathbb{A}^{n+1}$ is a closed cone satisfying $Z(f^h) \cap A_i = Z(f) \cap A_i$, and this means that $\pi(Z(f) \cap A_i) = Z_+(f^h) \cap D_+(x_i)$. ❑

Example 4.4 When trying to understand a variety in $\mathbb{P}^n$ it is often useful to consider the different "pieces" $X \cap D_+(x_i)$. Since $D_+(x_i)$ is isomorphic to an affine space $\mathbb{A}^n$, one my apply affine techniques to study $X \cap D_+(x_i)$.

For example this technique sheds considerable light on plane conics. The projective conic $xy - z^2 = 0$ becomes the hyperbola $(x/z)(y/z) = 1$ in the $\mathbb{A}^2$ which equals $D_+(z)$ and have x/z and y/z as coordinates, but it materializes as the parabola $y/x - (z/x)^2$ in $D_+(x)$ which is an $\mathbb{A}^2$ whose coordinates are y/x and z/x. So the difference between the hyperbola and the parabola is just a matter of perspective! They are both affine parts of the same projective curve. In other words and with Example 1.5 on page 17 in mind, all conics are up to the choice of coordinates the same when considered as living in the projective plane $\mathbb{P}^2$ over $\mathbb{C}$. ☆

Problems

4.3 Let $V \subseteq \mathbb{A}^{n+1}$ be a linear vector subspace of dimension $m+1$. Show that V is a cone and that the corresponding projective variety $\mathbb{P}(V)$ is isomorphic to $\mathbb{P}^n$. It is called a *linear subvariety* of $\mathbb{P}^n$. Show that if W is another linear subspace of dimension $m'+1$ and $m+m' \geqslant n$, then $\mathbb{P}(V)$ and $\mathbb{P}(W)$ has a non-empty intersection which is a linear subvariety.

Linear subvarieties (*linræe underbarieteter*)

4.4 Show that two different lines in $\mathbb{P}^2$ meet in exactly one point.

4.5 Show that n hyperplanes in $\mathbb{P}^n$ always have a common point of intersection. Show that n general hyperplanes meet in exactly one point.

4.6 Let $p_0, \ldots, p_n$ be $n+1$ points in $\mathbb{P}^n$ and let $v_0, \ldots, v_n$ be non-zero vectors lying on the corresponding lines in $\mathbb{A}^{n+1}$. Show that the p_i's lie on a hyperplane if and only if the v_i's are linearly dependent.

4.7 Let $f(x_0, x_1, x_2, x_3) = x_0^3 x_2 + x_3^2 x_1 + 1$. Determine f^h with respect to each of the four variables.

4.8 In this exercise $n = 2$ and the coordinates are x_0 and x_1. Let $f(x_0) = (x_0 - a)(x_0 - b)$. Determine f^h and make a sketch of $Z(f) \cap \mathbb{A}^1$ and the cone $Z(f^h)$

4.9 Show that any affine variety is quasi projective.

4.10 How doe the circles and the ellipses fit into the picture described in Examples 4.4 and 1.5?

★

The sheaf of regular functions on projective varieties.

Although polynomials on $\mathbb{A}^{n+1}$ do not descend to functions on $\mathbb{P}^n$, certain rational functions do. To describe these, assume that a and b are two polynomials both homogeneous of the same degree, say d. Although none of them define a function on the projective space $\mathbb{P}^n$, their fraction do, at least at points where the denominators do not vanish. Indeed, letting x and tx be two points on the same line through the origin, we find

$$\frac{a(tx)}{b(tx)} = \frac{t^d a(x)}{t^d b(x)} = \frac{a(x)}{b(x)}$$

whenever $b(x) \neq 0$. The function $a(x)/b(x)$ thus takes the same value at all points on the line $[x]$, and this common value is the value of $a(x)/b(x)$ at $[x]$.

4.11 This observation leads to the definition of the *sheaf of regular functions* on $\mathbb{P}^n$, or more generally to the notion of regular functions on any closed projective set $X \subseteq \mathbb{P}^n$, hence to the sheaf $\mathcal{O}_X$ of regular functions on X.

The sheaf of regular functions on $\mathbb{P}^n$ (knippet av regulære funksjoner)

A function f on an open subset U of X is said to be *regular* at a point $p \in U$ if there exists an open neighbourhood $V \subseteq U$ of p in X and homogenous polynomials a and b of the same degree such that $b(x) \neq 0$ throughout V and such that the equality

Regular functions (regulære funksjoner)

$$f(x) = \frac{a(x)}{b(x)}$$

holds for $x \in V$. For any open U in X, we let $\mathcal{O}_X(U)$ be set of functions regular at all points in U, and we let the restriction maps just be the restrictions. Sums and products of regular functions are regular so $\mathcal{O}_X(U)$ is a k-algebra, and the restriction maps are k-algebra homomorphisms. Moreover, a regular function in $\mathcal{O}_X(U)$ is invertible if and only if it does not vanish at any point in U.

EXAMPLE 4.5 If the index i is fixed, the functions x_j/x_i are regular on the basic open subset $D_+(x_i)$ of $\mathbb{P}^n$. ☆

Obviously $\mathcal{O}_X$ is a *presheaf* on X, and the first sheaf-axiom is trivially fulfilled (it always is, when the sections are set-theoretical functions with some extra properties). Also the second sheaf-axiom is easy to establish: If a regular function is given for each member of an open covering $\{U_i\}$ of an open subset of X and if they coincide on the intersection $U_i \cap U_j$, they patch together as continuous functions into $\mathbb{A}^1$, and since being regular is a local condition, the resulting function is regular (it restricts to regular functions on each of the members of the open covering $\{U_i\}$). Hence $\mathcal{O}_X$ is a *sheaf* on X.

PROBLEM 4.11 Show in detail that $\mathcal{O}_X(U)$ is a k-algebra. ★

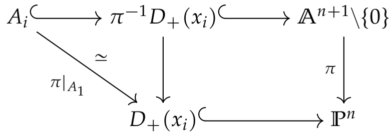
PROBLEM 4.12 Given an open $U \subseteq X$ and a continuous function $f\colon U \to \mathbb{A}^1$. Let $C_0(U)$ be punctured cone over U and denote by $\pi_U\colon C_0(U) \to U$ the (restriction of the) projection. Show that f is regular if and only if the composition $f \circ \pi$ is regular on $C_0(U)$. ★

4.12 It is of interest to compare regular functions on closed subsets of a closed projective set and on the surrounding set. At least locally there is a reasonable answer. Assume that $Y \subseteq X$ is a Zariski-closed subset of the closed projective set X and that $U \subseteq X$ is open. The following lemma is almost tautological:

LEMMA 4.13 (RESTRICTION AND LOCAL EXTENSION) *Restrictions of regular functions are regular and regular function can locally be extended: If f is a regular function in the open set $U \subseteq X$ and $Y \subseteq X$ is closed, the restriction $f|_{U \cap Y}$ is regular in $Y \cap U$. Any functions on $U \cap Y$ regular at a point $p \in Y$ extends to a regular function on some open neighbourhood V of p in X.*

Projective varieties are varieties

In the end of this section we shall establish that projective varieties are varieties. For this, there will be two steps, and in the first we content ourself to see that they are *prevarieties*. The projective varieties have been equipped with a topology and a sheaf of rings, so what is left, is to check they are locally affine. To this end, we shall show that the distinguished open subsets $D_+(x_i)$ are isomorphic to affine n-space $\mathbb{A}^n$—more precisely the regular functions x_j/x_i will serve as affine coordinates. This resolves the matter for the projective space itself, and for a closed subset $X \subseteq \mathbb{P}^n$ the sets $X \cap D_+(x_i)$ will do (closed subsets of affine space are affine).



4.14 In Paragraph 4.5 on page 60 we introduced the subvariety A_i of $\mathbb{A}^{n+1}$ where the i-th coordinate x_i equals one, and there we established that $\pi|_{A_i}$ is a homeomorphism between A_i and $D_+(x_i)$ (Proposition 4.10). Here we shall accomplish the description and prove it is an isomorphism of varieties. There is a natural inverse map α. If $x_i \neq 0$, we may send a point x with homogeneous coordinates $(x_0 : \cdots : x_n)$ to the point $(x_0x_i^{-1}, \ldots, x_nx_i^{-1})$ which obviously is independent of any scaling of the x_i's. It has the i-th coordinate equal to one, and hence lies in A_i.

Proposition 4.15 *The projection $\pi|_{A_i}$ is an isomorphism between A_i and $D_+(x_i)$. The inverse map is the map α above; that is, $\alpha(x_0, \ldots, x_n) = (x_0/x_i, \ldots, x_n/x_i)$.*

Notice, the last sentence says that the distinguished open $D_+(x_i)$ subset is an affine n-space on which the n functions $x_0/x_i, \ldots, x_n/x_i$ serve as coordinates.

Proof: The two maps $\pi|_{A_i}$ and α are clearly mutually inverse and both are homeomorphisms (Proposition 4.10 on page 64), so what is left, is to check that they are morphisms.

That $\pi|_{A_i}$ is a morphism is almost trivial: Let f be regular at p and represent f in some open neighbourhood U of p as $f(x) = a(x)/b(x)$ with a and b homogeneous polynomials of the same degree and with b being non-zero throughout U. One simply has $f \circ \pi|_{A_i} = a/b|_{A_i}$, which is regular on $A_i \cap \pi^{-1}U$ as b does not vanish along $\pi^{-1}U$.

To prove the α is a morphism, let f be regular on an open set $U \subseteq A_i$ and represent f as $f = a/b$ with a and b being polynomials and b not vanishing in U. To obtain $f \circ \alpha$ one simply plugs in $x_jx_i^{-1}$ in the j-th slot (this automatically inserts a one in slot i) and one arrives at the expression

$$f \circ \alpha(x_0 : \cdots : x_n) = a(x_0x_i^{-1}, \ldots, x_nx_i^{-1})/b(x_0x_i^{-1}, \ldots, x_nx_i^{-1}).$$

We already observed that the fractions $x_jx_i^{-1}$ are regular throughout $D_+(x_i)$, and as the regular functions form a ring with non-vanishing functions being invertible, it follows that $f \circ \alpha$ is regular in U. ❑

4.16 This was the warm up for $\mathbb{P}^n$, and the general case of a projective variety is not very much harder—in fact it follows immediately:

Proposition 4.17 *Assume that $X \subseteq \mathbb{P}^n$ is an irreducible closed projective set. Then $V_i = D_+(x_i) \cap X$ equipped with the sheaf $\mathcal{O}_X|_{V_i}$ is an affine variety.*

Proof: Closed irreducible subsets of affine varities are affine varities. ❑

We have almost establishes the following all important theorem:

Theorem 4.18 *Irreducible, closed projective sets are varieties when endowed with the Zariski topology and the sheaf of regular functions.*

Proof: Let the set in question by $X \subseteq \mathbb{P}^n$. The only thing that remains to be proven is that X satisfies the Hausdorff axiom. By Lemma 3.33 on page 49, if suffices to exhibit an open affine subset containing any two given points. This is no big deal: Given two distinct points in X, there is linear form λ on $\mathbb{A}^{n+1}$ that does not vanish at either. Hence both lie in the basic open subset $D_+(\lambda) \cap X$, which is affine by Proposition 4.17 above. ❑

The following corollary is with Proposition 4.15 above in mind, merely an observation

Corollary 4.19 *Let F be a homogeneous form on $\mathbb{A}^{n+1}$. The open set $D_+(F) \cap D_+(x_i)$ is affine. When the coordinates x_j/x_i on $D_+(x_i)$ are used, $D_+(F) \cap D_+(x_i)$ corresponds to the distinguish open set $D(F^d)$ of $\mathbb{A}^n$ where $F^d = F(x_0/x_i, \ldots, x_n/x_i)$.*

The projective closure

For any polynomial $f \in k[x_0 x_i^{-1}, \ldots, x_n x_i^{-1}]$ it holds true that $Z(f^h) \cap D_+(x_i) = Z(f)$. Assume that X is a subvariety of one of the basic open sets $D_+(x_k)$ and let $\mathfrak{a}$ is the ideal in $k[x_0 x_1^{-1}, \ldots, x_n x_k^{-1}]$ defining it. Let $\mathfrak{A}$ be the homogeneous ideal

$$\mathfrak{A} = \{ f^h \mid f \in \mathfrak{a} \}$$

where $f^h = x_k^d f$ is the homogenization of the polynomial f of degree d from $k[x_0 x_k^{-1}, \ldots, x_n x_k^{-1}]$. Then $Z_+(\mathfrak{A})$ is the closure of X in $\mathbb{P}^n$. Indeed,

4.2 *The projective Nullstellensatz*

4.20 The correspondence between homogeneous ideal in the polynomial ring $k[x_0, \ldots, x_n]$ and closed subsets of the projective space $\mathbb{P}^n$ is as in the affine case governed by a Nullstellensatz.

There are however, some differences. In the projective case the ideals must be homogeneous, and there is a slight complication concerning the ideals with empty zero locus. Just as in the affine case if $1 \in \mathfrak{a}$, the zero locus of $\mathfrak{a}$ is of course empty, but neither do ideals whose zero set in $\mathbb{A}^{n+1}$ is reduced to the origin (that is, $Z(\mathfrak{a}) = \{0\}$) have zeros in the projective space—the tuple $(0; \ldots; 0)$ is forbidden and is not the homogenous coordinates of any point. In other words, ideals $\mathfrak{a}$ whose radical equals the irrelevant ideal $\mathfrak{m}_+$, do not have zeros in $\mathbb{P}^n$.

4.21 A simple and down to earth and, not the least, a geometric way of thinking about the interplay between the affine and the projective Nullstellensatz, is via the affine cone $C(X) = \pi^{-1}X \cup \{0\}$ over a closed set $X \subseteq \mathbb{P}^n$ (recall the projection map $\pi\colon \mathbb{A}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ that sends a point to the line joining it to the origin). This sets up one-to-one correspondence between closed non-trivial[3] cones in $\mathbb{A}^{n+1}$ and non-empty closed subsets in $\mathbb{P}^n$;

[3] Formally, a cone in $\mathbb{A}^{n+1}$ is a subset closed under homothety; that is, if $x \in C$, then $tx \in C$ for all scalars $t \in K$. Clearly the singleton $\{0\}$ comply to this definition, so $\{0\}$ is a cone. It is called the *trivial cone* or the *null cone*.

Lemma 4.22 *Associating the affine cone $C(X)$ to X gives a bijection between closed non-empty subsets of $\mathbb{P}^n$ and closed non-trivial cones in $\mathbb{A}^{n+1}$. The bijection respects inclusions, intersections and unions.*

Proof: Let $C \subseteq \mathbb{A}^{n+1}$ be a non-trivial cone and denote by C_0 the punctured cone; that is, the intersection $C_0 = C(X) \backslash \{0\}$ of C and $\mathbb{A}^{n+1} \backslash \{0\}$. There are two points to notice; firstly, C_0 is nonempty, and if it is closed in $\mathbb{A}^{n+1} \backslash \{0\}$, its closure in $\mathbb{A}^{n+1}$ satisfies $\overline{C_0} = C$ (the origin can not be the only point on a line where a polynomial does not vanish), and secondly, $\pi^{-1}\pi(C_0) = C_0$. It follows that C is closed in $\mathbb{A}^{n+1}$ if and only if C_0 is closed in $\mathbb{A}^{n+1} \backslash \{0\}$, and by the definition of the Zariski topology on $\mathbb{P}^n$, we infer that C is closed if and only if $\pi(C_0)$ is closed in $\mathbb{P}^n$. This shows that the correspondence of the lemma is surjective, and it is injective since it $\pi(\pi^{-1}C) = C$ because π is surjective.

The last statement in the lemma is a general feature of inverse images. ❑

4.23 To any closed subset $X \subseteq \mathbb{P}^n$, we let $I(X)$ be the ideal in the polynomial ring $k[x_0, \ldots, x_n]$ generated by all homogeneous polynomials that vanish in X. It is clearly an homogeneous ideal, and one has $I(X) = I(C(X))$. Combining the bijection in Lemma 4.22 above with the bijection between homogeneous radical ideals and closed cones from the affine Nullstellensatz[4], one arrives at the following version of the Nullstellensatz in a projective setting:

[4] A closed subset $C \subseteq \mathbb{A}^{n+1}$ is a cone if and only if the ideal $I(C)$ is homogeneous. Indeed, C is a cone precisely when a polynomial $f(x)$ vanishes along X if and only if $f(tx)$ does for all $t \in k$.

Proposition 4.24 (Projective Nullstellensatz) *Assume that $\mathfrak{a}$ is a homogeneous ideal in $k[x_0, \ldots, x_n]$.*

- *Then $Z_+(\mathfrak{a})$ is empty if and only if $1 \in \mathfrak{a}$ or $\mathfrak{a}$ is $\mathfrak{m}_+$-primary; that is, $\mathfrak{m}_+^N \subseteq \mathfrak{a}$ for some N.*
- *If $Z_+(\mathfrak{a}) \neq \emptyset$, it holds true that $I(Z_+(\mathfrak{a})) = \sqrt{\mathfrak{a}}$.*
- *Associating $I(X)$ with X sets up a bijection between closed non-empty subsets $X \in \mathbb{P}^n$ and proper, radical homogeneous ideals I in $k[x_0, \ldots, x_n]$ different from the irrelevant ideal.*
- *The subset $Z_+(\mathfrak{a})$ is irreducible if and only if the radical $\sqrt{\mathfrak{a}}$ is a prime ideal.*

Proof: We already have argued for most of the statements; what remains, is to clarify when $Z_+(\mathfrak{a})$ is empty, and this happens precisely when $Z_+(\mathfrak{a}) \subseteq \{0\}$.

There are two cases: Either $Z(\mathfrak{a}) = \emptyset$ or $Z(\mathfrak{a}) = \{0\}$, which by the Affine Nullstellensatz correspond to respectively $1 \in \mathfrak{a}$ or $\sqrt{\mathfrak{a}} = I(\{0\}) = \mathfrak{m}_+$.

For the last statement, it is quite clear (and therefore left to the zealous students to verify) that X is irreducible if and only if the cone $C(X)$ over X is irreducible. ❑

4.3 Global regular functions on projective varieties

One of the fundamental theorem of affine varieties states that the space $\mathcal{O}_X(X)$ of global sections of the structure sheaf of an affine variety X —that is, the space of globally defined regular function—is equal to the coordinate ring $A(X)$. This is space is quite large and in many ways completely determines the structure of the variety.

For projective varieties the situation is quite different. The only globally defined regular functions turn out to be the constants (Theorem 4.26) below). True, one has the coordinate ring $S(X) = A(C(X))$ of the *cone* over X, but most elements there are not functions on X, not even the homogeneous ones.

By assumption X will be irreducible, and the same is then true for the cone $C(X)$. The ring $S(X) = A(C(X))$ is therefore an integral domain and has a fraction field which we shall denote by K. One calls $S(X)$ the *homogeneous coordinate ring* of X. It is a graded ring because the ideal $I(X)$ is homogeneous, and it has a decomposition into homogeneous parts $S(X) = \bigoplus_{i \geqslant 0} S(X)_i$, where $S(X)_i$ denotes the subspace of elements of degree i. The fraction field K of $S(X)$ is not graded, but the fraction of two homogeneous elements from $S(X)$ has a a degree, namely $\deg ab^{-1} = \deg a - \deg b$.

Homogeneous coordinate rings

PROBLEM 4.13 Let S be a graded ring. Show that the set T consisting of the homogeneous elements in S is a multiplicative system and that the localization S_T is a graded ring. Show S_T is an integral domain when S is, and in that case the homogeneous piece of degree zero $(S_T)_0$ is a field. ★

PROBLEM 4.14 Let $S = k[x_0, x_1]$ and let T be the multiplicative system $T = \{ x_1^i \mid i \in \mathbb{N} \}$. Show that the homogeneous piece $(S_T)_0$ of degree zero of S_T equals $k[x_0 x_1^{-1}]$. Show furthermore that the decomposition of S into homogeneous pieces is given as

$$S = \bigoplus_{i \in \mathbb{Z}} k[x_0 x_1^{-1}] \cdot x_1^i.$$

★

PROBLEM 4.15 If $X \subseteq \mathbb{P}^n$ is a projective variety. Show that the rational function field $K = k(X)$ equals $(S(X)_T)_0$ with notation as in Problem 4.14. ★

All regular function on open sets in $C(X)$ are elements of K, and two are equal as functions on an open if and only if they are the same element in K. The ground field k is contained in K as the constant functions on $C(X)$.

As gentle beginning let us consider the case of the projective space $\mathbb{P}^n$ itself. So let f be a global regular function on $\mathbb{P}^n$. Composing it with the projection $\pi\colon \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ we obtain a regular function $f\circ\pi$ on $\mathbb{A}^{n+1}\backslash\{0\}$. In example 3.6 on page 42 in Notes 3 we showed that every regular function on $\mathbb{A}^{n+1}\backslash\{0\}$ is a polynomial, hence $f\circ\pi$ is a polynomial. But $f\circ\pi$ is also constant on lines through the origin, and therefore must be constant. We thus arrive at the following:

Proposition 4.25 *It holds true that $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$.*

For a general projective variety the same is true, but considerably more difficult to prove. One has

Theorem 4.26 *The only globally defined regular functions on a projective variety $X\subseteq\mathbb{P}^n$ are the constants. In other words, it holds true that $\mathcal{O}_X(X) = k$.*

Proof: Let f be a global regular function on X. Composed with the projection it gives a global regular function one the punctured cone $C(X)\backslash\{0\}$ which we still denote by f. It is an element in the function field K of the cone $C(X)$.

Let D_i be the distinguished open set in $C(X)$ where $x_i \neq 0$; that is, in earlier notation $D_i = C(X)_{x_i}$. We know[5] that the coordinate ring $A(D_i)$ satisfies $A(D_i) = S(X)_{x_i}$ so that for each index i the function f has a representation $f = g_i/x_i^{r_i}$ for some $g_i \in S(X)$ and some natural number r_i. The function f being constant along lines through the origin, it must be homogeneous of degree zero; in other words, g_i is homogeneous and $\deg g_i = r_i$.

[5] Lemma 3.3 one page 6 in notes 2.

So we have $x_i^{r_i} f = g_i$, and the salient point is that g_i lies in the homogeneous part $S(X)_{r_i}$. It follows that $hx_i^{r_i} f \in S(X)_{r_i+j}$ for all elements h of $S(X)$ that are homogeneous of degree j.

Now, chose an integer r so big that $r > \sum_i r_i$. Then any monomial M of degree r contains at least one of the variables, say x_i, with an exponent larger than r_i, and consequently $Mf \in S(X)_r$. In other words, multiplication by f leaves the finite dimensional vector space $S(X)_r$ invariant. It is a general fact (for instance, use the Cayley-Hamilton theorem), that f then satisfies a relation of the type

$$f^m + a_{m-1}f^{m-1} + \ldots + a_1 f + a_0 = 0$$

where the a_i are elements in the ground field k. This shows that $f \in K$ is algebraic over k, and since k is algebraically closed by assumption, f lies in k and is constant. ❑

4.27 An important consequence of the theorem is that morphisms of projective varieties into affine ones necessarily are constant. Indeed, if $X\subseteq\mathbb{P}^n$ is projective and $Y\subseteq\mathbb{A}^m$ is affine, the component functions of a morphism $\phi\colon X\to Y\subseteq\mathbb{A}^m$ must all be constant according to the theorem we just proved. Hence we have

COROLLARY 4.28 *Any morphism from a projective variety to an affine one is constant.*

Another consequence is the following:

COROLLARY 4.29 *A variety X which is both projective and affine, is reduced to a point.*

PROOF: The coordinate functions are regular functions on a subvariety $X \subseteq \mathbb{A}^n$, and according to the theorem they must be constant when X is projective. ❑

4.4 *Morphisms from quasi projective varieties*

When it comes to morphism between affine varities the picture is quite clear . The main theorem (Theorem 3.27 on page 48) tells us they are just given as ring homomorphism between the coordinate rings; or if the target variety is contained in the affine space $\mathbb{A}^n$ a morphism is simply given by n regular component functions on the source.

When it comes to morhisms between projective (or quasi projective ones) the picture is not so clear. However many morphisms are easily defined as set-theoretical maps, and for experienced geometers it is pretty obvious they are morphism, but at least once in a lifetime one should check in detail it is the case. So we offer a little simplistic lemma in that direction.

A simplistic lemma

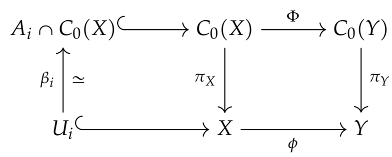
4.30 Let X and Y be two quasi-projective varieties and let $\phi\colon X \to Y$ be a set-theoretical map (but highly suspected to be a morphism). Assume we know that ϕ fits into a diagramme shaped like

$$\begin{array}{ccc} C_0(X) & \xrightarrow{\Phi} & C_0(Y) \\ {\scriptstyle \pi_X}\downarrow & & \downarrow{\scriptstyle \pi_Y} \\ X & \xrightarrow[\phi]{} & Y \end{array} \tag{4.2}$$

where $C_0(X)$ and $C_0(Y)$ are the punctured cones over respectively X and Y, and where the two vertical maps are the usual projections and where Φ is a morphism; in other words, ϕ lifts to a morphism between the punctured cones. Then it follows that ϕ is a morphism. Usually it is not difficult to check this case by case, but it is worthwhile doing it once and for all, hence the following little lemma:

LEMMA 4.31 *With the setting as in the diagramme (4.2) above, the map ϕ is a morphism.*

PROOF: Assume that X is open in a subvariety of the projective space $\mathbb{P}^n$ and that $(x_0 : \ldots : x_n)$ are homogenous coordinates in $\mathbb{P}^n$. Then the sets $U_i = D_+(x_i) \cap X$ form an open covering of X, and it suffices to see that each restriction $\phi|_{U_i}$ is a morphism. Earlier (in Paragraph 4.5 on page 60) we defined a natural map $\alpha_i \colon D_+(x_i) \to \mathbb{A}^{n+1} \backslash \{0\}$, which is a section of the canonical projection π over $D_+(x_i)$. Its basic property is being an isomorphism between $D_+(x_i)$ and and the (affine) linear subvariety A_i of $\mathbb{A}^{n+1}$ where $x_i = 1$. (Recall that $\alpha_i(x_0 : \ldots : x_n) = (x_0 x_i^{-1}, \ldots, x_n x_i^{-1})$). Restricted to the open subset U_i of X the section α_i gives an isomorphism $\beta_i \colon U_i \simeq A_i \cap C_0(X)$ satisfying $\pi_X \circ \beta_i = \mathrm{id}_X$. It follows that



$$\phi|_{U_i} = \phi \circ \pi_X \circ \beta_i = \pi_Y \circ \Phi \circ \beta_i,$$

and since the three maps to the right are morphisms, $\phi|_{U_i}$ is one as well. ❑

4.32 Examples of maps frequently met in projective algebraic geometry and fitting into the scenario of Lemma 4.31 are when the morphism Φ is given as $\Phi(x) = (f_0(x), \ldots, f_m(x))$ with the components f_i's homogeneous polynomials of the same degree. The set X can be the open subset of $\mathbb{P}^n$ where the f_i's do not vanish simultaneously; that is, $X = \mathbb{P}^n \backslash Z_+(f_0, \ldots, f_m)$, or any quasi projective set contained therein. And Y might be the entire projective space $\mathbb{P}^m$, or any quasi projective subvariety Y so that the cone $C_0(Y)$ contains the image of Φ.

On X the morphism Φ descends to the map $\phi([x]) = (f_0(x) : \ldots : f_m(x))$ between the quasi-projective varieties X and Y. Because the f_i's all have the same degree, say d, it holds true that

$$(f_0(tx), \ldots, f_n(tx)) = (t^d f_0(x), \ldots, t^d f_n(x))$$

for any non-zero scalar t, and therefore $\phi([x])$ does not depend on the representative of $[x]$. Moreover, the homogeneous coordinates $(f_0(x) : \ldots : f_m(x))$ are legitimate because in X not all of the f_i's vanish at the same time.

4.33 Different morphisms Φ might fit into the diagramme (4.2) paired with the same map ϕ. The components of Φ may for instance be changed by a common factor, and this does not change the map ϕ. Notice however, that the set where ϕ is defined; that is, the variety X in the diagramme, is susceptible to change. It might grow, and it might shrink, depending on the behaviour of the common factor. Certainly, a common factor might introduce common zeros, in which case the set X will shrink.

But the situation might also improve, and the set where ϕ is defined can grow. When the f_i's are rational functions, the morphism Φ is not defined where one of them has a pole, but multiplying through by the least common multiple of their denominators, yields a lifting Φ whose components are polynomials and thus extends ϕ beyond the set of poles.

Linear projections

4.34 As one can guess from the name, projections are central in projective geometry, and they are examples of rational maps coming out from the scenario of the little lemma.

A *projection* is a surjective, rational map $\pi\colon \mathbb{P}^n \dashrightarrow \mathbb{P}^m$ which in the staging of the little lemma 4.31 is induced by a surjective, linear map $\Phi\colon \mathbb{A}^{n+1} \to \mathbb{A}^{m+1}$. One assumes $m < n$, since the case $m = n$ would go under another label and rather be called a linear isomorphism or automorphism. The set where the projection π is not defined, is the projective linear space $\mathbb{P}(\ker\Phi)$ corresponding to the kernel of Φ. So the punctured *Projection*

4.35 The archetype of a projection is induce by the map $\mathbb{A}^{n+1} \to \mathbb{A}^{m+1}$ that forgets some $n - m$ of the coordinates, say the last ones. The projection is then described by $(x_0 : \ldots : x_{n+1}) \mapsto (x_0 : \ldots : x_m)$, and the common zeros of the components is the linear subspace V where the $m+1$ first coordinates vanish; that is, where $x_i = 0$ for $0 \leqslant i \leqslant m$. The subspac $\mathbb{P}(V)$ is often called the *centre* $\mathbb{P}(V)$ of the projection. *Centre of a projection*

In its simplest form—projection from a point— a projection just forgets one of the coordinates, the archetype being $(x_0 : \cdots : x_n)$ to $(x_0 : \cdots : x_{n-1})$, which is well defined away from the point $(0\colon \ldots : 0\colon 1)$.

4.36 Common practice is to identified the target space $\mathbb{P}^m$ with the linear subvariety of $\mathbb{P}(W)$ of $\mathbb{P}^n$ where $W \subseteq \mathbb{A}^{n+1}$ is given by the equations $x_{m+1} = x_{m+2} = \ldots = x_n = 0$. Notice that this a complementary subspace to the centre V; that is, $k^{n+1} = V \oplus W$.

The geometric interpretation of the projection from a point p onto a $\mathbb{P}(W)$, is as follows. Take a point in $\mathbb{P}^n$ not in the centre $\mathbb{P}(V)$; that is, a one dimensional subspace L of k^{n+1} not lying within V. The subspace $V + L$ of k^{n+1} spanned by V and L intersects W in a line; indeed, this follows from the classical dimension formula from linear algebra which yields, since $W + V + L = k^{n+1}$, that $\dim(V + L) \cap W = \dim(V + L) + \dim W - (n+1) = 1$. And $\pi(L)$ is that intersection.

In particular, if one projects from a point p, the target variety is a hyperplane on which p does not lie, and the image of a point $x \in \mathbb{P}^n$ is the intersection of the line throug x and p with H.

4.37 When one wants to study a variety by means of projections, it is of course of decisive importance to be able to describe the projected variety. It is in general difficult to find the equations of the projected variry in terms of the equations of the variety, this amounts to eliminating the variables that the projection forgets.

However, if the varity is given on parametric form, it is trivial to describe the projection.

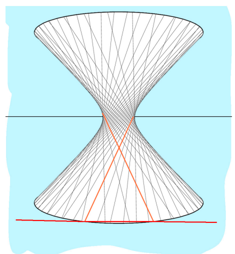
Examples

4.6 *(Projecting the twisted cubic)* The twisted cubic C it the image of $\mathbb{P}^1$ under the map $(u : v) \to (u^3 : u^2v : uv^2 : v^3)$ (which is a morphism according to Lemma 4.31). Consider the projection from $\mathbb{P}^3$ to $\mathbb{P}^2$ with centre $(0 : 0 : 0 : 1)$ which just forgets the last coordinate. The image of a point $(u^3 : u^2v : uv^2 : v^3)$ is $(u^3 : u^2v : uv^2)$ when $u \neq 0$, but when $u = 0$ the point coincides with the centre and the projection is not defined. However, one may discard the common factor u and obtain a genuine parametrization $(u : v) \to (u^2 : uv : v^2)$ of the projection of C. This is the conic $y^2 = xz$. Observe that the projection decreases the degree by one, due to the fact that the centre of projection lies on C.

4.7 We continue with the twisted cubic but change the centre of he projection to $(0 : 0 : 1 : 0)$; that is, the projection forgets the third coordinate. The effect on a point $(u^3 : u^2v : uv^2 : v^3)$ is to send it to the point $(u^3 : u^2v : v^3)$ in $\mathbb{P}^2$. This time the projection is defined all along $\mathbb{P}^1$, and one easily checks that the the equation of the image is $y^2 = x^2z$; that is, the image is the well-known standard cusp. Notice that the degree is conserved, but the image aquires a singular point.

4.8 *(The quadric in $\mathbb{P}^3$)* In this example we project the quadric Q in $\mathbb{P}^3$ with equation $xy - zw = 0$ from the point $p = (0 : 0 : 0 : 1)$, which lies on Q. The lines $z = x = 0$ and $z = y = 0$ both lie on Q, and they intersect in the point p. The entire first line is mapped to the point $p_1 = (0 : y : 0) \in \mathbb{P}^2$ and the second to the point $p_2 = (x : 0 : 0)$. So each of these two lines are collapsed to a point. Off these two lines the projection is one to one. If $q = (x : y : z)$ is a point in $\mathbb{P}^2$ with $z \neq 0$, there is exactly one point on the quadric Q projecting to q the coordinates of which are $(x : y : z : xy/z)$.

To summarize, projecting a quadric Q from a point q on it collapses the two lines L_1 and L_2 on Q passing through q to two different points p_1 and p_2 in $\mathbb{P}^2$. The projection induces an isomorphism from $Q \backslash L_1 \cup L_2$ to $\mathbb{P}^2 \backslash L$; that is, $\mathbb{P}^2$ deprived of the line L through p_1 and p_2, but the image of $Q \backslash \{q\}$, but includes the two points p_1 and p_2 as well.



☆

Problems

4.16 Let the projection $\mathbb{P}^3$ to $\mathbb{P}^2$ be given as $(x : y : z : w) \mapsto (x : x + z : w + y)$. Determine the centre and describe the projection of the twisted cubic parametrized as $(u : v) \mapsto (u^3 : u^2v : uv^2 : v^3)$. Hint: The key words is "rational node" (see Problem 3.8 on page 56).

4.17 Find points in $\mathbb{P}^4$ such that the projection of the rational normal quartic $(u^4 : u^3v : u^2v^2 : uv^3 : v^4)$ projects onto a twisted cubic.

4.18 Describe (by giving an equation) the image of the rational normal quartic under the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ that forgets the third and the forth coordinate. Accomplish the same task but with the projection that forgets the second and the forth coordinate.

4.19 Let C_d be the rational normal curve in $\mathbb{P}^d$ whose parametrization is

$$\phi_d(u : v) = (u^d : u^{d-1}v : \ldots : uv^{d-1} : v^d).$$

Let $\pi\colon \mathbb{P}^d \backslash \{q\} \to \mathbb{P}^{d-1}$ be the projection with centre $q = (0 : 0 : \ldots : 0 : 1)$. Prove that $q \in C_d$ and that the closure in $\mathbb{P}^{d-1}$ of $\pi(C_d \backslash \{q\})$ is equal to C_{d-1}.

★

4.5 *Two important classes of subvarieties*

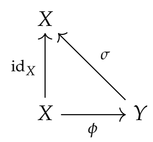
Closed embeddings

A *closed embedding* of a variety X in another variety Y is a morphism $\iota\colon X \to Y$ whose image is closed and which induces an isomorphism between X and its image $\iota(X)$.

Example 4.9 The parametrization of the conic $C = Z_+(y^2 - zx)$ in $\mathbb{P}^2$ is a closed embedding of $\mathbb{P}^1$ in $\mathbb{P}^2$ given as $(u, v) \mapsto (u^2 : uv : v^2)$. Indeed, it has the map that sends $(x : y : z)$ to $(yz^{-1} : yx^{-1})$ as inverse. It is good exercise to check this in detail (use the little Lemma 4.31). ☆

Another simple lemma

The next lemma will be useful at a few occasions when checking that certain morphisms are closed embeddings. The proof is almost trivial and is left as an exercise. It is however a special case of a more general lemma (which we offer as a side dish, see lemma **??** below).



Sections of morphisms

4.38 Given a map $\phi\colon X \to Y$. A map $\sigma\colon Y \to X$ the other way around is said to be a *left section*, for short a section, of ϕ if $\sigma \circ \phi = \mathrm{id}_X$. When ϕ is a morphism, we will also require σ to be a morphism.

Lemma 4.39 *Assume that the morphism $\mathbb{A}^n \to \mathbb{A}^{n+m}$ has a section that is a projection. Then ϕ is a closed embedding.*

Proof: After a coordinate change on $\mathbb{A}^{n+m}$, the map ϕ appears as the graph of a morphism $\mathbb{A}^n \to \mathbb{A}^m$. The details are left to the zealous students. ❑

4.6 The Veronese embeddings

This is a class morphisms of $\mathbb{P}^n$ into a larger projective space $\mathbb{P}^N$ given by all the monomials of a given degree d in $n+1$ variables. There are $\binom{n+d}{d}$ such monomials, so the number N is given as $N = \binom{n+d}{d} - 1$. These Veronese embeddings[6] deserve their name; they are *closed embeddings*. They depend on two natural numbers n and d, and the corresponding embedding will be denoted by $\Phi_{n,d}$, or most often just by Φ with n and d tacitly understood.

[6] They also go under the monstrosity of a name the *d-uple embeddings*.

EXAMPLE 4.10 We already met some morphisms of this type. The parametrizations of the rational normal curves are of this shape with $n = 1$. They are morphisms ϕ from $\mathbb{P}^1$ into $\mathbb{P}^d$ which are expressed as

$$\phi(x_0 : x_1) = (x_0^d : x_0^{d-1}x_1 : \ldots : x_0x_1^{d-1} : x_1^d)$$

in the homogeneous coordinates $(x_0 : x_1)$ of $\mathbb{P}^1$. Conics in $\mathbb{P}^2$ and the twisted cubic for instance, are prominent members of this clan. ☆

EXAMPLE 4.11 The *Veronese surface* is another example of the sort with $n = 2$ and $d = 2$. In this case the embedding of $\mathbb{P}^2$ into $\mathbb{P}^5$ is given as

The Veronese surface (*Veronese-flaten*)

$$\phi(x_0 : x_1 : x_2) = (x_0^2 : x_1^2 : x_2^2 : x_0x_1 : x_0x_2 : x_1x_2)$$

in terms of the homogeneous coordinates $(x_0 : x_1 : x_2)$ on $\mathbb{P}^2$. Notice that the maps in both these examples are morphisms according to the little lemma (Lemma 4.31 on page 72). ☆

The definition

4.40 To fix the notation let $\mathcal{I}$ be the set of sequence $I = (\alpha_0, \ldots, \alpha_n)$ of non-negative integers such that $\sum_i \alpha_i = d$; there are $N+1$ of them. The sequences I from $\mathcal{I}$ will serve as indices for the monomials of degree d; that is, when I runs through $\mathcal{I}$, the polynomials $M_I = x_0^{\alpha_0} \ldots x_n^{\alpha_n}$ run through the monomials of degree d in the x_i's. We let (m_I) for $I \in \mathcal{I}$, in some order, be homogeneous coordinates on the projective space $\mathbb{P}^N$.

4.41 The *Veronese embedding* with parameters n and d is then the map $\Phi_{n,d}\colon \mathbb{P}^n \to \mathbb{P}^N$ that sends the point $x = (x_0 : \ldots : x_n)$ to the point in $\mathbb{P}^N$ whose homogeneous coordinates are given as $m_I(\phi(x)) = M_I(x)$; that is,

Veronese embeddings (*Veronese-embeddingene*)

$$m_I(\Phi_{n,d}(x)) = x_0^{\alpha_0} \ldots x_n^{\alpha_n},$$

when $I = (\alpha_0, \ldots, \alpha_n)$. The monomials M_I are homogeneous of the same degree d and do not vanish simultanously anywhere. The mapping $\Phi_{n,d}$ is a therefore morphism as follows from Lemma 4.31 on page 72. But much more is true:

PROPOSITION 4.42 *The Veronese map $\Phi_{n,d}$ is a closed morphism and induces an isomorphism between $\mathbb{P}^n$ and its image $\Phi_{n,d}(\mathbb{P}^n)$; that is, $\Phi_{n,d}$ is a closed embedding.*

PROOF: To simplify notation we let Φ stand for $\Phi_{n,d}$. There are three salient points in the proof.

Firstly, the basic open subset $D_i = D_+(x_i)$ of $\mathbb{P}^n$ where the i-th coordinate x_i does not vanish, maps into one of the distinguished open subsets of $\mathbb{P}^N$, namely the one corresponding to the pure power monomial x_i^d. To make the notation simpler we let m_i denote the corresponding homogeneous coordinate[7] on $\mathbb{P}^N$; so that Φ maps D_i into $D_+(m_i)$. The first of these distinguished open subsets D_i is isomorphic to $\mathbb{A}^n$, with the fractions $x_j x_i^{-1}$ as coordinates, and the second to $\mathbb{A}^N$ with $m_I m_i^{-1}$ as coordinates.

[7] That is, m_i corresponds to m_I with I being the sequence $I = (0, 0, \ldots, d, \ldots, 0)$ having a d in slot i and zeros everywhere else.

Secondly, although the $n+1$ basic open subsets $D_+(m_i)$ do not cover the entire $\mathbb{P}^N$, they cover the image $\Phi(\mathbb{P}^n)$. Hence it suffices to see that for each index i the restriction $\Phi|_{D_i}$ has a closed image and is an isomorphism onto its image.

The third salient point is that the restrictions $\Phi|_{D_i} \colon \mathbb{A}^n \to \mathbb{A}^N$ have sections that are linear projections. Once this is establishes, we are through in view of Lemma 4.39 above. To exhibit a section, we introduce the n monomials $M_{ij} = x_j x_i^{d-1}$ where $j \neq i$ and we denote the corresponding homogenous coordinates by m_{ij}. Then $m_{ij}(\Phi(x)) m_i(\Phi(x))^{-1} = x_j x_i^{-1}$, and the projection onto the affine space $\mathbb{A}^n$ corresponding to the coordinates $m_{ij} m_i^{-1}$ is a section of the map $\Phi|_{D_i}$, and we are through! ❑

Two corollaries

4.43 Even if the Veronese varieties are specific varieties, they are of general theoretical interest. As an illustration we offer two corollaries.

COROLLARY 4.44 *Let $f(x_0, \ldots, x_n)$ be a non-zero homogeneous polynomial. Then the distinguished open subset $D_+(f)$ of $\mathbb{P}^n$ is affine.*

PROOF: Let d be the degree of f, and let $\Phi = \Phi_{n,d}$ be the Veronese embedding of $\mathbb{P}^n$ in $\mathbb{P}^N$. The point is that the locus $Z_+(f)$ in $\mathbb{P}^n$ becomes a *linear section* in $\Phi(\mathbb{P}^n)$. It will be equal to $Z_+(L) \cap \Phi(P^n)$ where L is the linear expression in the m_I's sharing coefficients with the expression of f in terms of the M_I's; that is, if $f = \sum_{I \in \mathcal{I}} \alpha_I M_I$ one has $L = \sum_{I \in \mathcal{I}} \alpha_I m_I$. Indeed, it then holds true that $f(x_0, \ldots, x_n) = L(x : \ldots : x_n)$, at least up to non-zero scaling.

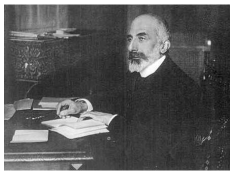
The distinguished open set $D_+(f)$ is the complement of $Z_+(f)$ and hence the intersection of $\Phi(\mathbb{P}^n)$ with the complement of $Z_+(L)$ which is the distinguished open subset $D_+(L)$ of $\mathbb{P}^N$ and isomorphic to $\mathbb{A}^N$. Consequently $\Phi(D_+(f))$, which is isomorphic to $D_+(f)$, is closed in $\mathbb{A}^N$, and hence it is affine. ❑

COROLLARY 4.45 *Let $X \subseteq \mathbb{P}^n$ be a subvariety which is not a point, and let $f(x_0, \ldots, x_n)$ be a homogeneous polynomial. Then $Z_+(f) \cap X$ is not empty.*

PROOF: Assume that $X \cap Z_+(f) = \emptyset$. Then $X \subseteq D_+(f)$; but $D_+(f)$ is affine and therefore X being closed in $D_+(f)$ is affine. So X is both affine and projective. Corollary 4.29 on page 72 applies, and X is a point. ❑

4.7 *The Segre embeddings*

4.46 The second kind of closed embeddings we shall describe are named after one of the great Italian geometers Corrado Segre. They are embeddings of the products $\mathbb{P}^n \times \mathbb{P}^m$ of two projective spaces into the projective space $\mathbb{P}^{nm+n+m}$. These products are thus projective, and subsequently we get the important corollary that products of projective varieties are projective.



Corrado Segre (1863–1924) Italian mathematician.

4.47 With $(x_0 : \ldots : x_n)$ and $(y_0 : \ldots : y_m)$ being homogeneous coordinates on the projective spaces $\mathbb{P}^n$ and $\mathbb{P}^m$ respectively, the *Segre maps* (or the *Segre embeddings* as they also are called as they turn out to be embeddings) are the maps with component functions all the products $x_i y_j$. In other words, they are the maps

Segre maps (Segre-avbildningens)

$$S\colon \mathbb{P}^n \times \mathbb{P}^m \to \mathbb{P}^{nm+n+m}$$

that send the pair $((x_0 : \ldots : x_n), (y_0 : \ldots : y_m))$ to the point whose homogeneous coordinates are the all possible products $x_i y_j$; one thus has

$$S(((x_0 : \ldots : x_n), (y_0 : \ldots : y_m))) = (x_0 y_0 : \ldots : x_i y_j : \ldots : x_n y_m),$$

with the products $x_i y_j$ listed in some order. This definition is legitimate since a simultaneous scaling of either the x_i's or the y_j's results in a simultaneous scaling of the product $x_i y_j$. Moreover, at any point of the product $\mathbb{P}^n \times \mathbb{P}^m$ at least one of the x_i's and one the y_j's do not vanish, and then the corresponding product does not vanish either. That S is a morphism is clear (?). The image of S is called a *Segre varity* and denoted $S_{n,m}$

Segre varities (Segre-variteter)

4.48 The double index invites the represent of the image points as the $(m+1) \times (n+1)$-matrix that is the product of the column vector formed by the y_i's and the row vector formed by the x_j's:

$$(y_0, \ldots, y_m)^t (x_0, \ldots, x_n) = \begin{pmatrix} x_0 y_0 & x_1 y_0 & \ldots & x_n y_0 \\ x_0 y_1 & x_1 y_1 & \ldots & x_n y_1 \\ \vdots & \vdots & & \vdots \\ x_0 y_m & x_1 y_m & \ldots & x_n y_m \end{pmatrix}. \qquad (4.3)$$

and subsequently it is natural to introduce homogeneous coordinates t_{ij} on $\mathbb{P}^{nm+n+m}$ indexed by pairs i, j with $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant m$ and think about the t_{ij} as the entries of a $(m+1) \times (n+1)$-matrix $M = (t_{ij})_{ij}$.

$$M = \begin{pmatrix} t_{00} & t_{10} & \dots & t_{n0} \\ t_{01} & t_{11} & \dots & t_{n1} \\ \vdots & \vdots & & \vdots \\ t_{0m} & t_{1m} & \dots & t_{nm} \end{pmatrix}. \tag{4.4}$$

Obviously the matrix in (4.4) has rank one, and its 2×2-minors are quadric polynomials which all vanish along the image $S_{n,m}$. We shall see that the Segre varieties are precisely the locus where these minors vanish. Even the stronger assertion that the homogeneous prime ideal $I(S_{n,m})$ is generated by the minors holds true, but we shall prove this. Finally, S will be an isomorphism between $\mathbb{P}^n \times \mathbb{P}^m$ and $S_{n,m}$

4.49 The matrix representation places the Segre varieties in a broader perspective. The are members of the larger family of so-called *determinantal varieties*. They are closed subvarieties defined by the minors of the matrix M of fixed given size r r.

Determinantal varieties (*determinatvarieter*)

(See also Example 2.11 on page 32 where we treated the affine case of 2×3-matrices: We embedded $\mathbb{P}^1 \times \mathbb{P}^2$ into $\mathbb{P}^5$.)

4.50

Proposition 4.51 *The Segre map S is a closed embedding of the product $\mathbb{P}^n \times \mathbb{P}^m$ into $\mathbb{P}^{nm+n+m}$. The image of S is the locus where the all the 2×2-minors of the matrix M in (4.4) vanish.*

Proof: Fix a pair of indices μ and ν with $0 \leqslant \mu \leqslant n$ and $0 \leqslant \nu \leqslant m$. On the set $U = D_+(x_\mu) \times D_+(y_\nu)$ the coordinate $t_{\mu\nu} = x_\mu y_\nu$ is non-zero and S sends U into the distinguished open set $D = D_+(t_{\mu\nu})$. It will be sufficient to prove that the restriction $S|_U$ is a closed embedding of U into D. Indeed, by an appropriate choice of coordinates, any pair points in $\mathbb{P}^n \times \mathbb{P}^m$ lies in $D_+(x_\mu) \times D_+(y_\nu)$, and for an injection being a closed embedding is a local question.

Now, U is an affine space $\mathbb{A}^{n+m}$ with coordinates $x_i x_\mu^{-1}$ and $y_j y_\nu^{-1}$ where $0 \leqslant i \leqslant n$ but $i \neq \mu$, and $0 \leqslant j \leqslant m$ but $j \neq \nu$. The distinguished open subset D is the affine space $\mathbb{A}^{mn+n+m}$ whose coordinates are $t_{ij} t_{\mu\nu}^{-1}$. Moreover, the restriction of the Segre map to the distinguished open subset U is given by the relations $x_i x_\mu^{-1} = (x_i y_\nu)(x_\mu y_\nu)^{-1} = t_{i\nu} t_{\mu\nu}^{-1}$ and $y_j y_\nu^{-1} = (x_\mu y_j)(x_\mu y_\nu)^{-1} = t_{j\nu} t_{\mu\nu}^{-1}$, so that the restriction $S|_U$ is a map

$$S|_U \colon \mathbb{A}^{n+m} \to \mathbb{A}^{nm+n+m}$$

having a projection as a section; namely the projection that forgets all coordinates but $t_{i\nu} t_{\mu\nu}^{-1}$ and $t_{\mu j} t_{\mu\nu}^{-1}$ for $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant m$ with $i \neq \mu$ and $j \neq \mu$. By Lemma 4.39 it follows that $S|_{U_{\mu\nu}}$ is a closed embedding and hence that S is one as well. ❑

4.52 A spin off of the proof is that the Segre variety $S_{n,m}$ is equal to the locus where all the 2×2-minors of M vanish. Take a point in $\mathbb{P}^{n+m+nm}$ where the all the 2×2-minors of M vanish, and assume that coordinate $t_{\mu\nu} \neq 0$. Putting $x_i = t_{i\nu} t_{\mu\nu}^{-1}$ and $y_j = t_{\mu j} t_{\mu\nu}^{-1}$ one finds $t_{ij} t_{\mu\nu} = t_{\mu j} t_{\mu\nu}$ and it follows that $t_{ij} t_{\mu\nu}^{-1} = t_{i\nu} t_{\mu\nu}^{-1} t_{\nu j} t_{\mu\nu}^{-1} = x_i y_j$ and our point lies in the image.

Corollary 4.53 *The product of two projective varieties is projective.*

Proof: Let the two projevtive varities be X and Y with X closed in $\mathbb{P}^n$ and Y in $\mathbb{P}^m$. By xxx $X \times Y$ is closed in $\mathbb{P}^n \times \mathbb{P}^m$ which in its turn is isomorphic to a closed subvariety of $\mathbb{P}^{mn+n+m}$ via the Segre embedding. ❑

Example 4.12 *(The quadric in $\mathbb{P}^3$)* The Segre map embeds the product $\mathbb{P}^1 \times \mathbb{P}^1$ into the projective space $\mathbb{P}^3$ as the quadric surface $Z_+(x_0x_3 - x_1x_2)$ (where the x_i's of course are homogeneous coordinates on $\mathbb{P}^3$). Indeed, using homogeneous coordinates $(t_0 : t_1)$ and $(u_0 : u_1)$ on the two factors, the Segre embedding will, after the products $t_i u_j$ have been ordered, send the point $((t_0 : t_1), (u_0 : u_1))$ to the point

$$(t_0u_0 : t_0u_1 : t_1u_0 : t_1u_1).$$

Points of this shape obviously satisfy the relation

$$x_0x_3 - x_1x_2 = (t_0u_0)(t_1u_1) - (t_0u_1)(t_1u_0) = 0. \tag{4.5}$$

and image is contained in the hypersurface $Z_+(x_0x_3 - x_1x_2)$. To see they are equal, let $(x_0 : x_1 : x_2 : x_3)$ be the homogeneous coordinates of a point p in the hypersurface. One of the coordinates x_i must be non-zero, and we may as well assume it to be x_0. Then p is image of the point $q = ((x_0 : x_1), (x_0 : x_2))$; indeed, in view of the relation (4.5) we find

$$S(q) = (x_0^2 : x_1x_0 : x_0x_2 : x_1x_2) = (x_0^2 : x_1x_0 : x_0x_2 : x_0x_3).$$

☆

Problem 4.20 This exercise is about a coordinate free approach to the Segre embeddings. Let V and W be two vector spaces ofer k. Inside $V \otimes_k W$ one has the subset D of decomposable tensors; *i.e.* those of shape $v \otimes w$ with $v \in V$ and $w \in W$, which also are called rank one tensors. Show that D a cone and that the projection of $D \backslash \{0\}$ into $\mathbb{P}(V \otimes_k W)$ is the Segre varaity $\mathbb{P}(V) \times \mathbb{P}(W)$. Hint: Under the isomorphism $\mathrm{Hom}_k(W, V) \simeq W^* \otimes_k V$ linear maps og rank one correspond to decomposable tensors. ★

Lecture 5

Dimension

Hot themes in Lecture 5: *Krull dimension and dimension of spaces—Finite maps, Lying–Over and Going–Up—Noether's Normalization Lemma—Transcendence degree and dimension—The dimension of $\mathbb{A}^n$—Maximal chains in varities—Varieties are catenary—The dimension of a product—Krull's Principal Ideal Theorem—reduction to the diagonal—Intersections in projective space*

For general topological spaces if can be surprisingly subtle to define the concept of dimension, and there is no completely satisfying notion. Manifolds of course, have a dimension (or at least each connected component has). They are locally homeomorphic to open sets in some euclidean space, and the dimension of that space is constant along connected components, and can serve as the dimension of the component.

Noetherian topological spaces as well, have a *dimension*, and in many instances it is very useful. It is fundamental invariant of a variety, equally important as the dimension is for a vector space. General topological spaces may have an infinite dimension, even Noetherian spaces might, but for varieties the dimension behaves well and stays finite.

The notion is inspired by the concept of the *Krull dimension* of a Noetherian ring, and it resembles vaguely one of our naive conception of dimension. For example, in three dimensional geometric gadgets, called threefolds, we may imagine increasing chains of subgadgets of length three; points in curves, curves in surfaces and surfaces in the threefold.

The definition we shall give works for any topological space, but the ensuing dimension carries not much information unless the topology is "Zariski-like".

A very useful tool when establishing the basic theory of dimension is one of Emmy Noehter's grand theorems, the Normalization Lemma which states that every variety is a finite cover of an affine space. Coupled with with the Lying–Over and the Going–Up Theorems of Irvin S. Cohen and Abraham Seidenberg the Normalization Lemma leads to the result that the dimension of a variety X and the transcendence degree of the function field $K(X)$ coincide. We'll formulate and prove the Normalization Lemma in the geometric context we work; that is, over an algebraically closed field. However ita remains true, and the poof is *mutatis mutandis* the same, over any field.

5.1 *Definition of the dimension*

5.1 The deliberations of the introduction materialize in the following definition. In the topological space X we consider strictly increasing chains of non-empty *closed* and *irreducible* subsets:

$$X_0 \subset X_1 \subset \ldots \subset X_r, \tag{5.1}$$

and we call r (that is, the number of inclusions in the chain) *the length* of the chain. The *dimension* $\dim X$ of X is to be the supremum of the set of r's for which there is a chain as in (5.1).

The dimension of a topological space (dimensjonen til et topologisk rom)

One says that the chain is *saturated* if there is no closed irreducible subset strictly in between any two of the X_i's; that is, if Z is a closed and irreducible subset such that $X_i \subseteq Z \subseteq X_{i+1}$, then $Z = X_i$ or $Z = X_{i+1}$. Moreover, we shall call a saturated chain *maximal* if it neither can be lengthened upwards nor downwards. Clearly, the supremum over the lengths of saturated chains, or for that matter, of the maximal chains, will be equal to the dimension of the space. But be aware that maximal chains are not necessarily of maximal length, there might be others that are longer, so their length is not necessarily equal to the dimension of X.

Saturated chains (mettede kjeder)

Maximal chains (maksimale kjeder)

5.2 Possibly the dimension of X can be equal to ∞, and in fact, there are Noetherian spaces for which $\dim X = \infty$, although we shall not meet many. There are even Noetherian rings whose Krull dimension is infinite; the first example was constructed by Masayoshi Nagata, the great master of counterexamples in algebra.

Masayoshi Nagata (1927–2008) Japanese mathematician

EXAMPLE 5.1 One do not need to go far to find Noetherian spaces of infinite dimension. The following weird topology on the set $\mathbb{N}$ of natural numbers is one example. The closed sets of this topology apart from the empty set and the entire space, are the sets defined by $Z_a = \{ x \in \mathbb{N} \mid x \leqslant a \}$ for $a \in \mathbb{N}$. They form a strictly ascending infinite chain and are irreducible, hence the dimension is infinite. On the other hand, any strictly descending chain is finite so the space is Noetherian. We leave it as an exercise for the interested student to check these assertions. ☆

Problems

5.1 The notion of dimension we introduced is only useful for "Zariski-like" topologies. Show that any Hausdorff space is of dimension zero. HINT: What are the irreducible subsets?

5.2 Show that the only irreducible and finite topological space of dimension one is the so-called Sierpiński space which is named after the Polish mathematician Wacław Sierpiński. It has two points η and x with $\{\eta\}$ open and $\{x\}$ closed.

5.3 Assume that $Y = Y_1 \cup \ldots \cup Y_r$ is the decomposition of the Noetherian space Y into irreducible components. Show that $\dim Y = \max \dim Y_i$.

5.4 *(Nagata's example.)* Let $B = k[x_1, x_2, \ldots]$ be the polynomial ring in countably many variables, and decompose the set of natural numbers as a union $\mathbb{N} = \bigcup_i J_i$ of disjoint finite sets J_i whose cardinality tends to infinity with i.

Let $\mathfrak{n}_i$ be the ideal in B generated by the x_j's for which $j \in J_i$, and let S the multiplicative closed subset $\bigcap_i B \backslash \mathfrak{m}_i$ of B. Moreover, let $\mathfrak{m}_i$ be $\mathfrak{n}_i A$. Nagata's examples is the localized ring $A = S^{-1}B$, and the aim of the exercise is to prove that A is Noetherian, but of infinite Krull dimension.

We shall need the rational function field $K_i = k(x_j | j \notin J_i)$ in the variables whose index does not lie in J_i, and the polynomial ring $K_i[x_j | j \in I_i]$ over K_i in the remaining variables; that is, those x_j for which $j \in J_i$. Moreover, the ideal $\mathfrak{a}_i$ will be the ideal in $K_i[x_j | j \in I_i]$ generated by the latter; that is, $\mathfrak{a}_i = (x_j | j \in J_i)$.

a) Show that $B_{\mathfrak{n}_i} \simeq K_i[x_j | j \in I_i]_{\mathfrak{a}_i}$.

b) Prove that $A_{\mathfrak{m}_i} = B_{\mathfrak{n}_i}$ and conclude that each $A_{\mathfrak{m}_i}$ is Noetherian with $\dim A_{\mathfrak{m}_i} = \# J_i$ and hence that $\dim A = \infty$.

c) Show that A is Noetherian. HINT: Any ideal is contained in finitely many of the of the $\mathfrak{m}_i$'s, and therefore finitely generated.

★

Basic properties of dimension

5.3 One immediately establishes the following basic properties.

LEMMA 5.4 *Assume that X is a topological space and that $Y \subseteq X$ is a closed subspace. Then $\dim Y \leqslant \dim X$. Assume furthermore that Y is irreducible and that X is of finite dimension. If $\dim Y = \dim X$, then Y is a component of X.*

PROOF: Any chain as (5.1) in Y will be one in X as well; hence $\dim Y \leqslant \dim X$. For the second assertion, assume that $\dim Y = \dim X = r$, and let

$$Y_0 \subset Y_1 \subset \ldots \subset Y_r = Y$$

be a maximal chain in Y. In case Y is not a component of X, there is a closed and irreducible subset Z of X strictly bigger than Y, and we can extend the chain to

$$Y_0 \subset Y_1 \subset \ldots \subset Y_r \subset Z.$$

Hence $\dim X \geqslant r + 1$, and we have a contradiction. ❑

5.5 Our concept of dimension coincides, when X is a closed irreducible subset of $\mathbb{A}^m$, with the Krull dimension of the coordinate ring $A(X)$. Indeed, the correspondence between closed irreducible subsets of X and prime ideals in $A(X)$ implied by the Nullstellensatz, yields a bijective correspondence between chains

$$X_0 \subset X_1 \subset \ldots \subset X_r$$

of closed irreducible subsets and chains

$$I(X_r) \subset \ldots \subset I(X_1) \subset I(X_0)$$

of prime ideals in $A(X)$. Hence the suprema of the lengths in the two cases are the same, and we have:

Proposition 5.6 *Let $X \subseteq \mathbb{A}^m$ be a closed algebraic subset. Then* $\dim X = \dim A(X)$.

Wolfgang Krull (1899–1971) German mathematician

Given that the polynomial ring $k[x_1, \ldots, x_n]$ is of Krull dimension equal to n, we know that $\dim \mathbb{A}^n = n$. This is of course what it should be, but it is astonishingly subtle to establish, and this reflects the fact that if R is a ring which is not Noetherian, the polynomial ring $R[t]$ may have a Krull dimension other than $\dim R + 1$. It holds tue that $\dim R + 1 \leqslant \dim R[t] \leqslant 2 \dim R + 1$ and there are examples showing that all possible values occur.

We shall give a proof that $\dim \mathbb{A}^n = n$ depending on Noether's Normalization Lemma; see Theorem 5.34 on page 94 below.

5.7 Dense open subsets do not necessarily have the same dimension as the surrounding space even when the surrounding space is irreducible, but it cannot be bigger:

Proposition 5.8 *Assume that X is an irreducible topological space and that U is an open dense subset. Then* $\dim U \leqslant \dim X$.

Proof: We are to see that $\dim U \leqslant \dim X$, so let

$$U_0 \subset U_1 \subset \ldots \subset U_r$$

be a chain of closed irreducible subsets of U. By Lemma 2.10 on page 24 the closures $\overline{U_i}$ are irreducible closed subsets of X and they satisfy $\overline{U_i} \cap U = U_i$. Hence the chain $\{\overline{U_i}\}$ form a strictly ascending chain of closed irreducible sets in X, and it follows that $r \leqslant \dim X$. Hence $\dim U \leqslant \dim X$ since the chain was arbitrary. ❏

5.9 In general strict inequality might be a in force in Lemma 5.8. The Sierpiński space from Problem 5.2 on the previous page is a stupidly simple example. It has merely two points, η which is open, and x which is closed. Clearly $\{\eta\}$ is an open dense subset of dimension zero, whereas the Sierpiński space itself is of dimension one since it has the maximal chain $\{x\} \subseteq X$ of closed irreducible subsets. Typically, strict inequality in 5.8 occurs when all the chains of maximal length are concealed from U by lying in the complement.

However, the situation is satisfactory for for varieties. As we shall establish later, their dimension coincides with the transcendence degree of their function field, and consequently equality prevails in Proposition 5.8; open dense subsets have the same function field as the surrounding variety (Corollary 5.36 on page 95).

5.10 The following little easy lemma turns out to be useful at a few occations.

LEMMA 5.11 *Assume that X is an irreducible topological space and that*

$$Z_0 \subset Z_1 \subset \ldots \subset Z_r$$

is a chain of irreducible closed subsets in X of length r. If U is a non-empty open subset of X such that $U \cap Z_0 \neq \emptyset$ the the intersections $\{Z_i \cap U\}$ form a chain in U of length r.

PROOF: The sets $Z_i \cap U$ are irreducible (Lemma 2.10 on page 24) and closed in U. Of course it holds true that $Z_i \cap U \subseteq Z_{i+1} \cap U$, and the assertion of the lemma amounts to these inclusions being strict. Now, $Z_i \cap U$ is open in Z_i, and by Lemma 2.10 again, it is dense whenever non-empty. But since U by assumption meets the smallest of the Z_i's, it meets all, and we conclude that $\overline{Z_i \cap U} = Z_i$. From this ensues that each $Z_i \cap U$ is strictly contained in $Z_{i+1} \cap U$; were they equal, the closures would be equal as well, but they are not. ❑

We do not yet know that open subsets of a variety X have the same dimension as X, but the lemma tells us that if $\dim X < \infty$, there are as least some open affine subsets with the same dimension as X. Indeed, any open affine meeting the smallest member of a chain of maximal length will do. In (the hypothetical) case that X is of infinite dimension, one may similarly find open affine subsets of arbitrary large dimension.

PROBLEM 5.5 Find an example of an irreducible topological space X other than the Sierpiński space that has an irreducible open subset U so that $\dim U < \dim X$. ★

The height of a prime ideal

5.12 Recall that if A is a ring and $\mathfrak{p} \subseteq A$ is a prime ideal, the *height* of $\mathfrak{p}$ is the length r of the longest strictly increasing chain of prime ideals *The height of an ideal*

$$\mathfrak{p}_0 \subset \ldots \subset \mathfrak{p}_r = \mathfrak{p}$$

ending at $\mathfrak{p}$. It equals the Krull dimension of the localization $\dim A_{\mathfrak{p}}$.

5.13 For any ring one has the inequality

$$\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} \leqslant \dim A. \tag{5.2}$$

Any saturated chain of prime ideals in A where the prime ideal $\mathfrak{p}$ occurs, can be split in two. The first part consists of the ideals in the chain contained in $\mathfrak{p}$ (this includes $\mathfrak{p}$ itself); and the supremum of lengths of such chains is $\dim A_{\mathfrak{p}}$. The second part consists of the remaining ideals; that is, those containing $\mathfrak{p}$ (so $\mathfrak{p}$ apperas in both parts), and the supremum of the lengths of those chains equals $\dim A/\mathfrak{p}$.

5.14 In many good cases there is even an equality

$$\dim A_{\mathfrak{p}} + \dim A/\mathfrak{p} = \dim A. \tag{5.3}$$

However to establish such an equality is slightly subtle—it requires that $\mathfrak{p}$ occurs in a chain of prime ideals in A of maximal length. Even for local Noetherian integral domains this is not necessarily true (the first example was found by Masayoshi Nagata). These examples are exotic creatures living on the fringe of the Noetherian society. One would hardly meet them practising mainstream algebraic geometry, and in the world of varieties they are absent.

As we shall prove later on, the equality (5.3) holds true for prime ideals in the coordinate rings of affine (irreducible) varieties. Even more is true, all maximal chains of closed irreducible subsets in an (irreducible) variety will have the same length. However, if the closed algebraic set X has irreducible components of different dimensions, the equality (5.3) does trivially not hold for all prime ideals in $A(X)$. The dimension $\dim X$ being the larger of the dimensions of the components, chains in the smaller components will be shorter than $\dim X$ (see problems 5.6 and 5.7 below).

Problems

5.6 Let $X = Z(zx, zy) \subseteq \mathbb{A}^3$. Describe X and determine $\dim X$. Exhibit two maximal chains of irreducible subvarieties of different lengths. Exhibit a hypersurface Z so that $Z \cap X$ is of dimension zero.

5.7 Let $X = Z(xy, y(y-1)) \subseteq \mathbb{A}^2$. Describe all chains of subvarieties in X.

5.8 Let $A = k[x_1, x_2, x_3]$ and let $\mathfrak{p}$ and $\mathfrak{q}$ be the two primeideals $\mathfrak{p} = (x_1)$ and $\mathfrak{q} = (x_2, x_3)$. Let S multiplicative system $S = A \backslash (\mathfrak{p} \cup \mathfrak{q})$. Show that $B = A_S$ is a Noetherian semi-local domain with the two maximal ideals $\mathfrak{m} = \mathfrak{p} A_s$ and $\mathfrak{n} = \mathfrak{q} A_S$. Show that $\dim A_{\mathfrak{m}} = 1$ and $\dim B_{\mathfrak{n}} = 2$, and conclude that A is a Noetherian domain with two maximal chains whose lengths differ.

★

5.2 *Finite polynomial maps*

In the midst of this chapter devoted to dimension we insert a section about finite morphisms or the slightly more general concept of finite polynomial maps. These maps play a significant role when the theory of dimension is developed, in that two varieties related by a finite morphism have the same dimension.

Images and fibres

5.15 Let $\phi\colon X \to Y$ be a polynomial map between closed algebraic sets (for some unclear reason the notion *morphism* is reserved for polynomial maps between varieties). To understand a map it is of course important to understand the fibres, and the following lemma gives a simple criterion for a point to lie in a given fibre.

LEMMA 5.16 *Let $\phi\colon X \to Y$ be a polynomial map between the two closed algebraic sets X and Y and let $x \in X$ and $y \in Y$ be two points. Then $\phi(x) = y$ if and only if $\phi^*\mathfrak{m}_y \subseteq \mathfrak{m}_x$.*

PROOF: One has $\phi(x) = y$, if and only if $f(\phi(x)) = 0$ for all $f \in \mathfrak{m}_y$; that is, if and only if $\phi^*(f) = f \circ \phi \in \mathfrak{m}_x$ for all $f \in \mathfrak{m}_y$. ❑

In other words, the fibre $\phi^{-1}(y)$ of ϕ over the a point $y \in Y$ is the closed algebraic set in X given by the ideal $\phi^*\mathfrak{m}_y$. The fibre can of course be empty, in which case $\phi^*\mathfrak{m}_y = A(X)$, and the ideal $\phi^*\mathfrak{m}_y$ need in general not be radical.

Problems

5.9 Let $\phi\colon \mathbb{A}^1 \to \mathbb{A}^1$ be the map $\phi(t) = t^n$. For each point $a \in \mathbb{A}^1$ determine the ideal $\phi^*\mathfrak{m}_a$ and the fibre $\phi^{-1}(a)$.

5.10 Let $\psi\colon \mathbb{A}^2 \to \mathbb{A}^2$ be the map $\psi(x,y) = (x,xy)$. Determine the ideals $\psi^*\mathfrak{m}_{(a,b)}$ and the fibres $\psi^{-1}(a,b)$ for all points $(a,b) \in \mathbb{A}^2$.

5.11 Let $\psi\colon \mathbb{A}^3 \to \mathbb{A}^3$ be given as $(x,y,z) \mapsto (yz, xz, xy)$. Find all fibres of ψ and their ideals.

★

Dominant maps

Morphisms between varieties whose image is *dense* in the target, are called *dominant*. They are somehow easier to handle than general morphisms between varieties, and often proofs are reduced to the case of dominant maps.

Dominant morphisms (dominerende avbildninger)

5.17 Suppose that X and Y are varieties and that $\phi\colon X \to Y$ is a dominant morphism. Let f be a regular function on Y that does not vanish identically. Then we may find an open dense set in U in Y where f does not have any zeros. Since by assumption the image $\phi(X)$ is dense in Y, the intersection

$U \cap \phi(X)$ is non-empty, and it ensues that $f \circ \phi$ does not vanish identically on $\phi^{-1}(U)$. In other words, the composition map $\phi^*\colon A(Y) \to A(X)$ is *injective*. This leads to

Lemma 5.18 *A morphism $\phi\colon X \to Y$ between affine varieties is dominant if and only if the corresponding homomorphism $\phi^*\colon A(Y) \to A(X)$ is injective.*

Proof: Half the proof is already done. For the remaining part, suppose that the image $\phi(X)$ is not dense. Then its closure Z in Y is a proper closed subset, and $I(Z)$ is a non-zero ideal. Any function f in $I(Z)$ vanishes along $\phi(X)$, and hence $\phi^*(f) = f \circ \phi = 0$. ❑

Finite maps

5.19 A polynomial map $\phi\colon X \to Y$ between two closed algebraic sets X and Y is said to be *finite* if the composition map $\phi^*\colon A(Y) \to A(X)$ makes $A(X)$ into a finitely generated $A(Y)$-module.

Finite polynomial maps (*endelige avbildninger*)

More generally a morphism $\phi\colon X \to Y$ between two varieties is said to be *finite* if every point $y \in Y$ has an affine neighbourhood U such that the inverse image $\phi^{-1}(U)$ is affine and the restriction $\phi|_{\phi_{-1}(U)}$ is a finite polynomial map. One easily checks that the composition of two finite morphisms (or polynomial maps) is finite.

Finite morphisms (*endelige morfismer*)

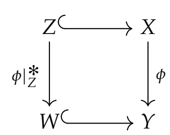
5.20 Finite morphisms have the virtue of being closed, and hence they are surjective when they are dominating. Equally important, their fibres are finite. Moreover, as alluded to above, two affine varieties which are related by a dominant finite morphism, have the same dimension.

5.21 The famous triplet of theorems of Irvin Cohen and Abraham Seidenberg—with the suggestive names Lying–Over, Going-Up and Going–Down—are results about integral extensions of rings, and because finite extensions of rings are integral, they imply at once the results about finite maps we are about to present. These strong theorems are valid in a much broader context than ours. Integral extensions are by no means always finite, just think about the integral closure of the integers $\mathbb{Z}$ in the field $\overline{\mathbb{Q}}$ of algebraic numbers. Even the integral closure of a Noetherian domain in its fraction field needs not be a finite module[1]. However, in our modest context of finite polynomial maps, no heavier artillery than Nakayama's lemma is needed.

[1] Finally, it turns out that the integral closure of a domain of finite type over a field is finite.

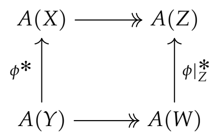
5.22 Here comes the basic property of finite ploynomial maps. It is frequently referred to as Going–Up although Lying–Over would be the proper name in the Cohen–Seidenberg nomenclature.

Proposition 5.23 (Lying–Over) *Let $\phi\colon X \to Y$ be a finite polynomial map. Then ϕ closed. If it is dominating, it is surjective.*



PROOF: We begin with proving that ϕ is surjective when it is dominating. So assume there is a y in Y not belonging to the image of ϕ. Then by Lemma 5.16 above, it holds true that $\mathfrak{m}_y A(X) = A(X)$. Now $A(X)$ being finite as an $A(Y)$-module, it follows from Nakayama's lemma that $A(X)$ is killed by an element of the shape $1 + a$ with $a \in \mathfrak{m}_y$. The assumption that ϕ be dominant ensures that ϕ^* is injective, and since $0 = (1 + a) \cdot 1 = \phi^*(1 + a)$, it follows that $a = -1$. This is absurd because a belongs to $\mathfrak{m}_y$ which is a proper ideal.

To see that ϕ is a closed map, let $Z \subseteq X$ be closed, and decompose Z into its irreducible components $Z = Z_1 \cup \ldots \cup Z_r$. Then the image $\phi(Z)$ satisfies $\phi(Z) = \phi(Z_1) \cup \ldots \cup \phi(Z_r)$, and it suffices to show that each $\phi(Z_i)$ is closed. That is, we may assume that Z is irreducible. Define W to be the closure of $\phi(Z)$, and observe that the restriction $\phi|_Z \colon Z \to W$ is a dominating and finite map (any generating set for $A(X)$ over $A(Y)$ reduces to one for $A(Z)$ over $A(W)$). Hence, by the first part of the proof, it is surjective. In other words, $\phi(Z) = W$, and $\phi(Z)$ is closed. ❑



5.24 Not only are finite maps surjective, but any closed irreducible subset of the target is dominated by a a closed irreducible subset of the source

PROPOSITION 5.25 (GOING–UP) *let $\phi \colon X \to Y$ be a dominating morphism between two varieties and let $Z \subseteq Y$ be a closed and irreducible subset. Then there exists a closed and irreducible subset $W \subseteq Y$ such that $\phi(W) = Z$.*

PROOF: Consider $\phi^{-1}(Z)$ which is non-empty since ϕ is surjective (proposition 5.23) and let $W_1, \ldots, W_r$ be its components. Again by Proposition 5.23 the images $\phi(W_i)$ are closed and of course, their union equals Z. Since Z is assumed to be irreducible, it ensues that for at least one index i it holds that $\phi(W_i) = Z$. ❑

LEMMA 5.26 *Let $\phi \colon X \to Y$ be a dominating finite morphism between varieties, and suppose that $Z \subset X$ is a proper and closed subset. Then $\phi(Z)$ is a proper subset of Y.*

PROOF: The proof is easily reduced to the case that both X and Y are affine. Assume that $\phi(Z) = Y$ and let f be any regular function on X vanishing along Z. Since ϕ^* makes $A(X)$ a finitely generated module over $A(Y)$, there is a relation

$$f^r + \phi^*(a_{r-1})f^{r-1} + \ldots + \phi^*(a_1)f + \phi^*(a_0) = 0$$

where the a_i's are elements of $A(Y)$, and we may assume that r is the least integer for which there is such a relation. Obviously the relation implies that $\phi^*(a_0) = a_0 \circ \phi$ vanishes along Z, but since $\phi(Z)$ is equal to Y, the composition map $\phi|_Z^*$ is injective, and hence $a_0 = 0$. The integer r being minimal and $A(X)$ being an integral domain, we conclude that $f = 0$, and $Z = X$. ❑

5.27 We have now come to the result that justifies the little excursion into the world of finite maps in the middle of a chapter dedicated to the Krull-dimension. We do not yet know that varieties are of finite dimension, so some care must be taken to include the case of (the *a posteriori* non-exsitent) infinite dimensional varieties, and we resort to Noetherian induction.

Proposition 5.28 (Going–Up II) *Let $\phi\colon X \to Y$ be a finite and dominating morphism between varieties. Then* $\dim X = \dim Y$.

Proof: To begin with we take any chain

$$W_0 \subset W_1 \subset \ldots \subset W_r \tag{5.4}$$

in X and push it down to Y with the help of ϕ. Each $\phi(W_i)$ is irreducible and closed in Y after Lying–Over, and Lemma 5.28 ensures that strict inclusions are preserved. Hence

$$\phi(W_0) \subset \phi(W_1) \subset \ldots \subset \phi(W_r)$$

is a chain of closed irreducible subsets of Y of length r. Taking the supremum of lengths of chains as (5.4), gives $\dim X \leqslant \dim Y$. To establish the reverse inequality, we shall lift chains in Y to chains in X by recursively climbing down[2] a given chain. Let a chain

$$Z_0 \subset Z_1 \subset \ldots \subset Z_r \tag{5.5}$$

in Y be given, and suppose we have found a chain

$$W_\nu \subset W_{\nu+1} \subset \ldots \subset W_r$$

in W with $\phi(W_i) = Z_i$. The restriction $\phi_\nu = \phi|_{W_\nu}$ is a finite map from W_ν to Z_ν and after Going–Up (Proposition 5.25) there is a closed irreducible subset of $W_{\nu-1}$ of X_ν such that $\phi_\nu(W_{\nu-1}) = Z_{\nu-1}$. In this way every chain (5.5) can be lifted to a chain of the same length, and we conclude that $\dim Y \leqslant \dim X$ ❑

[2] It may sound paradoxical that one uses Going–Up to climbe down, but it comes from the transition between ideals and subvarieties reversing inclusions.

Problems

5.12 We shall come back to a closer analysis of the fibres of finite polynomial maps, but for the moment we content ourselves with this exercise. Let $\phi\colon X \to Y$ be a finite morphism (or polynomial map). Show that all fibres of ϕ are finite. Hint: Pick a point y in Y and argue that the ring $A(X)/\mathfrak{m}_y A(X)$ is a finite dimensional vector space over k hence has only finitely many maximal ideals.

5.13 Show that the composition of two finite morphisms (or polynomial maps) is finite. Show that the composition of two dominant composable morphisms of varieties is dominant.

★

5.3 Noether's Normalization Lemma

The Normalization Lemma

We now turn to one of the most famous results of Emmy Noether's, her so-called Normalization Lemma. We shall stated in our context of varieties which means for algebras over an algebraically closed field. The proof however, works *mutatis mutandis* for any domain of finite type over any infinite field, and in fact, this general version will be useful for us at a later occasion. There is also a slight twist to the proof below making it valid over finite fields as well (which we shall not need).

5.29 The proof of the Normalization Lemma is an inductive argument, and the basic ingredient is the induction step as formulated in the following lemma:

LEMMA 5.30 *Let $X \subseteq \mathbb{A}^m$ be an affine variety whose fraction field $K(X)$ has transcendence degree at most $m-1$; then there is a linear projection $\pi\colon \mathbb{A}^m \to \mathbb{A}^{m-1}$ so that $\pi|_X\colon X \to \mathbb{A}^{m-1}$ is a finite morphism.*

PROOF: Let $A(X) = k[T_1, \ldots, T_m]/I(X)$ be the coordinate ring of X and denote by T_i the image of T_i in $A(X)$. Since the transcendence degree of $A(X)$ over k is less than m, the m elements $t_1, \ldots, t_m$ can not be algebraically independent and must satisfy an equation

$$f(t_1, \ldots, t_m) = 0,$$

where f is a polynomial with coefficients in k. Let d be the degree of f and let f_d be the homogeneous component of degree d. Put $u_i = t_i - a_i t_1$ for $i \geqslant 2$ where the a_i's are scalars to be chosen. This gives[3]

$$0 = f(t_1, \ldots, t_m) = f_d(1, a_2, \ldots, a_m)t_1^d + Q(u_1, \ldots, u_m)$$

where Q is a polynomial whose terms all are of degree less that d in t_1. Now, since the ground field is infinite, a generic choice of the scalars a_i implies that $f_d(1, a_2, \ldots, a_m) \neq 0$ indeed, the complement of $Z(f_d(1, t_2, \ldots, t_m))$ in $\mathbb{A}^{m-1}$ is even dense (see exercise for the case that k is merely assumed to be infinite 5.14 below). Hence the element t_1 is integral over $k[u_2, \ldots, u_m]$ and by consequence, $A(X)$ is a finite module over the algebra $k[u_2, \ldots, u_m]$. The projection $\mathbb{A}^m \to \mathbb{A}^{m-1}$ sending $(t_1, \ldots, t_m)$ to $(u_2, \ldots, u_m)$ does the trick. ❏

[3] Recall that for any polynomial $p(x)$ it holds true that $p(x+y) = p(x) + yq(x,y)$ where q is a polynomial of total degree less than the degree of f.

PROBLEM 5.14 Let k be an infinite field and $f(t_1, \ldots, t_n)$ a non-zero polynomial with coefficients from k. Show that $f(a_1, \ldots, a_n) \neq 0$ for infinitely many choices of a_i from k. HINT: Use induction on n and expand f as $f(t_1, \ldots, t_n) = \sum_i g_i(t_1, \ldots, t_{n-i})t_n^i$. ★

5.31 By induction on m one obtains the full version of the Normalization Lemma:

Theorem 5.32 (Noether's Normalization Lemma) *Assume that $X \subseteq \mathbb{A}^m$ is a closed subvariety and that the function field $K(X)$ is of transcendence degree n over k. Then there is a linear projection $\pi\colon \mathbb{A}^m \to \mathbb{A}^n$ such that the projection $\pi|_X\colon X \to \mathbb{A}^n$ is a finite map.*

Proof: We keep the notation from the lemma with the coordinate ring of X being $A(X) = k[T_1, \ldots, T_m]/I(X)$ and the t_i's being the images of the T_i's in $A(X)$, and proceed by induction on m. If $m \leqslant n$, the elements $t_1, \ldots, t_m$ must be algebraically independent since they generate the field $K(X)$ over k. But any non-zero polynomial in $I(X)$ would give a dependence relation among them, so we infer that $I(X) = 0$, and hence that $X = \mathbb{A}^m$.

Suppose then that $m > n$. By Lemma 5.30 above, there is a finite projection $\phi\colon X \to \mathbb{A}^{m-1}$. The image $\phi(X)$ is closed by Proposition5.23 on page 90 and of the same transcendence degree as X since $K(X)$ is a finite extension of $K(\phi(X))$. Applying the induction hypothesis to $\phi(X)$, we may find a finite projection $\pi\colon \phi(X) \to \mathbb{A}^n$. The composition $\pi \circ \phi$ is then a finite map $X \to \mathbb{A}^n$. ❑

5.33 An important corollary of the Normalization lemma is that the dimension of a variety coincides with the transcendence degree of its rational function field over the ground field.

Theorem 5.34 *Let X by any variety. Then* $\dim X = \operatorname{trdeg}_k K(X)$.

In particular, the theorem states that the affine n-space $\mathbb{A}^n$ is of dimension n. Indeed, the function field $K(\mathbb{A}^n)$ of the affine space is the field of rational function $k(x_1, \ldots, x_n)$ in n variables which is of transcendence degree n over k. We may also infer that the dimension of X is finite which was not *a priori* clear. However, the field K(X) is a finitely generated extension of the ground field k and we know *a priori* that the transcendence degree $\operatorname{trdeg}_k K(X)$ is finite.

Proof: There are two parts of the proof; the case of $\mathbb{A}^n$ and the general case, and the latter is easily reduced to former bu way of the Normalization Lemma and the Going–Up Theorem. Indeed, replacing X by some open dense and affine subset that has the same dimension as X (which exists according to Lemma 5.11 on page 87), we may assume that X is affine. Let $n = \operatorname{trdeg}_k K(X)$. By the Normalization Lemma there is a finite map $X \to \mathbb{A}^n$, hence $\dim X = \dim \mathbb{A}^n = n$ in view of the Going–Up Theorem.

The case of $\mathbb{A}^n$ is done by induction on n; obviously it holds that $\mathbb{A}^1$ is one dimensional (the ring $k[t]$ is a PID). So assume that $n > 1$ and let $Z \subset \mathbb{A}^n$ be a maximal proper and closed subvariety sitting on top of a chain of maximal length. Then $\dim Z = \dim \mathbb{A}^n - 1$ and $\operatorname{trdeg}_k K(Z) \leqslant n - 1$ because $I(Z) \neq 0$. Noether's Normalization Lemma gives us a finite and dominating morphism $Z \to \mathbb{A}^m$, where $m = \operatorname{trdeg}_k K(Z)$. By induction it holds true that $\dim \mathbb{A}^m = m$

and thus $\dim Z = \operatorname{trdeg}_k K(Z)$. This yields

$$\dim Z = \dim \mathbb{A}^n - 1 = \operatorname{trdeg}_k k(Z) \leqslant n - 1,$$

and therefore $\dim \mathbb{A}^n \leqslant n$. The other inequality is trivial; there is an obvious ascending chain of linear subspaces of length n in $\mathbb{A}^n$. ❑

5.35 As promised in Pargraph 5.9 we now can give the following:

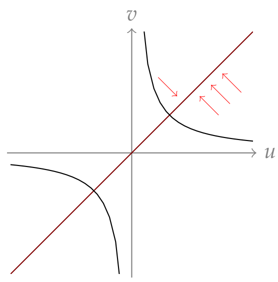
COROLLARY 5.36 *Any dense open subsets of a variety has the same dimension as the surrounding variety.*

PROOF: The variety and the open subset have the same function field. ❑

EXAMPLE 5.2 A good illustration of the the perturbation process that the proof of the Normalization Lemma is based on, is the classical hyperbola X with equation $uv = 1$ in the affine plane $\mathbb{A}^2$. The coordinate ring of X equals $k[u,v]/(uv-1)$ which may be identified with the extension $k[u,1/u]$ of $k[u]$, the hyperbola being the graph of the function $1/u$. The inclusion $k[u] \subseteq k[u,1/u]$ corresponds to the projection of X onto the u-axis.

This map is not finite although its non-empty fibres consist of one point; indeed, any relation $1/u^n = \sum_{i<n} f_i(u)/u^i$ with $f_i \in k[u]$ would result in the relation $1 = \sum_{i<n} f_i(u)u^{n-1}$ whose right side vanishes for $u = 0$.

However, perturbing u slightly, we obtain a subring over which $k[u,1/u]$ is finite. The subring $k[u+1/u]$ will do the job; indeed, $k[u,1/u] = k[u,u+1/u]$ is generated by u over $k[u+1/u]$ and one has the integral dependence relation



$$u^2 - u(u + 1/u) + 1 = 0.$$

that shows that it is generated by 1 and u.

It is remarkable that almost *any* perturbation of u will work; that is, $k[u,1/u]$ is finite over $k[au + b/u]$ as long as both the scalars a and b are non-zero. ☆

PROBLEM 5.15 Show that $k[u,1/u]$ is a finite module over $k[au + b/u]$ for any scalars a and b both being different from zero. ★

Maximal chains in varieties

Our second application of the Noether's Normalization Lemma we is to establish that the Krull dimension of varieties, in contrast to that of many other rings, behave descently in that all maximal chains have the same length. In particular they will be catenary; all saturated chain connecting to given irreducible closed subsets have the same length.

5.37 We start out by preparing the gorund with two lemmas. The first asserts the highly expected fact that hypersurfaces in affine space $\mathbb{A}^n$ are of codimension one, a forerunner of the general Hauptidealsatz of Krull's. In particular they are maximal, closed irreducible and proper subsets.

Lemma 5.38 *The zero locus $X = V(f)$ in $\mathbb{A}^n$ of an irreducible polynomial f is of dimension $n-1$.*

Proof: The coordinate ring $A(X)$ is given as $A(X) = k[T_1, \ldots, T_n]/(f)$ and the function field $K(X)$ as $K(X) = k(t_1, \ldots, t_n)$ with the relaison $f(t_1, \ldots, t_n) = 0$ holding among the t_i's (with our usual convention that lower case letters denote the classes of the upper case versions in force). At least one of the variables occurs in f, and we may as well suppose it is T_1. Thence $t_2, \ldots, t_n$ will be algebraically independent over k, and consequently we have $\operatorname{trdeg}_k K(X) = n-1$. Indeed, any polynomial $g(T_2, \ldots, T_n)$ that satisfies $g(t_2, \ldots, t_n) = 0$, is a multiple of f and must therefore depend on T_1. ❑

Lemma 5.39 *Let X be a variety and Z a maximal proper irreducible subset. Then* $\dim Z = \dim X - 1$

Proof: By Noether's Normalization Lemma there is a finite surjective morphism $\pi\colon X \to \mathbb{A}^n$ with $n = \dim X$. The image $\pi(Z)$ is irreducible and closed (by Lying–Over), and we content that it is a maximal proper subset of this kind. Indeed, if the closed and irreducible subset W were strictly contained between $\pi(Z)$ and $\mathbb{A}^n$ our set Z would be contained in one of the components of $\pi^{-1}(W)$, say W_0. By Lying–Over there is no inclusion relation between closed irreducible sets that have the same image under a finite map; hence W_0 lies strictly between Z and X which contradicts the hypothesis that Z is maximal. Hence $\pi(Z)$ is maximal in $\mathbb{A}^n$ and of dimension $n-1$ by Lemma 5.38 above. We finish the proof by Lying–Over which asserts that $\dim Z = \dim \pi(Z)$ since the restriction $\pi|_Z$ is finite. ❑

5.40 With the two previous lemmas up our sleeve the main theorem of the section is easy to prove.

Theorem 5.41 *All maximal chains in a variety are of same length.*

Proof: The proof goes by induction on the dimension of X (which we have shown is finite) and the case $X = 0$ is trivial. Let $Z_r \subset X$ be the largest member of a maximal chain

$$Z_0 \subset Z_1 \subset \ldots \subset Z_{r-1} \subset Z_r \subset X$$

of length r in X. The top member Z_r is a maximal proper closed subvariety of X, and therefore $\dim Z = \dim X - 1$ after Lemma 5.39 above. The induction hypothesis then takes effect and implies that $\dim Z_r = r-1$, and we can conclude that $\dim X = r$. ❑

5.42 Localization of catenary rings are catenary so that rings of essentially finitey type over k are catenary. In particular, such rings that are local will have maximal chains all of the same length. However, if the rings not local this not true any more. Problem xxx describes a semi-local ring with two maximal ideals, one height one and the other of height two.

The dimension of a product

The Normalization Lemma also gives an easy proof of the formula for the dimension of a product. It hinges on the fact that the product of two finite maps is finite, and by The Normalization Lemma the proof is then reduced to the case of two affine spaces.

PROPOSITION 5.43 *Let X and Y be two varieties. Then* $\dim X \times Y = \dim X + \dim Y$.

LEMMA 5.44 *Let X, Y, Z and W be affine varieties. Let $\phi\colon X \to Y$ and $\psi\colon Z \to W$ be two finite morphisms. Then the morphism $\phi \times \psi\colon X \times Z \to Y \times W$ is finite.*

PROOF: We first establish the lemma in the special case when $W = Z$ and $\psi = \mathrm{id}_Z$. In that case the map $(\phi \times \mathrm{id}_Z)^*\colon A(Y)\otimes A(Z) \to A(X)\otimes A(Z)$ is just $\phi^* \otimes \mathrm{id}_{A(Z)}$. If $a_1, \ldots, a_r$ are elements in $A(X)$ that generates $A(X)$ as an $A(Y)$-module, the elements $a_i \otimes 1$ generates $A(X)\otimes A(Z)$ as a module over $A(Y)\otimes A(Z)$, and we are through.

One reduces the general case to this special case by observing that $\phi \times \psi$ is equal to the composition

$$X \times Z \xrightarrow{\phi\times \mathrm{id}_Z} Y \times Z \xrightarrow{\mathrm{id}_Y \times \psi} Y \times W,$$

and using that the composition of two finite maps is finite. ❑

PROOF OF PROPOSITION 5.43: By replacing X by an open affine subset U and Y by an open affine subset V we assume that X and Y affine; indeed $U \times V$ is dense in $X \times Y$, and dense open subsets have the same dimension as the surrounding variety (Corollary 5.36 on page 95).

So assume that X and Y are affine. According to The Normalization Lemma there are finite and surjective maps $\phi\colon X \to \mathbb{A}^n$ and $\psi\colon Y \to \mathbb{A}^m$ with $n = \dim X$ and $m = \dim Y$. Then $\phi \times \psi\colon X \times Y \to \mathbb{A}^n \times \mathbb{A}^m$ is finite by lemma 5.44 above, and it is clearly surjective, hence $\dim X \times Y = n + m$ after Going–Up (Proposition 5.28 on page 92). ❑

5.4 *Krull's Principal Ideal Theorem*

5.45 This is another great German theorem, whose native name is *Krull's Hauptidealsatz*, but unlike the Nullstellensatz, it is mostly referred to by its English name in the Anglo-Saxon part of the world. The simplest version concerns the dimension of the intersection of a hypersurface with a variety X in $\mathbb{A}^m$, and confirms the intuitive belief that the hypersurface cuts out a space in X of dimension one less than $\dim X$. This statement must be taken with a grain of salt since the intersection could be empty, and of course, the variety X could be entirely contained in the hypersurface in which case the intersection equals X, and the dimension does not drop. If X is not irreducible, the situation is somehow more complicated. The different components of X can be of different dimensions and they may or may not meet the hypersurface.

5.46 The Hauptidealsatz applies to general Noetherian ring which are enormulsy more delicate beings than the ones we meet in the world of varieties. We state it in tis generality, but shall only prove a milder version relevant for varieties.

Theorem 5.47 (The Principal Ideal Theorem) *Let A be Noetherian ring and let $f \in A$ be a non-zero element which is not a unit. Then the height of a minimal prime of the principal ideal (f) is a most one.*

There is standard proof found in most text books on commutative algebra; for instance in Peskine's book[4] (Lemma 10.22 on page 138). There is also nice 4
proof of quite another flavour in Kaplansky's text[5] (xxx). 5

The Geometric Principal Ideal Theorem

5.48 We proceed directly to the version of the Principal Ideal Theorem most actual for us. In his colourful and beautiful book Red Book[6] David Mumford 6
gives a simple and straightforward proof of Krull's Principal Ideal Theorem in the context of varieties. It relies on Noether's Normalisation lemma and uses of *the norm* to relate hypersurfaces in the source and the target of a finite map between affine varieties; a reminiscence of the so-called elimination theory which was high fashion a century or so ago. Mumford attributes the proof to John Tate. Since student sare supposed to have a background in commutative algebra, we present this proof only in an Appendix, but here comes the theorem:

Theorem 5.49 (Geometric Principal Ideal Theorem) *Let X be a variety and let f be regular function on X that does not vanish identically. If $Z(f)$ is not empty, it holds true for every component Z of $Z(f)$ that* $\dim Z = \dim X - 1$.

$$\begin{array}{ccc} B & \subset & L \\ \cup & & \cup \\ A & \subset & K \end{array}$$

5.50 The theorem asserts that irreducibel hypersurfaces are of codimension one, but be aware that the converse is not true in general. There are plenty varieties having irreducible subvarieties of codimension one that are not hypersurfaces. For example in the coordinate ring $A = k[x, y, z, w]/(xy - zw)$ of $X = Z(xy - zw)$ in $\mathbb{A}^4$ one has the primary decomposition

$$(x) = (x, z) \cap (x, w),$$

so that $Z(x) \cap X$ is the union of the two planes $Z(x, w)$ and $Z(x, z)$. One may prove that any hypersurface containing either of those planes can not irreducible (see Exercise 5.16 below). The point is that A is not a UFD. From Kaplansky's criterion that a domain is a UFD if and only if "every prime (ideal) contains a prime (element)" one infers that in any variety whose coordinate ring is not UFD, one may find irreducible subsets of codimension one that are not hypersurfaces. Indeed, in the coordinate ring there will be prime ideals of height one that are not principal.

5.51 There is no statement as clear and uniform as in Theorem 5.49 valid for closed algebraic sets X (which may be reducible). It is not very difficult to exhibit examples of situations where the codimension of $Z(f)$ in X is equal to any prescribed number.

Problems

5.16 Refering to the staging in Paragraph 5.50 above, let $X \subseteq \mathbb{A}^4$ be $X = Z(xy - zw)$. Show that for any hypersurface $Z(f)$ in $\mathbb{A}^4$ containing the plane $Z(x, w)$, the intersection $X \cap Z(f)$ is reducible. Clearly Z(x,w) is a component of $Z(f) \cap X$, the point is to show that there are others.

5.17 Let $X \subseteq P^n$ be a closed subvariety and assume that the cone $C(X)$ is a UFD. Show that any $Z \subseteq X$ of codimension one is of shape $Z(f) \cap X$ for some hypersurface $Z(f)$ in $\mathbb{P}^n$.

5.18 Give examples of a closed algebraic set and a regular function f on X such that the codimension of $Z(f)$ in X is equal to any prescribed number. HINT: Use disjoint unions.

★

The case of several functions vanishing

5.52 The Principal Ideal Theorem is about the dimension of the locus with one constraint; that is, the intersection of a variety X with one hypersurface. However, it generalizes to the intersections of a variety X with a sequence of hypersurfaces, or in a slightly more general staging, to the locus where several regular functions vanish. Since imposing the vanishing of each one of the functions increases the codimension with at most one, induction on the number of functions easily gives the following:

THEOREM 5.53 *Suppose that X is a variety and that $f_1, \ldots, f_r$ are regular functions on X. Then every component Z of the zero locus $Z(f_1, \ldots, f_r)$ is of codimension at most r in X; that is* $\dim Z \geqslant \dim X - r$.

PROOF: The proof goes by induction on r. Let W be a component of the locus $Z(f_1, \ldots, f_{r-1})$ that contains Z. By induction W is of codimension at most $r - 1$ in X; that is, $\dim W \geqslant \dim X - r + 1$. Moreover, Z must be a component of of $W \cap Z(f_r)$, and therefore either f_r vanishes identically on W or $\dim W =$

$\dim Z - 1$ by the Principal Ideal Theorem (Theorem 5.49 above). In the latter case obviously $\dim Z \geqslant \dim X - r$, and in the former, we find $Z = W$ and $\dim Z = \dim X - r + 1 \geqslant \dim X - r$. ❑

System of parameters and fibres of morphisms

5.54 In commutative algebra one has the notion of a *system of parameters* in a local ring A. If the Krull dimension of A is n and the maximal ideal is $\mathfrak{m}$, such a system is a sequence of elements $f_1, \ldots, f_n$ of n elements in $\mathfrak{m}$ that generate a $\mathfrak{m}$-primary ideal; or expressed in symbols, such that $\sqrt{(f_1, \ldots, f_n)} = \mathfrak{m}$.

system of parameters (*parametersystemer*)

5.55 Translating this into geometry we let X be an affine variety of dimension n and A the local ring $\mathcal{O}_{X,x}$ of X at a point $x \in X$. The elements $f_1, \ldots, f_n$ are regular functions on X that vanish at x, and requiring them to constitute a system of parameters in $\mathcal{O}_{X,x}$ is to ask that $\sqrt{(f_1, \ldots, f_n)\mathcal{O}_{X,x}} = \mathfrak{m}_x\mathcal{O}_{X,x}$.

By the Nullstellensatz this is equivalent to asking that in a neighbourhood of x the only common zero of the f_i's is the point x. The zero locus $Z(f_1, \ldots, f_n)$ thus has an irredundant decomposition into irreducibles shaped as $Z(f_1, \ldots, f_n) = \{x\} \cap Z_1 \cap \ldots \cup Z_r$, or in other words, x is an isolated point of the zero set $Z(f_1, \ldots, f_n)$.

5.56 A careful application of Krull's Principal Ideal Theorem yields that for affine varieties system of parameters are always exatnt.

Proposition 5.57 *Let X be an affine variety of dimension n and $x \in X$ a point. Then there exists regular functions $f_1, \ldots, f_n$ on X such that x is an isolated point in $Z(f_1, \ldots, f_n)$.*

Proof: We shall recursively construct a sequence of regular functions $f_1, \ldots, f_n$ on X all vanishing at x so that for all ν with $1 \leqslant \nu \leqslant n$, every component of $X_\nu = Z(f_1, \ldots, f_\nu)$ that contains x is of codimension ν . Clearly this suffices to establish the theorem; indeed, when $\nu = n$ that statement reads: all components of $Z(f_1, \ldots, f_n)$ containing x are of dimensions zero; hence there can only be one component which must be equal to $\{x\}$.

Assume then that the functions $f_1, \ldots, f_\nu$ are found and let $Z_1, \ldots, Z_r$ be the components of $X_\nu = Z(f_1, \ldots, f_\nu)$ that contain x. Let furthermore $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the corresponding prime ideals in $A(X)$. When $\nu < n$ every one of the Z_i's is of dimension at least one, and the $\mathfrak{p}_i$'s are strictly contained in $\mathfrak{m}_x$. Citing the prime avoidance lemma we infer that $\mathfrak{p}_1 \cup \ldots \mathfrak{p}_\nu \subsetneq \mathfrak{m}$, and hence we may find functions $f_{\nu+1}$ vanishing at x, but which do not vanish identically along any of the Z_i's. It follows from the Principal Ideal Theorem that all components of $Z(f_{\nu+1}) \cap Z_i$ are of codimension one in Z_i; that is, they are all of codimension $\nu + 1$ in X. ❑

Example 5.3 At any singularity it is (by definition) impossible to find a system of parameters that generates the maximal ideal; the ubiquitous example being the plane cusp C given by the equation $y^2 - x^3 = 0$. In the coordinate ring $A = k[x,y]/(y^2 - x^3)$ it holds $y^2 = x^3$ and the maximal ideal $\mathfrak{m}$ of the origin is (x,y). One finds $\mathfrak{m}^2 = (x^2, xy, y^2) = (x^2)$.

Any function g suspect of being a generator for $\mathfrak{m}$ can not vanish to the second order at the origin, and hence it must be congruent $ax + by$ modulo $\mathfrak{m}^2$ for some scalars a and b not both being zero. Thence $(g) = (ax + by)$ in $A/\mathfrak{m}^2$.

In case a is non-zero, we may as well assume that $b = 1$. When also $b \neq 0$, we find $(y + ax) = (x^3 + ax) = (x^2(x + a))$, and locally near the origin, that is in the local ring $\mathcal{O}_{C,O}$ at the origin, the function $x + a$ is invertible and the ideal $(y + ax)$ becomes the ideal (x^2). Hence $(g)\mathcal{O}_{C,O} = \mathfrak{m}^2\mathcal{O}_{C,O}$. The other cases when either a or b vanishes, are left to the zealous students. ☆

PROBLEM 5.19 Assume that $f_1, \ldots, f_n$ is a system of parameters at the point x of the affine variety X. Show that any component Z of $Z(f_1, \ldots, f_r)$ has codimension r for any $1 \leqslant r \leqslant n$ (regardless of the construction in the proof). Show that any sequence of regular functions $f_1, \ldots, f_r$ that vanish at x and with all components Z of $Z(f_1, \ldots, f_r)$ satisfying $\operatorname{codim} Z = r$, can be completed to a system of parameters at x. ★

5.58 According to Theorem 5.53 above, the fibre over the origin of a dominant map $\phi \colon X \to \mathbb{A}^r$ has all its components of dimension at least equal to $\dim X - r$. Indeed, the map has regular functions $f_1, \ldots, f_r$ as components, the fibre over the origin is just $Z(f_1, \ldots, f_r)$ and then 5.53 gives $\dim X - \dim Z \leqslant r$.

This observation can be generalised to fibres over any point of any variety, based on the fact that affine varieties unconditionally possess systems of parameters at all points, and yields the estimate below. Strict inequality commonly occur, but merely for special points belonging to a "small" subset. In xxx we shall give a more precise statement

PROPOSITION 5.59 *Let $\phi \colon X \to Y$ be a dominant morphism of varieties. For every point x in Y and every component Z of the fibre $\phi^{-1}(x)$ it holds true that*

$$\dim Z \geqslant \dim X - \dim Y.$$

PROOF: Replacing Y by a neighbourhood of x need is, we may assume that Y is affine. Let $r = \dim Y$. In Proposition 5.57 on the preceding page we showed that affine varieties have a systems of parameters at every one of their points; hence there are regular functions $f_1, \ldots, f_r$ on Y such that x is isolated in $Z(f_1, \ldots, f_r)$. By further shrinking Y, we may assume that $\{x\} = Z(f_1, \ldots, f_r)$. The fibre is then described as $\phi^{-1}(x) = Z(f_1 \circ \phi, \ldots, f_r \circ \phi)$, and by Krull's Principal Ideal Theorem every one of its components have a codimension at most equal to r; that is,

$$\dim X - \dim Z \leqslant r = \dim Y,$$

and this gives the inequality in the proposition. ❏

Problem 5.20 Show that $x_0 - x_1, x_2, x_3$ is a system of parameters at the origin for $Z(x_0x_1 - x_2x_3)$. ★

Problem 5.21 Gerenarlize the previous exercise in the following direction. Let $X \subseteq \mathbb{P}^n$ be a subvariety of dimension d. Assume that $L_1, \ldots, L_{d+1}$ are linear forms whose intersection is a subspace of codimension $d+1$ that does not meet X. Show that the L_i's form a system of parameters for the cone $C(X)$ at the origin. Finally, prove that such linear forms always can be found. ★

Problem 5.22 Assume that X is a (irreducible) variety and that Y is a curve. Show that all components of all fibres of a dominant morphism $\phi \colon X \to Y$ are of codimension one in X. ★

5.5 *Applications to intersections*

5.60 The Principal Ideal Theorem has some important consequences for the intersections of subvarieties both in the affine space $\mathbb{A}^n$ and the projective spaces $\mathbb{P}^n$. It allows us to give upper estimates for the dimension of an intersection of two closed subvarieties expressed in terms of their dimensions.

And in the projective case, it also ensures that the intersection is non-empty once a natural condition on the dimensions of the two is fulfilled.

A head on application of the Principal Ideal Theorem is futile since varieties require several more equations than there codimension indicates, however an interplay with a fabulous trick called the "Reduction to the diagonal" paves the way.

5.61 The most striking result is that the intersection of two closed subvarieties of $\mathbb{P}^n$ will be non-empty once their dimensions comply to the following very natural condition. If X and Y designate the two subvarieties, then $X \cap Y \neq \emptyset$ once

$$\operatorname{codim} X + \operatorname{codim} Y \leqslant n. \tag{5.6}$$

One can even say more, for any component Z of the intersection $X \cap Y$, the following inequality holds

$$\operatorname{codim} Z \leqslant \operatorname{codim} X + \operatorname{codim} Y. \tag{5.7}$$

For intersections of subvarities of the affine space $\mathbb{A}^n$ a similar inequality holds true, but under the assumption that the intersection is non-empty; so in that case, no common point is garanteed.

Problem 5.23 Give examples of projective varieties X and Y not satisfying he inequality (5.6) and having an empty intersection. Hint: Take two linear subvarieties $\mathbb{P}(V)$ and $\mathbb{P}(W)$ of $\mathbb{P}^n$ with $\dim \mathbb{P}(V) + \dim \mathbb{P}(W) < n$ (*e.g.* two skew lines in $\mathbb{P}^3$). ★

PROBLEM 5.24 Give examples of two closed subvarieties in affine space $\mathbb{A}^n$ satisfying 5.6 but having an empty intersection. ★

Reduction to the diagonal

5.62 The trick named "Reduction to the diagonal" is based the following observation. Let X and Y be two subvarieties of $\mathbb{A}^n$. The product $X \times Y$ lies of course as a closed subvariety of the affine space $\mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$, and clearly the subset $X \cap Y$ is equal to the intersection $\Delta \cap X \times Y$, where Δ is the diagonal in $\mathbb{A}^n \times \mathbb{A}^n$. Moreover, it is not difficult to check that the two closed algebraic sets are isomorphic; either use their respective defining universal properties or resort to considering the defining ideals.

The salient point is that the diagonal is cut out by a set of very simple equations. If the coordinates on corresponding to the left factor in $\mathbb{A}^n \times \mathbb{A}^n = \mathbb{A}^{2n}$ are $\{x\}_i$ and those of the right factor $\{y_i\}$ the diagonal is given by the vanishing of the n functions $x_i - y_i$. Hence we can conclude by Krull's Principal Ideal Theorem that any (non-empty) component Z of $X \cap Y$ satisfies $\dim Z \geqslant \dim X \times Y - n$ but $\dim X \times Y = \dim X + \dim Y$ and we find

$$\dim Z \geqslant \dim X + \dim Y - n.$$

Summing up we formuate the result as a lemma

PROPOSITION 5.63 *Let X and Y be two subvarieties of $\mathbb{A}^n$ then any (non-empty) component of the intersection $X \cap Y$ satifies*

$$\operatorname{codim} Z \leqslant \operatorname{codim} X + \operatorname{codim} Y.$$

Of course, it might very well happen that $X \cap Y$ is empty, even for hypersurfaces. As well, the strict inequality might hold; for example it could happen that $X = Y$!

EXAMPLE 5.4 An inequality as in 5.63 does hold for subvarieties of general varieties. For example, the two planes $Z_1 = Z(x, y)$ and $Z_2 = Z(z, w)$ in $\mathbb{A}^4$ intersect only in the origin. They are both contained in the quadratic cone $X = Z(xz - yw)$ which is three-dimesional. Considered as subvarieties of X they are of codimension one, but their intersection just being the origin, is of codimension three. ☆

The projective case

5.64 The proof of the intersection theorem for projective space applies the affine version to the affine cones over the involved varieties. We therefore begin with a few observations about them. The natural equality $C(X \cap Y) = C(X) \cap C(Y)$ is obvious, and if Z is a component of the intersection $X \cap Y$, the cone $C(Z)$ will be a component of $C(X \cap Y)$. Passing to cones increases the dimensions by one; that is, for any variety X it holds that $\dim C(X) = \dim X + 1$. Then of course, it holds true that $\operatorname{codim}_{\mathbb{P}^n} X = \operatorname{codim}_{\mathbb{A}^{n+1}} C(X)$; that is, the codimenion of X in $\mathbb{P}^n$ is the same as the codimension of its cone in $\mathbb{A}^{n+1}$. And thirdly, the most salient point is that intersection of cones always is non-empty; they meet at least in the origin.

5.65 The following theorem is one of the cornerstones in projective geometry. Whether two varieties intersect or not is as much a question of their size as of their relative position: If they are "large enough", they intersect.

Proposition 5.66 *Let X and Y be two projective varieties in the projective space $\mathbb{P}^n$. Assume that $\dim Y + \dim X \geqslant n$. Then the intersection $X \cap Y$ is non-empty, and any component Z of $X \cap Y$ satisfies*

$$\operatorname{codim} Z \leqslant \operatorname{codim} X + \operatorname{codim} Y.$$

Proof: Firstly, if $\dim X + \dim Y \geqslant n$ then $\dim C(X) + \dim C(Y) \geqslant n + 2$ and, as already noticed, the salient point is that the intersection $C(X) \cap C(Y)$ is always non-empty: The two cones both contain the origin! Moreover, the dimension of any component W of $C(X) \cap C(Y)$ satisfies $\dim W \geqslant \dim C(X) + \dim C(Y) - n - 1 = \dim X + \dim Y - n + 1 \geqslant 1$, and one deduces that the intersection $C(X) \cap C(Y)$ is not reduced to the origin, and hence is the cone over a non-empty subset in $\mathbb{P}^n$.

Since the cone over over a projective variety and the variety itself have the same codimension, respectively in $\mathbb{P}^n$ and $\mathbb{A}^{n+1}$, we deduce directly from Proposition 5.63 that

$$\operatorname{codim} Z \leqslant \operatorname{codim} X + \operatorname{codim} Y.$$

❏

Problems

5.25 Given an example of a projective variety W of a given arbitrary dimension and two subvarieties X and Y of W with empty intersection, but which satisfy

$$\dim X + \dim Y = \dim W.$$

5.26 Let $X = Z(\mathfrak{p})$ and $Y = Z(\mathfrak{q})$ be two closed subsets of the algebraic set W and let ι_X and ι_Y denote the inclusion maps. Show that the intersection $X \cap Y = Z(\mathfrak{p} + \mathfrak{q})$ is characterised by the universal property that a pair of polynomial maps $\phi_X \colon Z \to W$ and $\phi_Y \colon Z \to W$ factors through $X \cap Y$ if and only if $\iota_X \circ \phi_X = \iota_Y \circ \phi$.

5.27 Give a categorical proof of the isomorphism between $X \cap Y$ and $\Delta \cap X \times X$; that is, a proof only relying on universal properties (and hence is valid in any category where the involved players exist).

5.28 Give a more mundane proof the isomorphism between $X \cap Y$ and $\Delta \cap X \times X$ using the ideals of the involved varieties.

★

5.6 *Appendix: Proof of the Geometric Principal Ideal Theorem*

The proof use the norm, so we begin with a general property related to finite extension, principal ideal and norms.

5.67 For any finite extension $A \subseteq B$ of domains with A integrally closed in its fraction field, there is a multiplicative map $N\colon B \to A$ called the *norm* (see Appendix 5.68 on the following page). If K and L designate the fraction fields of respectively A and B, the norm $N(f)$ is the determinant of the K-linear map $L \to L$ just being multiplication by f; *i.e.* it sends a to fa. We shall need three of its basic properties.

- The norm is multiplicative: $N(fg) = N(f)N(g)$;
- For elements $f \in A$ it holds that $N(f) = f^{[L:K]}$;
- f is a factor of $N(f)$; that is $N(f) = bf$ for some $b \in B$.

5.68 The proof of Principal Ideal Theorem the hinges upon the following lemma:

Lemma 5.69 *In the setting just described, it holds that $\sqrt{f} \cap A = \sqrt{N(f)}$ for any element $f \in B$,*

Proof: In view of f being a factor of $N(f)$, one inclusion is obvious, namely that $\sqrt{N(f)} \subseteq \sqrt{f} \cap A$. For the other, assume that $g \in \sqrt{f} \cap A$; that is $g \in A$ is an element is on the form $g^s = af$ for some $a \in B$. Then $g^{ns} = N(g^s) = N(a)N(f)$, and hence g belongs to $\sqrt{N(f)}$. ❑

Proof of the Principal Ideal Theorem: We begin with proving the theorem under the additional assumption that $Z(f)$ is irreducible, and subsequently reduce the general theorem to that case.

Citing the Normalizaton Lemma and putting $n = \dim X$, there is finite surjective map $\pi\colon X \to \mathbb{A}^n$ which on the level of coordinate rings is manifested as a finite extension $A(\mathbb{A}^n) \subseteq A(X)$. Let $N\colon A(X) \to A(\mathbb{A}^n)$ be the corresponding norm map.

The map π clearly sends $Z(f)$ into $Z(N(f))$, and the crux of the proof is that $Z(f)$ dominates $Z(N(f))$. Once this is established, Going–Up and Lemma 5.38 above about hypersurfaces in affine spaces yield that $\dim Z(f) = \dim Z(N(f)) = n - 1$, and we shall be through. To prove that $\pi|_{Z(f)}$ is dominating, it suffices to show that the corresponding map

$$(\pi|_{Z(f)})^*\colon A(\mathbb{A}^n)/(\sqrt{N(f)}) \to A(X)/(\sqrt{f})$$

between the coordinate rings is injective (Lemma 5.18 on page 90), but this is to say that $\sqrt{f} \cap A(\mathbb{A}^n) = \sqrt{N(f)}$, which is exactly the assertion in Lemma 5.69 above.

So to the reduction. Decompose $Z(f)$ into the union of the irreducible components

$$Z(f) = Z_1 \cup \ldots \cup Z_r,$$

and let Z_ν be one of the them. Pick an affine open suset U of X so that $U \cap Z_i = \varnothing$ for $i \neq \nu$ but $Z_\nu \cap U \neq \varnothing$. Clearly the restriction $f|_U$ persists being neither invertibel nor indetically zero, and obviously $Z(f|_U) = Z(f) \cap U = Z_\nu \cap U$ is irreducible. In view of the first part, the equalities $\dim U = \dim X$ and $\dim Z_\nu = \dim Z_n \cap U$ then accomplish the proof. ❑

Appendix appendicularis: The trace and the norm

5.70 The given in this appendix is a finite extension of domains $A \subseteq B$ with A being integrally closed in its fraction field K. The fraction field L of B is then a finite extension of K. Every element in L has a minimal polynomial which us the monic polynomial of leats degree having f as a root. But just like linear operators, it also has a characteristic polynomial, whose coefficients are interesting invariants of the element. Especially the sub-leading (the leading coefficient is one) and the constant term, up to sign, they are trace and the norm of the elements.

5.71 So let f be an eøement in L. It induces a K-linear map $[f]\colon L \to L$ simply by multiplication; that is, the map that is given as $[f](b) = fb$. Of course, when f belongs to B, the subring B of L is invariant under $[f]$, and $[f]$ becomes an A-linear endomorphism of B.

The field L is a finite dimensional vector space over K, and the multiplication map $[f]$ being K-linear, it has a characteristic polynomial:

$$P_f(t) = \det(t \cdot I - [f]) = t^n + a_{n-1}t^{n-1} + \ldots + a_1 t + a_0$$

The coefficients of $P_f(t$ lie in K, and the degree equals the degree $[L : K]$ of the field extension. The coefficients a_{n-1} and a_0 attract special attention and turn out to particular useful. The standard notation is $\operatorname{tr} f = -a_{n-1}$ and $N(f) = \det[f] = (-1)^n a_0$, and yhey are called respectively the *trace* and the *norm* of the element f. Both the trace and the norm depend on the extension $K \subseteq K$, and when emphasis on the extension is required one writes $\operatorname{tr}_{L/K}(f)$ and $N_{L/K}(f)$ for the trace and the norm.

The trace of elements (sporet til et element)

The norm (normen til elementer)

5.72 Well-known properties of linear maps translate into the basic properties of the the norm and the trace

PROPOSITION 5.73 *One has*

- ❑ *The norm is multiplicative; that is* $N(f \cdot g) = N(f) \cdot N(g)$ *and* $N(1) = 1$. *If* $f \in K$, $N(d) = f^{[L:K]} \cdot N(g)$

❏ *The trace is K linear, and* $\operatorname{tr}(1) = [L : K]$.

PROOF: Since obviously $[fg] = [f] \circ [g]$ the norm is multiplicative, and if $f \in K$, the multiplication map $[f]$ is just f times the identity id_L. The trace is as alway the sum the giagonal elements of the matrix of $[f]$ in any K-basis for L grom which the addititivity ensues, and also that the trace of the identiy equals the dimension of L over K. ❏

In terms of eigenvalues (in some extension field of K) the trace equals the sum and the norm the product.

LEMMA 5.74 *When A is Noetherian and integrally closed in K, the characteristic polynomial of* $[f]$ *has coefficients in A whenever* $f \in B$. *In particular, the norm* $N(f)$ *and the trace* $\operatorname{tr} f$ *of an element from B belongs to A.*

PROOF: If B is a free A-module, this is pretty clear since in that case L has a basis over K consisting of elements from B which also constitute a basis for B over A. The matrix M of $[f]$ with respect to that basis has entries in A and hence the determinant $\det(t \cdot I - M)$ belongs to the polynomial ring $A[T]$. Now, the remaining two salient points of the proof are firstly that A equals the intersection of all the DVR's in K shaped like $A_{\mathfrak{p}}$ where $\mathfrak{p}$ is a height one prime in A, and secondly, that any torsion free module over a DVR is free. ❏

The three properties of the norm described in Paragraph 5.67 on page 105 are just translations of properties of the determinant well known from linear algebra. The determinant is multiplicative and hence the same holds for the norm; *i.e.* $N(fg) = N(f)N(g)$. When the element f belongs to A, the multiplication map $[f]$ is just $f \cdot \mathrm{id}_L$ whose matrix in any basis is diagonal with all entries along the diagonal equal to f, and it holds that $\det[f] = f^{[L:K]}$. For the third feature in Paragraph 5.67, observe that if $P(T) = T^n + a_{n-1}T^{n-1} + \ldots + a_1T + (-1)^n N(f)$ is the characteristic polynomial of $[f]$, the Caley–Hamilton theorem yields

$$N(f) = (-1)^{n+1} \cdot f \cdot (f^{n-1} + a_{n-1}f^{n-2} + \ldots + a_1). \tag{5.8}$$

Separable and unseparable

We shall work in general situation where the polynomials have coefficients in an arbitrary field K. Alølthough our ground field k is algebraically closed, most of the application of this section will be to *e.g.* functions fields K(X) of varieties, which are far from being algebraically closed. So we considers the roots of $f(t)$ in an algebraic closure $\bar{K}$ of K

In the litterature a polynomial is said to be *separable* if the roots (in $\bar{K}$) are distinct — in a rather vague etymology the usage is explained by that they can be separated.

There is standard way of detecting multiple roots by use of the derivative. A root a of the polynomial $f(t)$ is not simple precisely when the derivative $f'(t)$ vanishes at a as well. Indeed, one may write $f(t) = (t-a)^m g(t)$ where $g(a) \neq 0$. Leibnitz' rule then yields

$$f'(t) = m(t-a)^{m-1} + (t-a)^m g'(t),$$

and we see that $f'(a) = 0$ if and only if $m \geqslant 2$.

A non-constant *irreducible* polynomial $f(t)$ cannot share a common factor with any other polynomial which is not a scalar multiple $af(t)$. In particular if a is a multiple root, $(x-a)$ would be a common factor of f and f', And unless f' vanishes identically, this is impossible since their degrees are different.

Over fields of caracteristic zero the only polynomials with vanishing derivative are the constants, so in that case an irreducible f has distinct roots and is hence separable. However, when K has characteristic p, this is not true any more. For instance, the power t^p has derivative pt^{p-1} which vanishes since $p = 0$ in K. More generally one finds applying the chain rule that

$$(f(t^{p^\nu}))' = f'(t^{p^\nu}) \cdot p^\nu t^{p^v - 1} = 0.$$

Hence polynomials shaped like $f(t^{p^m})$ have vanishing derivatives. And in fact these are all.

Lemma 5.75 *Let $f(t)$ be a polynomial in $K[t]$ and assume that $f'(t)$ is the zero polynomial. Then K is of charavcteristic p and $f(t) = g(t^{p^\nu})$ for some $g \in K[t]$ and $\nu \in \mathbb{N}_0$.*

Proof: Write $f(t) = \sum_{i \in I} a_i t^i$ where $I \subseteq \mathbb{N}$ are the indices with $a_i \neq 0$. Then $f'(t) = \sum_{1 \leqslant i \leqslant n} i \cdot a_i t^{i-1}$, and since powers of t are linearly independent, it follows that $ia_i = 0$ for all $i \in I$. Hence $i = 0$, that is i is divisible by p and we can write $i = p^{\nu_i} m_i$. Letting ν be the smaller of the ν_i's and $g(t) = \sum a_i t^{p^{\nu_i - \nu} m_i}$ or $a_i = 0$. ❑

Proposition 5.76 *If $k \subseteq L$ is a separable extesnion, thet trace $\operatorname{tr}\colon L \to K$ is surjective. Henve the bilonear form $\operatorname{tr}(xy)$ is non degenerate on K.*

Proof: By the Primitive Element Theorem there is an element a such that $L = K(a)$. Let $Q(t)$ be the minimal polynomial so that $L = K[t]/(Q(t))$. Separability means that the roots of $Q(t)$ in sn slrgsntriic clodsure $\bar{K}$ are distinct (they can be "separated"). It follows that $(Q(t)) = (t-\beta_1) \cap \ldots (t - b_n)$ and The Chinese Remainder theorem gives an isomorphism a

$$L \otimes_K \bar{K} = \bar{K}[t]/(Q(t)) \simeq \prod_i \bar{K}.$$

Now, the multiplication map $[f]$ is a $\bar{K}$ map of $L \otimes_K \bar{K}$, and a basis $\{e_i\}$ induces a basis $e_i \otimes 1$ of $L \otimes_K \bar{K}$, so and the matrix of $[f]$ in the two are of course equal, and the trace $\mathrm{tr}_{L/K} \otimes id = \mathrm{tr}_{L \otimes_K \bar{K}/\bar{K}}$ is same whether f is considered a map of K or of $L \otimes_K \bar{K}$.

But the chinese basis shows that ❑

Lecture 6

Rational Maps and Curves

Hot themes in Lecture 6: *Rational maps—maximal set of definition—function fields—birational maps and birational equivalence—normalisation—non-singular curves—extension of rational maps on non-singular curves—non-singular models of curves–function fields of transcendence degree one.*

Two varieties X and Y over the field k are said be *birationally equivalent* if they have isomorphic non-empty open subsets, that is one may find open dense subsets $U \subseteq X$ and $V \subseteq Y$ and an isomorphism $U \simeq V$. An open dense subsets of a variety has the same function field as the surrounding variety, two birationally equivalent varieties have isomorphic function field (as algebras over the ground field), and during this lecture we shall see that the converse also holds. The theory of varieties up to birational equivalence is thus basically equivalent to the theory of fields finitely generated over the ground field.

Birationally equivalent varieties (*Birasjonalt ekvivalente varieteter*)

6.1 Birational geometry did almost dominate algebraic geometry at a certain period. The classification of varieties up to birational equivalence is a much courser classification than classification up to isomorphism, and hence it is *a priori* easier task (but still, challenging enough). However, for non-singular projective curves, as we later shall see, the two are equivalent. Two such curves are isomorphic if and only if they are birationally equivalent—that is, if and only if their function fields are isomorphic over k.

Shigefumi Mori (1951–)
Japanese Mathematician

6.2 Already for projective non-singular surfaces, the situation is completely different. There are infinitely many non-isomorphic surface in the same birational class (see example 6.4 below for a simple example of two), and they can form a very complicated hierarchy. For varieties of higher dimension, the picture is even more complicated, but the so called *Mori Minimal Model Program* that as evolved during the last twenty years, shed some light on the situation.

6.3 Another important question is wether there are non-singular[1] varieties in every birational class; or phrased differently, whether every variety X is birationally equivalent to a non-singular one—or has a *non-singular model* as one also says. For curves this is no big deal. A curve X is non-singular if and only if all its local rings are integrally closed in the function field $K(X)$,

[1] For the moment, we have not spoken about non-singular varieties, but we shall shortly do; see xxx

and by a normalization procedure, one achieves a non-singular model of X. For surfaces it is substantially more complicated, but it was proven by Zariski and xxx. In general it is not known whether it is true or not. When the ground field is of characteristic zero, however, it holds true as demonstrated by Heisuke Hironaka.

Heisuke Hironaka (1931–) Japanese Mathematician

6.1 *Rational and birational maps*

Just like we spoke about rational functions on a variety being function defined and regular on a non-empty open subset, one may speak about *rational maps* from a variety X to another Y. Strictly speaking, this is a pair consisting of an open subset $U \subseteq X$ and a morphism $\phi\colon U \to Y$. Commonly a rational map is indicate by a broken arrow like $\phi\colon X \dashrightarrow Y$.

rational maps (rasjonale avbildniger)

6.4 If V is another open subset of X containing U, an *extension* of ϕ to V is a morphism $\psi\colon V \to Y$ such that $\psi|_U = \phi$; it is common usage to say that ϕ is defined on V. An open subset $U \subseteq X$ is called a *maximal subset of definition* for ϕ if ϕ is defined on U and cannot be extended to any strictly larger open subset. The next proposition tells us that every rational map has a unique maximal set of definition:

Extension of morphisms (utvidelse av morfier)

Maximal subsets of definition (maksimale definisjons mengder)

Proposition 6.5 *Let X and Y be two varieties, and $\phi\colon X \dashrightarrow Y$ a rational map. Then ϕ has a unique maximal set of definition.*

Proof: Since X is a Noetherian topological space, any non-empty collection of open subsets has a maximal element. Hence maximal sets of definition exists, and merely the unicity statement requires some work.

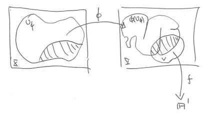
Let $U \subseteq X$ be an open subset where ϕ is defined. Assume that V_1 and V_2 are open subsets of X containing U and that both are maximal sets of definition for ϕ. Let the two extensions be ϕ_1 and ϕ_2. Both restrict to morphisms on the intersection $V_1 \cap V_2$, and the salient point is that these two restrictions coincide. Indeed, both ϕ_1 and ϕ_2 restrict to ϕ on U, and because Y is a variety the Hausdorff axiom holds for Y. Consequently, the subset of $V_1 \cap V_2$ where ϕ_1 and ϕ_2 coincide, is closed; and since they coincide on U, which is dense in $V_1 \cap V_2$, they coincide along the entire intersection $V_1 \cap V_2$. This means that ϕ_1 and ϕ_2 can be patched together to give a map defined on $V_1 \cup V_2$, which is a morphism (being a morphism is a local property). My maximality, it follows that $V_1 = V_2$. ❑

Problem 6.1 Give an example to show that the proposition does not hold when X merely is a prevarietry. Hint: Take a new look at "the bad guy" (Example 3.9 on page 49). ★

Problem 6.2 Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the map defined as $\phi(x : y : z) = (x^{-1} : y^{-1} : z^{-1})$. Show that ϕ is birational. ★

Problem 6.3 Consider the map $(u : v) \mapsto (u^2v^{-2} : u^3v^{-3} : 1)$. Prove that it is morphism on $\mathbb{P}^1\backslash\{(1:0)\}$. Prove that it can be extended to a morphism $\mathbb{P}^1 \to \mathbb{P}^2$. ★

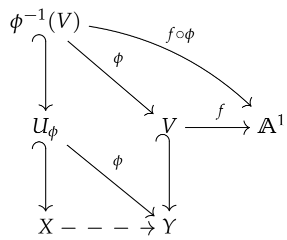
Problem 6.4 Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map given by $\phi(x : y : z) = (y^2 : xy : x^2)$. Show that ϕ is a morphism away from the point $p = (0 : 0 : 1)$, and that image is the conic C parametrized by $(u^2 : -uv : v^2)$. Show that line $ax + by$ passing by p is mapped to the point $(a^2 : -ab : b^2)$. ★



Functoriality

6.6 Dominant rational maps enjoy a weaker but similar functorial property as morphisms do. "By composition" they induce in a contravariant way a k-algebra homomorphism, but merely between *the function fields* of the two involved varieties.

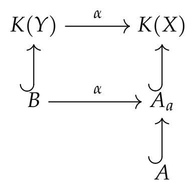
To be precise, assume that $\phi\colon X \dashrightarrow Y$ is the dominant, rational map, and that ϕ is defined on the open set U_ϕ; that is, $\phi\colon U_\phi \to Y$ is a morphism. For any open $V \subseteq Y$, the inverse image $\phi^{-1}(V)$ is non-empty and open in U_ϕ. A member f of the function field $K(Y)$ is a regular function defined on some open set V_f of Y. The composition $f \circ \phi$ is a regular function on $\phi^{-1}(V_f)$ and hence defines an element in the function field $K(X)$. In this way we obtain a homomorphism $\phi^*\colon K(Y) \to K(X)$. Of course constant functions map to constant functions, and thence ϕ^* is a k-homomorphism.



An important property is that this construction is reversible:

Theorem 6.7 (Main Theorem on Rational maps) *Given two varieties X and Y and a k-algebra homorphism $\alpha\colon K(Y) \to K(X)$. Then there exists a (unique) dominant rational map $\phi\colon X \dashrightarrow Y$ such that $\phi^* = \alpha$.*

Notice that the map α is not merely any field homorphism, but it must act trivially on the field of constants k. The map ϕ will be unique if one demands that its source be the maximal subset to which it can be extended.



Proof: We begin by choosing an open and affine set in each of the varieties X and Y. Call them U and V, with $U \subseteq X$ and $V \subseteq Y$. They have coordinate rings $A = \mathcal{O}_X(U)$ and $B = \mathcal{O}_Y(V)$, and the function fields $K(X)$ and $K(Y)$ are the fraction fields of A and B respectively. As U and V were randomly chosen, there is no reason for the homomorphism α to send B into A, but replacing B by an appropriate localization, we may arrange the situation for it to be true.

The k-algebra B is finitely generated over k and has generators $b_1, \ldots, b_s$. The images $\alpha(b_i)$ are of the form $\alpha(b_i) = a_i a^{-1}$ with the a_i's and a all belonging to A (the field $K(X)$ is the fraction field of A and a a common denominator for the $\alpha(b_i)$'s). But then α sends B into the localized ring A_a.

Translating this little piece of algebra into geometry will finish the proof. The localization A_a is the coordinate ring of the distinguished affine open subset U_a of U, and by the main theorem about morphisms between affine varieties, there is a morphism $\phi\colon U_a \to V$ with ϕ^* equal to $\alpha|_V$. Hence ϕ represents a rational and dominating map with the requested property that $\alpha = \phi^*$ ❑

6.8 More generally, a dominant rational map can be composed with any other rational map (whose source is the target of the given rational map) not merely with rational functions. Indeed, assume that $\phi\colon X \dashrightarrow Y$ is dominant and $\psi\colon Y \dashrightarrow Z$ is a rational map. Since the image of ϕ is dense, it meets any non-empty open set, in particular the largest set of definition U_ψ of ψ. Hence $\phi^{-1}(U_\psi)$ is an open and non-empty subset of U_ϕ. It follows that the composition $\psi \circ \phi$ is a well-defined morphism on $\phi^{-1}(U_\psi)$, hence defines a rational map $X \dashrightarrow Z$.

6.9 Thus varieties and dominant rational maps form a category. There is also a modified category Rat_k whose morphisms are equivalence classes of rational and dominating maps. Two such being equivalent if they have the same target and source and being equal on a open dense subset. Theorem 6.7 above tells us that this category is almost equivalent to the category of finitely generated field extensions of k and k-algebra homomorphisms.

6.10 A *birational map* is a rational map which has a rational inverse. To give a birational map between two varieties X to Y is to give open sets $U \subseteq X$ and $V \subseteq Y$ and an isomorphism $\phi\colon U \to V$, and when such a map is extant, one says that X and Y are *birationally equivalent*. Be aware that the open set U might be smaller than the maximal set of definition U_ϕ as in Example 6.1 below.

Birational maps (birasjonale avbildninger)

Birationally equivalent varieties (birasjonalt ekvivalente varieteter)

In the vernacular of category theory one would express this by saying they are isomorphic in the category Rat_k. The main theorem (Theorem 6.7 above) tells us that X and Y are birationally equivalent if and only if their functions fields $K(X)$ and $K(Y)$ are isomorphic as k-algebras.

Examples

6.1 *(A quadratic transform)* It is pretty obvious that the map σ from Problem 6.2 on page 112 given as $\sigma(x : y : z) = (x^{-1} : y^{-1} : z^{-1})$ is birational—it is even its own inverse. Indeed, it is regular on the open set $D_+(xyz)$ where no coordinate vanishes, and it maps $D_+(xyz)$ into itself. Clearly σ^2 is the identity on $D_+(xyz)$. The map σ is called a *quadratic transform*.

Quadratic transforms (kvadratisk transformasjon)

It is worth while understanding the map σ better. Multiplying all components by xyz we obtain the expression $\sigma(x : y : z) = (yz : xz : xy)$, which reveals that σ is defined away from the three "coordinate points" $e_z = (0;0;1)$, $e_y = (0;1;0)$ and $e_x = (1;0;0)$, since if two coordinates do not vanish neither does their product. It also reveals that each of the three lines $L_x = V(x)$, $L_y = V(y)$ and $L_z = V(z)$ are collapsed to the corresponding coordinate point.

For instance, points on L_x are of the form $(0 : y : z)$ and σ maps them all to the point $(yz : 0 : 0) = (1 : 0 : 0) = e_x$ (the equalities being valid whenever $yz \neq 0$).

The map σ can not be extended beyond $\mathbb{P}^2 \backslash \{e_x, e_y, e_z\}$. Indeed, through each of coordinate points pass two of the lines that are collapsed, and the two lines are mapped to *different* points by σ. Therefore, by continuity, we are left no chance of defining σ at coordinate points. So $\mathbb{P}^2 \backslash \{e_x, e_y, e_z\}$ is the maximal set of definition for σ.

6.2 *(Cremona groups)* The map σ form th previous example plays a main role in the study of the group of birational automorphisms of $\mathbb{P}^2$; that is, the group of birational maps from $\mathbb{P}^2$ to $\mathbb{P}^2$. In view of Theorem 6.7 on page 113 this group is nothing but the Galois group of $k(x_1, x_2)$ over k, and it is one of the so-called *Cremona groups* named after the Italian mathematician Luigi Cremona. These are the Galois groups of the rational function fields $k(x_1, \ldots, x_n)$; often denoted by $Cr_n(k)$. Except for $n = 1$ and $n = 2$ nothing much is known about these groups. For $n = 1$ it is just the group $\mathrm{PGl}(2, k)$—every automorphism $\mathbb{P}^1$ is linear. For $n = 2$ there is a famous theorem of Max Noether's (the father of Emmy Noether) that any birational automorphism of $\mathbb{P}^2$ is a composition of quadratic transforms and linear automorphisms; in other words, the Cremona group $Cr_2(k)$ is generated by $\mathrm{PGl}(3, k)$ and the map σ.

Antonio Luigi Gaudenzio Giuseppe Cremona (1830–1903) Italian mathematician

Cremona groups (*Cremona-grupper*)

6.3 *(Maps from $\mathbb{P}^1$)* Any rational map $\mathbb{P}^1 \to \mathbb{P}^n$ is defined everywhere[2]; in other words, the maximal set of definition U_ϕ of ϕ is equal to the entire projective line $\mathbb{P}^1$. Chose homogeneous coordinates $(x_0 : \ldots : x_n)$ on $\mathbb{P}^n$ and let $D_+(x_i)$ be one of the distinguished open sets that meet the image of U_ϕ under ϕ. The variety $D_+(x_i)$ is an affine n-space with coordinates $x_j x_i^{-1}$ with $0 \leqslant j \leqslant n$ and $j \neq i$.

The inverse image $V = \phi^{-1}(D_+(x_i) \cap \phi(U_\pi))$ is an open set, and the n component functions of $\phi|_V$ are rational functions on $\mathbb{P}^1$ regular on V. They may be brought on the form f_j/f_i with $0 \leqslant j \leqslant n$ and $j \neq i$, where the polynomial f_i is a common denominator of the f_j's and does not vanish on V. At points in V the relation $x_j x_i^{-1} = f_j f_i^{-1}$ holds.

The idea is to use the f_k's (now including f_i) as the homogenous components of a morphism Φ of $\mathbb{P}^1$ into $\mathbb{P}^n$ and define Φ by the assignment $\Phi(x) = (f_0(x) : \ldots : f_n(x))$. It might be that the $n + 1$ polynomials f_k have a common factor, but it can be discarded, and we may assume that the f_k's are without common zeros. Then it is easily checked (remember Paragraph 4.32 on page 73) that Φ is a morphism that extends ϕ.

Max Noether (1844–1921) German mathematician

[2] This example is a forerunner for the general theorem asserting that any rational map from a regular curve into a projective space is in fact regular everywhere.

6.4 *(The quadratic surface)* In this example we use homogeneous coordinates $(x : y : z : w)$ on the projective space $\mathbb{P}^3$. The quadric $Q = Z_+(xz - yw) \subseteq \mathbb{P}^3$ is birationally equivalent to the projective plane $\mathbb{P}^2$, but the two are not isomorphic. This is one of the simplest example of two non-isomorphic projective and non-singular surfaces being birationally equivalent.

To begin with, the two are not isomorphic. They are not even homeomorphic since any two curves[3] in $\mathbb{P}^2$ intersect, but on the quadric there are families of disjoint lines (in fact, there are two such). For example the two disjoint lines $x = y = 0$ and $x + z = y + w = 0$ both lie on Q.

Next, we exhibit a birational map $\phi\colon Q \dashrightarrow \mathbb{P}^2$. It will be defined on the open set $U = D_+(w) \cap Q$. In $D_+(w) \simeq \mathbb{A}^3$, where we by mildly abusing the language, use coordinates named x, y and z, the equation of Q becomes, $y = xz$. It is almost obvious that the projection $\mathbb{A}^3 \to \mathbb{A}^2$ sending (x, y, z) to (x, z) induces an isomorphism from $Q \cap D_+(w)$ to $\mathbb{A}^2$, but it is a rewarding exercise for the students to check all details. HINT: The inverse map is given as $(x, z) \mapsto (x, xz, z)$.

[3] A *curve* is a closed subset of Krull dimension one; notice that this is a purely topological notion.

☆

PROBLEM 6.5 In this exercise the previous example is elaborated. Show that the projection $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ with centre $p = (0 : 0 : 0 : 1)$ is well defined on $Q\backslash\{p\}$ and when restricted to $D_+(w) \cap Q$, becomes the isomorphism in the example. Show that the plane $Z_+(y)$ meets Q along two lines passing by p, and that these lines under the projection are collapsed to two different points in $\mathbb{P}^2$. HINT: The projection is given as $(x : y : z : w) \mapsto (x : y : z)$. ★

Problems

6.6 Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map that sends $(x_0 : x_1 : x_2)$ to $(x_2^2 : x_0x_1 : x_0x_2)$. Determine largest set of definition. Show that ϕ is birational , and determine what curves are collapsed.

6.7 Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map that sends $(x_0 : x_1 : x_2)$ to $(x_2^2 - x_0x_1 : x_1^2 : x_1x_2)$. Determine the set largest set where ϕ is defined. Show that ϕ is birational, and determine what curves are collapsed.

6.8 Let $\phi\colon \mathbb{A}^2 \to \mathbb{A}^4$ be the map defined by

$$\phi(x, y) = (x, xy, y(y-1), y^2(y-1)).$$

a) Show that $\phi(0, 0) = \phi(0, 1) = (0, 0, 0, 0)$, and that ϕ is injective on $U = \mathbb{A}^2\backslash\{(0, 0), (0, 1)\}$.

b) Show that $\phi|_U$ is an isomorphism between U and its image. HINT: $\phi|_U$ takes values in $V = \mathbb{A}^4\backslash Z(x)$ and the map $V \to \mathbb{A}^2$ sending (u, v, w, t) to (u, v) is a left section for $\phi|_U$.

c) Show that the image of ϕ is given by the polynomials $ut - vw$, $w^3 - t(t - w)$ and $u^2w - v(v - u)$.

★

Blowing up

There is in some sense a "atomic" way of modifying a varietry but still obtaining a birational one. Called the blowing up.

The simples example is the case of a point in $\mathbb{P}^2$. After having chosen coordinaytes we assuem the point is $(0 : 0 : 1)$ and we call the coordinates $(x_0 : x_1 : x_2)$. Now the blown up plane will be a closed subvariety of the product $\mathbb{P}^2 \times \mathbb{P}_1$. To define it we introduce coordinates y_0 and y_1 on $\mathbb{P}^1$ and we consider the subset $y_1 x_0 - y_0 x_1$. In the Segre embedding of $\mathbb{P}^2 \times \mathbb{P}^1$ in $\mathbb{P}^5$ this is just the linear section with the hyperplane which in n Sgere coodinates (as in xxx) has equation $t_{01} - t_{10}$.

PROPOSITION 6.11 *The projection $\pi\colon \widetilde{\mathbb{P}^2} \to \mathbb{P}^2$ is birational. The fibre $E = \pi^{-1}(p)$ is isomorphic to $\mathbb{P}^1$ and π is an isomorphism $\widetilde{(}\mathbb{P}^2)\backslash E$ and $\mathbb{P}^2\backslash\{p\}$.*

PROOF: Indeed, the rational map $\psi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^1$ that sends $(x_0 : x_1 : x_2)$ to $(x_0 : x_2)$ is a well defined morphism in $\mathbb{P}^2\backslash\{p\}$ and hence gives a morphism into $\mathbb{P}^2 \times \mathbb{P}^1$ sending x to $(x, \psi(x))$ (which by the way is just the graph of ψ). In coordinates

$$\psi(x_0 : x_1 : x_2) = \big((x_0 : x_1), (x_0 : x_1 : x_2)\big)$$

so ovbiously the Segre form $t_{10} - t_{01}$ vanishes along the image. If $q = \big((/y_0 : y_1), (x_0 : x_1 : x_2)\big)$ is a point on the product where the form $t_{01} - t_{10}$ vanishes, it holds true that $y_1 x_0 = y_0 x_1$, so that for instance if $x_0 \neq 0$ one may solve to get

$$q = \big((y_0 : y_0 x_1 x_0^{-1}), (x_0 : x_1, : x_2)\big) = \big((x_0 : x_1), (x_0 : x_1 : x_2)$$

since $x_0 \neq 0$ implies that $y_0 \neq 0$, and thus q lies in $\widetilde{\mathbb{P}^2}$.

❑

6.2 Curves

In this section X will denote a curve; that is, a variety of dimension one. We aim at establishing the basic facts of the birational geometry of curves.

Anticipating the general notion of non-singularity we shall say that a point $P \in X$ is a non-singular point if the local ring $\mathcal{O}_{X,P}$ is integrally closed in the function field $K(X)$. For one-dimensional Noetherian rings, as the ring $\mathcal{O}_{X,P}$ is, being integrally closed in its the field of fractions is equivalent to being a regular ring. This is one of the reasons why the theory for curves is substantially easier than for general varieties—one obtains a non-singular model just by normalizing X. Another reason is the fact rational maps from a non-singular curve into a projective space is defined everywhere, that is to say, it is a regular map. This result, which is a sort of upper class l'Hôpital's rule, has a counterpart in theory the general, similar to a theorem in complex analysis called Hartog's theorem, which asserts that rational maps from normal varieties are defined off closed subset of codimension at least two.

This implies that birational maps between projective non-singular curves are isomorphisms, and consequently there is up to isomorphism only one non-singular and projective curve in a birational class. In a nutshell, there is no distinction between biregular and birational geometry of curves.

Another consequence of the extension property of rational maps between curves is that every non-singular curve is isomorphic to an open set of a non-singular *projective* curve (it could of course be equal to the whole). In particular, any finitely generated field of transcendence degree one over an algebraically closed field k is the function field of a projective and non-singular curve.

Discrete valuation rings

6.12 A Noetherian local ring A with $\dim A = n$ and with maximal ideal $\mathfrak{m}$ is said to be *regular* if the maximal ideal can be generated by n elements; that is, by as many elements as the dimension indicates. Nakayama's lemma tells us that the minimal number of generators of $\mathfrak{m}$ equals the so called embedding dimension $\dim_{A/\mathfrak{m}} \mathfrak{m}/\mathfrak{m}^2$ of A, so A is regular precisely when the Krull dimension and the embedding dimension coincide. A general ring A is *regular* if all the local rings $A_\mathfrak{p}$ are regular.

Regular local rings (*regulære lokale ringer*)

6.13 When it comes to one-dimensional rings, which is our main concern in this section, A is regular if and only if $\mathfrak{m}$ is principal. This has many equivalent formulations, we cite the few we shall need.

Proposition 6.14 *Let A be a Noetherian local domain with maximal ideal $\mathfrak{m}$ of dimension one. Then the following are equivalent*

- *The maximal ideal $\mathfrak{m}$ is principal;*
- *A is a PID and all ideals are powers of $\mathfrak{m}$;*
- *A is integrally closed.*

Proof: With begin with establishing that the first assertion implies the second, so let x a generator for the maximal ideal $\mathfrak{m}$ and let $\mathfrak{a} \subseteq A$ be a non-zero ideal. Let n be the largest integer such that $\mathfrak{a} \subseteq \mathfrak{m}^n$. Krull's intersection theorem asserts that $\bigcap_i \mathfrak{m}^i = 0$, and the ideal $\mathfrak{a}$ is therefore not contained in all powers of $\mathfrak{m}$ and such an n exists. Since $\mathfrak{a} \nsubseteq \mathfrak{m}^{n+1}$, there is an $a \in \mathfrak{a}b$ such that $a = bx^n$ with $b \notin \mathfrak{m}$; that is, b is a unit since the ring is local. It follows that $(x^n) \subseteq \mathfrak{a}$, and we are done.

Every PID is a UFD and all UFD's are integrally closed, and the third assertion follows from the second.

Finally, assume that A is integrally closed in its fraction field K and let $x \in \mathfrak{m}$ be any element. Since A is Noetherian and of dimension one, there is an element $y \in A$ not in (x) such that $(x : y) = \mathfrak{m}$. This means that $yx^{-1}\mathfrak{m} \subseteq A$. We contend that $yx^{-1} = A$. If not, one would have $yx^{-1}\mathfrak{m} \subseteq \mathfrak{m}$ and since $\mathfrak{m}$ is a finitely generated and faithful A-module it would follow that yx^{-1} is integral over A. Hence it holds that $yx^{-1} \in A$ since A is integrally closed, and therefore also $y \in (x)$, which is not the case. ❑

6.15 A ring as in the proposition is also a *discrete valuation ring*. If t is a generator for the maximal ideal $\mathfrak{m}$, one calls t a *uniformizing parameter* of A. All non-zero ideals in A are of the form (t^ν) with $\nu \in \mathbb{N}_0$, and therefore any non-zero element in the fraction field K may be written as αt^ν with α a unit in A and ν an integer. Indeed, if $f \in A$ and $f \neq 0$, we let $v(f)$ be the unique non-negative integer such that $(f) = \mathfrak{m}^{\nu(f)}$, then $f = \alpha t^{v(f)}$ with α being a unit, and for a general non-zero element fg^{-1} of the function field, one finds $fg^{-1} = \alpha t^{\nu(f)-\nu(g)}$ with α a unit.

Uniformizing parameters (*uniformiserende parameter*)

6.16 The function $v\colon A\backslash\{0\} \to \mathbb{Z}$ sending f to the unique integer such that $f = \alpha t^{v(f)}$ with α a unit, is called the *valuation* associated to A. It resembles the well-known order function from complex function theory (recall that every meromorphic function has an order at a point, positive if its a zero and negative in case of a pole), and it share several of its properties. For instance, the two following property hold true

Valuations (*valuasjoner*)

- ❑ $\nu(fg) = \nu(f) + \nu(g)$;
- ❑ $\nu(f+g) \geqslant \min\{\nu(f), \nu(g)\}$,

with equality in the latter when $\nu(f) \neq \nu(g)$. Any function $A\backslash\{0\} \to \mathbb{Z}$ satisfying these to properties is called a *discrete valuation* on (K).

Discrete valuations

6.17 Coming back to the geometric context, we let P be a non-singular point of the curve X. A uniformizing parameter t at P is a rational function on X which is regular at P and generates the maximal ideal $\mathfrak{m}_P$. Another common way of phrasing this is to say that t is regular and vanishes to first order at P, or that t has a simple zero at P. Every function f can be expressed as $f = \alpha t^{\nu_P(f)}$ with α a rational function on U that is regular and non-vanishing at P. One may think about the valuation $v_P(f)$ as the order of f at P, either the order of vanishing if f is regular at P or the oder of the pole if not.

Examples

6.5 Consider the elliptic curve C in $\mathbb{A}^2$ given by the equation $y^2 = x(x^2-1)$, and let $P = (0,0)$. The local ring $\mathcal{O}_{C,P}$ is then regular; in fact, its maximal ideal is generated by the coordinate function y. The maximal ideal $\mathfrak{m}_P$ is generated by x and y, and in the local ring $\mathcal{O}_{C,P}$ the function $x-1$ is invertible. So it holds true that

$$x = y^2/(x^2-1). \tag{6.1}$$

6.6 On the other hand, the local ring $\mathcal{O}_{D,P}$ is not regular if D the *rational node* with equation $y^2 - x^2(x-1)$. To see this let $\mathfrak{n}_p$ be the ideal of $P = (0,0)$ in $\mathcal{O}_{\mathbb{A}^2,P}$ and $\mathfrak{m}_P$ that of P in $\mathcal{O}_{D,P}$. They are both generated by x and y, and restriction of functions induces a surjection $\theta\colon \mathfrak{n}_P/\mathfrak{n}_P^2 \to \mathfrak{m}_P/\mathfrak{m}_P^2$. We contend that this is an isomorphism. Therefore $\mathfrak{m}_P$ requires both generators x and y and is not principal. Indeed, the kernel of θ equals $(y^2 - x^2(x-1))\mathcal{O}_{\mathbb{A}^2,P} \cap \mathfrak{n}_P$ which is simply contained in the square $\mathfrak{n}_P$ since $y^2 - x^2(x-1)$ lies there. The situation is well illustrated with the short exact sequence

$$0 \longrightarrow (y^2 - x^2(x-1)) \longrightarrow \mathcal{O}_{\mathbb{A}^2,P} \longrightarrow \mathcal{O}_{C,P} \longrightarrow 0.$$

☆

Problems

6.9 Assume that ν is a discrete valuation on a field K. Show that the set $A = \{\, x \in K \mid \nu(x) \geqslant 0 \,\}$ is discrete valuation ring by showing that $\{\, x \in K \mid \nu(x) > 0 \,\}$ is a maximal ideal generated by one element.

6.10 Let q denote the origin in $\mathbb{A}^1$. The induces an injective map map $\mathcal{O}_{\mathbb{A}^1,q} \to \mathcal{O}_{C,P}$. Show that the rextriction of ν_P to $\mathcal{O}_{\mathbb{A}^1,q}$ is twice ν_q.

★

The extension lemma

6.18 We start with establishing the main property of curves in the present context that any rational map from a curve into a projective varletry is defined at all non-singular points of the curve.

One may think about this as an advanced form of "l'Hôpital's" rule. The tactics of the proof is first to realise the mapping in a neighbourhood of P as the composition $\pi \circ \Phi$ where $\pi\colon \mathbb{A}^{n+1}\backslash\{0\} \to \mathbb{P}^n$ is the canonical projection, and where the map Φ is represented as $\Phi = (g_0, \ldots, g_n)$ with the g_i's regular functions near P, and then cancel out the common factors of the g_i's that vanish at P.

Lemma 6.19 *Let U be a curve and $P \in U$ a non-singular point. Assume that $\phi\colon U\backslash\{P\} \to \mathbb{P}^n$ is a morphism. Then there exists a morphism $\psi\colon U \to \mathbb{P}^n$ extending ϕ.*

Proof: The first observation is that it suffices to find an open $U_0 \subseteq U$ containing P over which ϕ extends. Indeed, if $\psi_0\colon U_0 \to \mathbb{P}^n$ is such an extension, the two morphisms ψ_0 and ϕ coincide on $U_0\backslash\{P\}$, and hence they patch together to a morphism on U. It follows that we may assume U to be affine.

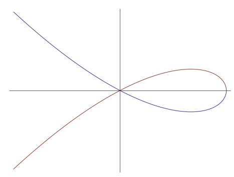
Secondly, we may, possible after having renumbered the coordinates, assume that the image $\phi(U\backslash\{P\})$ meets the basic open set $D = D_+(x_0)$; then the inverse image $V = \phi^{-1}(D)$ is a non-empty open subset of U. The basic open set D is an affine n-space with coordinates $x_1x_0^{-1}, \ldots, x_nx_0^{-1}$, and the map $\phi|_V$ is therefore given by n component functions regular on V. They are all rational function on U, and may therefore be written as fractions $f_i = g_i/g_0$ of regular functions on U.

Consider the morphism $\Phi(x) = (g_0, g_1, \ldots, g_n)$ from U into $\mathbb{A}^{n+1}$. It is well defined at the point P, but of course, it might be that it maps P to the origin. However, if this is not the case, the composition $\pi \circ \phi$ is defined at P and extends ϕ to the neighbourhood of P where the g_i's do not vanish simultaneously, and we will be done.

Now, the salient point is that we have the liberty to alter the morphism Φ by cancelling common factors of the g_i's without changing the composition $\pi \circ \Phi$: After such a modification the composition $\pi \circ \Phi$ and the original morphism ϕ coincide where they both are defined. Indeed, it holds true that $(hg_0; \ldots; hg_n) = (g_0; \ldots; g_n)$ where both sets of homogeneous coordinates are legitimate.

To get rid of common zeros the functions g_i's might have at the point P, we introduce a uniformizing parameter t at P; that is, a regular function t on some neighbourhood U_0 of P which generates the maximal ideal of the local ring $\mathcal{O}_{U,P}$. One may then write $g_i = \alpha_i(t)t^{\nu_i}$ with the $\alpha_i(t)$'s being regular functions on U_0 that do not vanish at P, and where the ν_i's are non-negative integers. Putting $\nu = \min_i \nu_i$, the differences $\mu_i = \nu_i - \nu$ will be non-negative, and at least one will be zero so that the corresponding g_i does not vanish at P. Hence replacing the g_i's by $g_it^{-\nu} = \alpha_i(t)t^{\nu_i - \nu}$ we arrive at the requested modification of Φ. ❑

EXAMPLE 6.7 The assumption that P be a non-singular point is essential. For instance, let $U \subseteq \mathbb{A}^2$ be the "ordinary double point" given as $U = Z(y^2 - x^2(1-x))$; and let $\phi(x, y) = yx^{-1}$. Then ϕ is defined on $U\backslash\{0\}$ but can not be extend. We have depicted the situation over the reals; when the absolute value $\|x\|$ is small, $\|1 - x\|$ is bonded away from zero, and the curve has two distinct (analytic) components parametrized as $y = x\sqrt{1-x}$ (the red one) and $y = -x\sqrt{1-x}$ (the blue one). The function yx^{-1} approaches 1 when x approaches zero along the red component, and it tends to -1 when x goes to zero while staying on the blue. This shows that there is not even a continuous extension. ☆



The extension Theorems

Most of the work is done in proving the lemma , and we can collect the fruits. Here comes the theorems:

Theorem 6.20 (The Extension theorem) *Let X be a curve and $P \in X$ a non-singular point. Any rational map $\phi\colon X \dashrightarrow Y$ where Y is a projective variety, is defined at P.*

Proof: Assume the projective variety Y is a closed subvariety of $\mathbb{P}^m$; that is, $Y \subseteq \mathbb{P}^m$. Let U be a neighbourhood of P such that ϕ is defined on $U \setminus \{P\}$. By the extension lemma (lemma 6.19 above), the map ϕ composed with the inclusion Y into $\mathbb{P}^m$ extends to P, and the extension takes values in Y since Y is closed in $\mathbb{P}^m$. ❑

As is illustrated in Example 6.7 above it is paramount that P be a non-singular point. If X has *e.g.* two different branches passing through P, the "limit" of ϕ at P along the two branches may be different.

Theorem 6.21 *Assume that X and Y are two projective and non-singular curves that are birationally equivalent. Then they are isomorphic.*

Proof: Let $U \subseteq X$ and $V \subseteq Y$ be two open sets such that there is an isomorphism $\phi\colon U \dashrightarrow V$. Since Y is projective and X is non-singular a repeated application of theorem 6.20 above gives a morphism $\Phi\colon X \to Y$ extending ϕ. Similarly, there is morphism $\Psi\colon Y \to X$ extending ϕ^{-1}. Finally, the Hausdorff axiom holds for both X and Y, and one infers that $\Phi \circ \Psi = \mathrm{id}_Y$ and $\Psi \circ \Phi = \mathrm{id}_X$ since they extend $\phi \circ \phi^{-1} = \mathrm{id}_V$ and $\phi^{-1} \circ \phi = \mathrm{id}_U$ respectively. ❑

Desingularization of curves

Every curves has a non-singular model. This is compairably easy due to the fact that for curves being normal is the same as being non-singular. Howevert it hinges on a non-trival result about normalization.

Theorem 6.22 (Finiteness of integral closure) *Let A be a domain finitely generated over the field k with fraction field K. For any finite field extension L of K, the the integral closure B of A in L is a finite module over A.*

In particular taking $L = K$, we see that the integral closure of A in K is finite over A. For domains other than those being finitely generated over a field, this theorem is subtle, and in positive characteristic it is not generally true, even for Noetherian domains.

6.23 Given X a variety we apply this to the coordinate ring $A(X)$ of X letting B be the integral closure of $A(X)$ in the function field $K(X)$. The theorem tells us that B is a finitely generated algebra over the ground field k, and thence there is an affine variety $\widetilde{X}$ whose coordinate ring equals B. The inclusion $A(X) \subseteq B$ induces a morphism $\widetilde{X} \to X$, which is finite because B is a finite $A(X)$-module, and since $A(X)$ and B have the same fraction field, this morphism is birational.

Proposition 6.24 *Any affine variety X has a normalisation $\widetilde{X}$; that is, a normal affine variety and a finite birational morphism $\pi\colon \widetilde{X} \to X$. It enjoys the universal property that any dominating morphism $\psi\colon Y \to X$ whose source Y is normal, factors through π; in other words, there is a morphism $\phi\colon Y \to \widetilde{X}$ such that $\pi = \psi \circ \phi$.*

Proof: Most is already accounted for, only the factorization remains. But ψ is dominant and induces an injection $K(X) \subseteq K(Y)$, and because $A(Y)$ is integrally closed in $K(Y)$, any element in $K(X)$ integral over $A(X)$ belongs to $A(Y)$. It follows that $A(\widetilde{X}) \subseteq A(Y)$ and the ensuing morphism $Y \to \widetilde{X}$ is the requested map. ❑

6.25 Recall that by Lying–Over (Proposition 5.23 on page 90) finite maps are closed, and they are surjective when dominating, so the same holds true for the normalisation morphisms $\pi_X\colon \widetilde{X} \to X$; they are closed and surjective.

6.26 Of course, one would desire a similar result for any variety not only for affine ones, and indeed there is one. The proof consists of patching together the normalizations of members of an open affine cover of X. We shall not go through this time consuming process in general, but confine our attention to curves. Owing to the extension theorem, there is a shortcut making the gluing very easy in that case, which we learned from the book[4] by Igor Shafarevich.

4

Theorem 6.27 *Let X be a quasi-projective curve. Then X has a normalization.*

Proof: The first reduction is to find a cover of X of just two affine opens, U_1 and U_2. The curve X lies in some projective space $\mathbb{P}^m$ and we may chose a hyperplane h_1 not containing the curve. Then h_1 cuts the curve in finitely many points, and we choose a second hyperplane h_2 avoiding those finitely many points. Putting $U_i = D_+(h_i) \cap X$ we obtain our two affine open subset that cover X.

The next step is to introduce the normalization V_i of the U_i's and the normalization maps $\pi_i\colon V_i \to U_i$. Moreover, we shall need an affine open subset $U \subseteq X$ contained in $U_1 \cap U_2$ all whose points are non-singular. Then U lies naturally in both the V_i's as an open dense subset and $\pi_i|_U = \mathrm{id}_U$ for $i = 1,2$.

The idea is now to "glue" V_1 and V_2 together along U, and this will amount to embedding them both in a variety W, with the the two embedding coinciding on U, and taking their union inside W.

To construct a suitable W we begin with embedding each of the V_i's in some projective space, as big as one needs, and closing them up there, we obtain two projective curves W_1 and W_2, having respectively V_1 and V_2 as open dense subsets. Our W will be the product $W_1 \times W_2$.

Now we come to the point where The Extension Theorem helps us. The Extension Theorem yields morphisms $\phi_1\colon V_1 \to W_2$ and $\phi_2\colon V_2 \to W_1$. Indeed, there are natural inclusions $U \subseteq V_1$ and $U \subseteq V_2$ and since V_1 is non-singular and W_2 projective, the inclusion of U into W_2 extends to a morphism $\phi_1\colon V_1 \to W_2$. And in an analogous way we find a morphism $\phi_2\colon V_2 \to W_1$.

It follows that we have morphism $V_1 \hookrightarrow W_1 \times W_2$ and $V_2 \hookrightarrow W_1 \times W_2$ both being injective, one of the component maps being the inclusion $V_i \subseteq W_i$. The first one, for instance, is the map sending x to $(x, \phi_1(x))$, and it factors as $V_1 \to V_1 \times W_2 \hookrightarrow W_1 \times W_2$ where the first map identifies V_1 with the graph of ϕ_1. The graph is a closed subvariety and V_1 is isomorphic to it. Closing up V_1 in the product $W_1 \times W_2$ gives closed subvariety $\overline{V_1}$ and since $\bar{V}_1 \cap V_1 \times W_2 = V_1$, we se that V_1 is an open subvariety of $\bar{V}_1$. Of course the same applies to V_2 and V_2 is an open subvariety of $\overline{V_2}$.

Since U is a common open dense set of V_1 and V_2, it holds true that $\bar{U} = \overline{V_1} = \overline{V_2}$.

We contend that $\widetilde{X} = V_1 \cup V_2$ is a normalization of X. Since both V_i's are normal and they cover $\widetilde{X}$, it is normal. It is birational to X and the two maps π_i coincide on U, and hence patch together to a map $\widetilde{X} \to X$. ❑

6.28 In proof from the previous paragraph we realized the normalization $\widetilde{X}$ as an open dense subset of the closed subvariety $\bar{U}$ of the projective variety $W_1 \times W_2$. The variety $\bar{U}$ is therefore a projective curve having $\widetilde{X}$ as a dense open subset, which can be all of $\bar{U}$, and in fact, when the original curve X is projective this will the case. This leads to:

Theorem 6.29 *If X is a projective curve, the normalization $\widetilde{X}$ is projective as well.*

Proof: Denote by $\pi_X \colon \widetilde{X} \to X$ the normalization map. In the remark preceeding the theorem, we described the inclusion $\widetilde{X} \subseteq \widetilde{U}$ of $\widetilde{X}$ as an open subset of a projective curve. Let $\widetilde{U}$ be the normalization of $\bar{U}$. The normalization map $\pi_{\bar{U}} \colon \widetilde{U} \to \bar{U}$ induces a rational map into X and since X is projective and $\widetilde{U}$ is non-singular it extends to morphism $\psi \colon \widetilde{U} \to X$. By the universal property of the normalization map $\phi_X \colon \widetilde{X} \to X$ the map ψ factors through $\widetilde{X}$; that is, there is a map $\phi \colon \widetilde{U} \to \widetilde{X}$ such that $\psi = \phi \circ \pi$. But $\pi_{\bar{U}}$ being surjective, it follow that $\widetilde{X} = \widetilde{U}$ and $\widetilde{X}$ is projective. ❑

The non-singular model

Theorem 6.30 (Fundamental theorem for curves) *Given a field K of transcendence degree one over k. Then there exists a non-singular projective curve X,unique up to isomorphism, such that $K \simeq K(X)$*

Proof: Once we have found one curve whose function field is K we are happy citing xxx, and to find one we appeal to the Theorem of the Primitive Element. However, this requires the extension to be separable, but one we some work one to find an appropriate $x \in K$, one may realize K as a finite separable extension of $k(x)$. Then there is an element $f \in X$ so that $K = k(x)[f]$. That is f satisfies an irreducible equation

$$y^n + a(x)y^{n-1} + \ldots + a_1(x)y + a_0(x) = 0 \tag{6.2}$$

over $k[x]$. When x and y are Interpreted as coordinates on the affine plane $\mathbb{A}^2$, the equation (6.2) is the equation of an irreducible curve whose function field is precisely K. ❑

Lecture 7

Structure of maps

7.1 *Generic structure of morhisms*

7.1 Among the many nice applications of Noether's Normalization lemma we offer in this section a structure theorem for dominating morphisms between varieties; or we should rather call it a generic structure theorem. Morphisms can be utterly intricate, but over a sufficiently small (but dense) open subset of the target they are to a certain extent well behaved and factors basically as the composition of a projection and a finite map. Of course, finite morphisms are complicated and complex creatures, and the complications are hidden in the finite map.

Theorem 7.2 (Generic structure of dominant morphisms) *Let $\phi\colon X \to Y$ be a dominant morphism. Then there exist open affine subsets $U \subseteq Y$ and $V \subseteq X$ such that V maps into U and such that $\phi|_V$ factors as $\phi|_V = \pi \circ \psi$ where $\pi\colon \mathbb{A}^n \times U \to U$ is the projection and $\psi\colon V \to \mathbb{A}^n \times U$ is a finite map.*

$$\begin{array}{ccccc} X & & \xrightarrow{\phi} & & Y \\ \uparrow & & & & \uparrow \\ V & \xrightarrow{\psi} & \mathbb{A}^n \times U & \xrightarrow{\pi} & U \end{array}$$

Finite maps preserve dimensions by Going–Up (Proposition 5.28 on page 92) from which ensues that $n + \dim U = \dim V$. Open dense sets have the same dimension as the surrounding spaces, and the integer n appearing in the theorem therefore equals the relative dimension $\dim X - \dim Y$.

Proof: We let L be the function field of X and K that of Y. Since ϕ is dominating, it gives rise to an extension $K \subseteq L$. Chose any open and affine subset $U \subseteq Y$ and denote its affine coordinate ring by A. Let $V \subseteq X$ be any open affine subset mapping into U. The coordinate ring B of V then contains A and is finitely generated as an A-algebra.

$$\begin{array}{ccc} K & \subset & L \\ \cup & & \cup \\ A_h & \subset & B_h \\ \cup & & \cup \\ A & \subset & B \end{array}$$

The algebra $B_K = B \otimes_A K$ is a finitely generated algebra over K as B is finitely generated over A. Noether's Normalization lemma applies, and there are elements $w_1, \ldots, w_n$ which are algebraically independent over K and are such that B_K is a finite module over K. We also pick generators $z_1, \ldots, z_r$ for B_K over $K[w_1, \ldots, w_n]$.

The basic trick is to replace U and V by smaller distinguished open affine subsets, U by U_h and V by $\phi|_V^{-1}(U_h) = V_{h \circ \phi|_V}$, where h is the product of all denominators that might occur in the w_i's or in the generators z_j's for B_K over

$K[w_1, \ldots, w_n]$; the coordinate ring of U_h will be the localized ring A_h and that of $V_{h\circ\phi|_v}$ will be B_h.

Each w_i may be written as $w_i = b_i s_i^{-1}$ with $b_i \in B$ and $s_i \in A\backslash\{0\}$, and the same for the z_i's, they are of the form $z_i = c_i t_i^{-1}$ with $c_i \in B$ and $t_i \in A\backslash\{0\}$. As our element h we take the product of all the s_i's and all the t_i's, then A_h is obtained by adjoining the denominators s_i^{-1} and t_i^{-1} to A, and the w_i's and the z_i's all lie in B_h. Moreover, $A_h[w_1, \ldots, w_n]$ is contained in B_h, and B_h is a finite module over it, and of course, the w_i's persist being algebraically independent so that $A_h[w_1, \ldots, w_n]$ is isomorphic to a polynomial ring over A_h. ❑

Actually, one has the slightly more general result.

THEOREM 7.3 *With the setting as in the theorem, there exists an open set U and a finite covering of the inverse image $\phi^{-1}(U)$ with affine open sets such each restriction $\phi|_{V_i}$ factors as $\phi|_{V_i} = \pi \circ \psi_i$ where $\psi_i \colon V_i \to \mathbb{A}^n \times U$ is finite.*

PROOF: Start with any finite affine and open covering V_i of $\phi^{-1}(U)$. Shrink U sufficiently to work for all V_i. ❑

7.4 We have seen several instances of dominating morphisms having rather complicates images. The projection of a quadratic surface in $\mathbb{P}^3$ from a point on it for example, has an image $\mathbb{P}^2 \backslash L \cup \{p_1, p_2\}$ where $L \subseteq \mathbb{P}^2$ is a line and p_1 and p_2 are two points on the line. However, the images of dominant maps will always contain a Zariski open set.

COROLLARY 7.5 *The image of a dominant morphism contains a Zariski open set.*

PROOF: The open set $U \subseteq Y$ that appears in the theorem is contained in the image of ϕ since both finite maps and a projections are surjective. ❑

The dimension of fibres

A good concept of dimension should comply to the principle of being "additive along maps". This holds for linear maps as we learned during courses of linear algebra; the dimensions of the kernel and the image add up to the dimension of the source. Since the dimension of differentiable manifolds is governed by tangent spaces, and the derivative of a map is expressed as a bunch of linear maps of tangent spaces, one expects a similar relation between fibres, image and source.

7.6 So also in our world of varieties; the dimension is "additive along dominant maps", at least for the fibres over generic points; that is, for points belonging to an open dense subset of the target variety. Without further hypotheses on the varieties , there are not many limitations for the fibre dimensions over the "bad" points, but there is one governing principle: The dimension is upper semicontinuous. A consequence is a theorem of Claude Chevalley asserting that the points whose fibres are of a given dimension, form a so-called constructible set. Thus morphisms are not too wild.

7.7 We begin with the generic case which is an easy corollary of the structure theorem above.

THEOREM 7.8 *If $\phi\colon X \to Y$ is a dominant morphism between varieties, there is an open set $U \subseteq Y$ so that for for every $x \in U$ and every component Z of the fibre $\phi^{-1}(x)$ it holds true that*

$$\dim Z = \dim X - \dim Y.$$

PROOF: Let U be an open dense subset of Y as in the general version 7.3 of the stucture theorem. For $x \in U$ and Z a component of $\phi^{-1}(x)$, at least one of the members V_i of the covering meets Z. Then $\dim Z = \dim Z \cap V_i$ and the latter is the fibre of $\phi|_{V_i}$ over x. Now one may factor ϕ as

$$V_i \longrightarrow \mathbb{A}^n \times U \longrightarrow U$$

with one map being finite and the other a projection. It follows that $\phi|_{Z \cap V_i}$ is a finite dominant map $Z \cap V_i \to \mathbb{A}^n \times \{x\}$, and by Going–Up (Proposition 5.28 on page 92) we infer that $\dim Z = n$. ❑

7.9 For general point one has

PROPOSITION 7.10 *Let $\phi\colon X \to Y$ be a dominant morphism of varieties. For every point x in Y and every component Z of the fibre $\phi^{-1}(x)$ it holds true that*

$$\dim Z \geqslant \dim X - \dim Y.$$

PROOF: Replacing Y by a neighbourhood of x need is, we may assume that Y is affine. Let $r = \dim Y$. In Proposition 5.57 on page 100 we showed that affine varieties have a systems of parameters at every one of their points; hence there are regular functions $f_1, \ldots, f_r$ on Y such that x is isolated in $Z(f_1, \ldots, f_r)$. By further shrinking Y, we may assume that $\{x\} = Z(f_1, \ldots, f_r)$. The fibre is then described as $\phi^{-1}(x) = Z(f_1 \circ \phi, \ldots, f_r \circ \phi)$, and by Krull's Principal Ideal Theorem every one of its components have a codimension at most equal to r; that is,

$$\dim X - \dim Z \leqslant r = \dim Y,$$

and this gives the inequality in the proposition. ❑

Semi-contninuity of fibre dimension

7.11 An important part of the analysis of a morphism $\phi\colon X \to Y$ is to understand the partition of Y into the subsets where the fibres of ϕ have a given dimension. These sets can have a rather intricate topology, but the sets $W_r(\phi)$ where the fibre dimension is at least a given value r, are topologically simpler. They turn out to be closed. Formally we define

$$W_r(\phi) = \{\, y \in Y \mid \dim \phi^{-1}(y) \geqslant r \,\}.$$

The dimension of an irreducible space is the supremum of the dimensions of the different components, so a point y lies in $W_r(\phi)$ when $\dim Z \geqslant r$ for at least one component Z of the fibre $\phi^{-1}(y)$. With Proposition 7.10 above in mind, it is clear that $W_r(\phi)$ fills up the entire target Y when r is less than the relative dimension $\dim X - \dim Y$

That $W_r(\phi)$ is a closed subsets of Y is commonly referred to by saying that the dimension is an *upper semi-continuous* function—or in the parlance of geometers, that the fibre dimension *increases upon specialization*.

PROPOSITION 7.12 (PRINCIPLE OF UPPER SEMI-CONTINUITY) *Let $\phi\colon X \to Y$ be a morphism. Then $W_r(\phi)$ is closed in Y.*

PROOF: We proceed by induction on $\dim Y$ and begin with picking an open dense subset U of Y as in the structure theorem. The complement of U is the union of a finite collection of closed irreducible sets $\{Z_i\}$, all of dimension less than $\dim Y$. Their inverse images $\phi^{-1}(Z_i)$ are again unions of closed irreducible subsets Z_{ij} of X. Clearly $W_r(\phi)$ is the union of the sets $W_r(\phi|_{Z_{ij}})$ which are all closed in Z_i by induction. They are thus closed in Y since the Z_i's are, and it follows that $W_r(\phi)$ is closed, being then a finite union of closed sets. ❑

Constructible sets

7.13 In a topological space X a *locally closed subset* is a subset that is the intersection of an open and a closed set or in other words it is closed in an open set or for that matter, open in a closed one. A subset of X is *constructible set* if is the union of finitely many locally closed sets. A *Boolean algebra* $\mathcal{B}$ in X is a collection of subsets of X that is closed under finite set-theoretical operations. It contains the entire space and the empty set, finite unions and intersections of members of $\mathcal{B}$ belong to $\mathcal{B}$ and the set theoretical difference of two members is a member as well. The constructible sets form the smallest Boolean algebra to which the opens sets belong.

Locally closed sets (*lokalt lukkede mengder*)

Constructible sets (*konstruktible mengder*)

Boolean algebras (*Boolske algebraer*)

COROLLARY 7.14 *The locus where the fibres of the morphism $\phi\colon X \to Y$ are of dimension r is constructible, that is the finite union of locally closed subsets.*

PROOF: By the theorem the locus of points y in Y where $\dim \phi^{-1}(y) = r$ equals $W_r(\phi)\backslash W_{r+1}(\phi)$ wich is open in $W_r(\phi)$. ❑

7.15 The induction procedure from the proof of Proposition 7.12 above, with minor modifications, yields the following characterization of images of morphisms, which is due to Claude Chevalley.

THEOREM 7.16 *Let $\phi\colon X \to Y$ be a morphism of varieties. Then the image $\phi(X)$ is constructible.*

PROOF: Induction on dim Y. Since locally closed subsets of a closed subset are locally closed in the surrounding space, we assume that the morphism ϕ is dominating. Pick an open subset U as in the structure theorem. The complements is a finite union of components Z_i, all having dimension less than Y, and each inverse image $\phi^{-1}(Z_i)$ is the finite union of components Z_{ij}. By induction the image of $\phi|_{Z_{ij}}$ is constructible, and as $\phi(X) = U \cup \bigcup_{i,j} \phi|_{Z_{ij}}(Z_{ij})$, we are through. ❑

PROBLEM 7.1 Verify that the constructible sets form a Boolean algebra and check that it is the smallest one containing the open sets. ★

7.2 *Properness of projectives*

A very convenient property of compact topological spaces is that they have closed images under continuous maps. In algebraic geometry the topologies are never Hausdorff and too weak for something like that to be true, but there is a very good substitute. Projective varieties behave like compact spaces, their images are always closed.

For projective curves, this was established already in xxx, and the proof we offer in the general case is a reduction to the curve case with the help of on the generic structure theorem.

7.17 The proof we offer is a simplistic variant of which in scheme-theory-speak is called "The valuative criterion for properness". The idea is rather simple. A very rough sketch skipping a few subtle points is as follows. The structure theorem we obtain good open set in the image, and if y is point not in that good open set, we take a curve in the good set whose closure passes through y. The curve can be lifted to the target since we exert strong control over the map over the good open set, and since the target is projective, the lifting extends to the entire closure of the curve, and knowing the result for curves, we obtain a point in the closure mapping to y.

THEOREM 7.18 *Let $\phi\colon X \to Y$ be a morphism and let $Z \subseteq X$ be a closed subvariety. If Z is projective, $\phi(Z)$ is closed.*

PROOF: Let $\phi\colon X \to Y$ be the morphism under scrutiny. Replacing Y by the closure of the image of ϕ we may safely assume that ϕ is dominating and our mission is then to prove that ϕ is surjective. Chose an open affine subset U of Y as in the structure theorem and denote by V an open affine subset of the inverse inpage $\phi^{-1}(U)$ as in the theorem. In other words, the restriction $\phi|_V$ factors as $\phi|_V = \pi \circ \psi$ as in the display

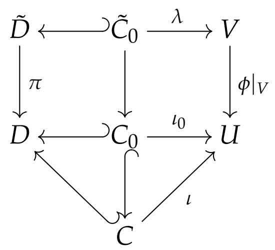
$$V \xrightarrow{\psi} U \times \mathbb{A}^n \xrightarrow{\rho} U, \qquad \phi|_V \colon V \to U$$

where ψ is finite and ρ the projection onto U.

Pick a point $x \in Y$; we have to exhibit a point $z \in X$ with $\phi(z) = y$. The idea is to take a regular curve C and a morphism $\iota\colon C \to Y$ to whose image is closed in Y, pass trough y and meets the open subset U. Lemma 7.19 below guarantees that such curves abound; just normalize the curves found there. To proceed, chose an $x \in C$ that maps to y, denote the inverse image of U in C by C_0 and let D be a projective and non-singular curve containing C.

The map $\iota|_{C_0}\colon C_0 \to U$ obviously lifts to an inclusion $\kappa\colon C_0 \to U \times \mathbb{A}^n$. We want to extend this lifting all the way up to V, but must replace C_0 by a finite cover.

To this end, consider $\psi^{-1}(\kappa(C_0))$ and let W be one its components. Then the function field $K(W)$ is a finite extension of the function field $K(C_0)$, hence the integral closure of C_0 in $K(W)$ maps into W and we have found the required lifting. The integral closure $\tilde{D}$ of D in $K(W)$ is regular and the induced normalization map $\pi\colon \tilde{D}$ to D is finite, hence surjective, and it extends the map $\tilde{C}_0$ to C_0. This situation is summarized in the following diagram.



Pick a point $z \in \tilde{D}$ such that $\pi(z) = x$. The crucial point is now that because the curve D is regular and the varietyX is projective, the map $\lambda\colon \tilde{C}_0 \to V$ extends to a map $\kappa\colon \tilde{D} \to X$. It follows that $\phi(\kappa(z)) = \iota(\pi(z)) = \iota(x) = y$, and y lies in the image of ϕ. ❑

Lemma 7.19 *Given a point x in the variety X and a closed subset Z containing x. Then there is an irreducible curve passing by x not contained in Z.*

Proof: Replacing X by an open affine neighbourhood od x, we may clearly assume that X is affine (just close up a curve found in that case).

Let f be a regular function with $Z \subseteq Z(f)$. It suffices to find a curve through x not lying in $Z(f)$. After Problem 5.19 on page 101 we may find a system of parameters $f_1, f_2, \ldots, f_n$ with $f_1 = f$ at x, and citing the same problem, we infer that all the components of $Z(f_2, \ldots, f_n)$ are of codimension $n-1$, so any one of them passing by x will be a curve as we search for. ❑

7.20 The as followinh corillary goes under the name "the Fundamental Theorem of Elimination Theory" The variety Z is given by a collection of polynomials $f(a,x)$ with coefficients in A and the image $\pi(Z)$ consist of points a so that $f(a,x) = 0$ can be solved in x; that we elinimated the x's in the set of equations.

Corollary 7.21 *Let X be a variety and let $Z \subseteq X \times \mathbb{P}^n$ be a closed subset. Let $\pi \colon X \times \mathbb{P}^n \to X$ denote the projection. Then $\pi(Z)$ is closed.*

Proof: Because every variety has an open cover whose members are quasi-projective (*e.g.* affine), the proposition is easily reduced to the case that X is quasi-projective. We may thus assume that X is an open subset of a projective variety W. Let $\overline{Z}$ denote the closure of Z in $W \times Z$. The theorem yields that $\pi(\overline{Z})$ is closed in W, but the equality $\pi(Z) = X \cap \pi(\overline{Z})$ holding true, $\pi(Z)$ will be closed in X. Indeed, we have $Z = \overline{Z} \cap (X \times \mathbb{P}^n)$ because Z is closed in the open set $X \times \mathbb{P}^n$, from which ensues that

$$\{y\} \times \mathbb{P}^n \cap \overline{Z} = \{y\} \times \mathbb{P}^n \cap \overline{Z} \cap X \times \mathbb{P}^n = \{y\} \times \mathbb{P}^n \cap Z.$$

This entails the inclusion$X \cap \pi(\overline{Z}) \subseteq \pi(Z)$, and the reverse inclusion being trivial, we are through. ❑

Problems

In a topological space X a *locally closed subset* is a subset that is the intersection of an open and a closed set or in other words it is closed in an open set or for that matter, open in a closed one. A subset of X is *constructible set* if is the union of finitely many locally closed sets. A *Boolean algebra* $\mathcal{B}$ in X is a collection of subsets of X that is closed under finite set-theoretical operations; It contains the entire space and the empty set, finite unions and intersections of members of $\mathcal{B}$ belong to $\mathcal{B}$ and the set theoretical difference of two members is a member as well.

Locally closed sets (lokalt lukkede mengder)

Constructible sets (konstruktible mengder)

Boolean algebras (Boolske algebraer)

7.2 Let X be a topological space. Prove that the collection of constructible sets in X is the smallest Boolean algebra containing the open (or the closed) sets. Prove that inverse image of constructible sets under continuous maps are constructible.

7.3 *(Chevalley's Nullstellensatz.)* Show that the image of a constructible set under a morphism between varieties is constructible.

7.4 Let $\phi \colon X \to Y$ be a morphism of varieties and $r \in \mathbb{N}_0$ a non-negative integer. Show that the set $\{\, y \in Y \mid \dim \phi^{-1}(y) \,\} = r$ is locally closed.

7.5 Let X be an affine variety. Let f and g be regular functions on $X \times \mathbb{P}^1$ such that $Z(f,g) \cap \{x\} \times \mathbb{P}^1$ is finite for all $x \in X$. Show that $\pi(Z) \subseteq X$ has all components of codimesnion one. Conclude that if $A(X)$ is a UFD, then $\pi(Z) = Z(h)$ for some $h \in A(X)$.

7.6 *(The resultant.)* Let A be a ring. Denote by R_n the A-module consisting of polynomials in $A[t]$ of degree strictly less and n. Then R_n is free of rank n

with the powers t^i for $i \leqslant n$ as a basis. Given two polynomials $f(t)$ and $g(t)$ in $A[t]$ of degree n and m respectively. Consider the k-linear map

$$R_n \times R_m \to R_{n+m}.$$

that sends the pair (p, q) to $qf + pg$, and let Φ be its matrix in the bases whose elements are the powers of t. The determinant $\det \Phi$ is called the resultant of f and g and commonly written as $\operatorname{Res}(f, g)$.

a) Assume that A is the field k. Prove that $\det \Phi = 0$ if and only if f and g has a common root in some field extension of k. HINT: Start by showing that Φ has a kernel if and only if f and g has a common factor.

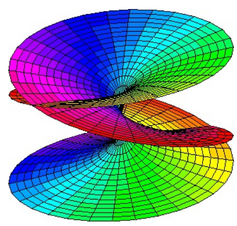
b) Let $\mathfrak{m}$ be a maximal ideal in A and let $k = A/\mathfrak{m}$. Denote by $\bar{h}$ the image in $k[t]$ of a polynomial $h \in A[t]$. Show that $\det \Phi \in \mathfrak{m}$ if and only $\bar{f}$ and $\bar{g}$ has a common root in an extension of k.

7.7 Show that $\operatorname{Res}(f, g)$ belongs to the ideal (f, g) in $A[t]$ generated by f and g.

★

7.3 *Finite maps*

Those who have followed course in complex function theory and Riemann surfaces have certainly seen holomorphic maps depicted as in the margin—which is typically how a finite map of complex curves look like near a branch point. Away from the branch points, locally they are just a bunch of stacked discs. Illustrations as those presuppose we have small open sets (that is, disks) to our disposal. We do not have that in the Zariski topology, and these pictures must as usual be taken with a grain of salt (when working in positive characteristic, the grain ought to be rather large). Anyhow, some features are general. We notice that the number of points in most fibres are the same (three in picture), with a correct interpretation of how to count points in the fibre, this is generally true. It is called the degree of the map.



However, there is a closed set of exceptional fibres. How their cardinality relate to the cardinality of the generic fibre depends havely on the target variety. I

In characteristic zero the counting is just the naive counting, but if the ground field has positive caracteristic p, a genuine multiplicity may occur.

There is multiplicity coming up. The example to have in mind is $\mathbb{A}^1 \to \mathbb{A}^1$ sending x to x^p. It is polynomial map corresponding to the map $t \to t^p$. The eauation $t^p = a$ has just one solution, and every fibre reduced to one point. However, the multiplicity will be p—which is in concordance with the usual way to assign multiplicities to roots of polynomials.

The degree

7.22 We start with a geometric set up with X and Y two varieties of the same dimension and $\phi\colon X \to Y$ a dominant morphism. As any dominant morphism, the morphism ϕ induces a field extension $K(Y) \subseteq K(X)$, which is algebraic since the two transcendence degrees over k are the same. Being function fields of varieties, both fields are finitely generated over k, and the extension is therefore finite. The degree $[K(X) : K(Y)]$ is called the *degree* of ϕ and denoted $\deg \phi$.

Degree of morphisms (*graden til avbildninger*)

Lemma 7.23 *Assume that ϕ and ψ are composable dominating maps between varieties of the same dimension. The the composition $\phi \circ \psi$ is dominating and the $\deg \phi \circ \psi = \deg \phi \cdot \deg \psi$. If both ϕ and ψ are finite, the composition $\psi \circ \phi$ is finite as well.*

Proof: Let the two morphisms be $\phi\colon X \to Y$ and $\psi\colon Y \to Z$, and let $U \subseteq Z$ be an arbitrary non-empty open subset. The inverse image $\psi^{-1}(U)$ is non-empty because ψ is dominating, and hence $\psi^{-1}(U)$ meets $\phi(X)$ since ϕ is dominating. But that is the same as to say that U meets $\psi(\phi(X))$.

The field extensions associated with the three involved maps constitute a tower of successive extensions

$$K(Z) \subseteq K(Y) \subseteq K(X),$$

and the degree of field extensions being multiplicative in towers, it follows $[K(X) : K(Z)] = [K(X) : K(Y)][K(Y) : K(Z)]$, or in other words that $\deg \psi \circ \phi = \deg \psi \cdot \deg \phi$. ❑

Example 7.1 The simplest example of the staging above, is the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^1$ that sends the point t to $f(t)$ where $f(t)$ is a given polynomial of degree n, say. In terms of a coordinate u on the second affine line, the extension of function fields corresponding to ϕ is the extension $k(u) \subseteq k(t)$ where $u = f(t)$. It is generated by t and the minimal equation is $f(t) - u = 0$ which is of degree n. The new degree thus passes the sanity test; in this simple case it coincides with the old definition.

As to the fibre of ϕ over a point a it is just given by the solutions of the equation $f(t) = a$. When f is separable, the derivative $f'(t)$ is a non-zero polynomial, and has finitely many roots, say $\{b_i\}$. For points a off the set $\{f(b_i)\}$ the derivative does not vanish at any of the solutions to $f(t) = a$, which therefore all are simple solutions. We conclude that for those a's there are exactly n-points in the fibre.

The inseparable case is more involved, and we illustrate that case by the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^1$ that sends t to t^p, where p now is the characteristic of k. The equation $t^p = a$ has exactly one solution, since for any $b \in k$ the equation $t^p - b^p = (t-b)^p$ holds true. All fibres of ϕ are therefore reduced to single point

point, but the degree of ϕ is p, so every points is counted with multiplicity p in its fibre. ☆

Multiplicities

The ideal behaviour of finite map $\phi\colon X \to Y$ is that the generic fibre have exactly n points. As examples show this is not true in general. The next best is an algebraic substitute, namely that the so-called "algebraic fibre" generically is of dimension n what kind of creature "algebraic fibre" ever is. This animal also enables us to defined the multiplicity of the points in a fibre.

7.24 To explain what is meant by the "algebraic fibre" we assume that both X and Y are affine, and as usual we shall denote their coordinate rings by $A(X)$ and $A(Y)$. Fix a point $y \in Y$. Points belonging to the fibre $\phi^{-1}(y)$ correspond to maximal ideals $\mathfrak{m}_x$ in $A(X)$ containing the ideal $\mathfrak{m}_y A(X)$. If $x_1, \ldots, x_r$ constitute the fibre, these maximal idels are $\mathfrak{m}_{x_1}, \ldots, \mathfrak{m}_{x_r}$. The ring $A(X)/\mathfrak{m}_y A(X)$ is Artinian, and the primary decomposition of $\mathfrak{m}_y A(X)$ takes the form

$$\mathfrak{m}_y A(X) = \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$$

where the $\mathfrak{q}_i$'s are $\mathfrak{m}_{x_i}$-primary ideals. Therefore, according to The Chinese Remainder Theorem, there is decomposition

$$A(X)/\mathfrak{m}_y A(X) = \prod_i A(X)/\mathfrak{q}_i,$$

and the factor $A(X)/\mathfrak{q}_i$ is supported at $\mathfrak{m}_{x_i}$. This is the motivates for babtizing the ring $A(X)/\mathfrak{m}_y A(Y)$ the "algebraic fiber" over y, and in lack of a better name, we shall call the dimesnion $\dim_k A(X)/\mathfrak{m}_y A(Y)$ the *algebraic cardinality* of the fibre.

The algebraic cardinality (*den algebariske kardinaliteten*)

7.25 Taking dimensions we arrive at multiplicities; the multiplicity of the point x_i in the fibre is defined as $\dim_k A(X)/\mathfrak{q}_i$. Observe that x_i is a simple point of the fibre, *i.e.* of multiplicity one, precisely when $\mathfrak{q}_i = \mathfrak{m}_i$. This holds along the entire fibre if and only if $A(X)/\mathfrak{m}_y A(X)$ is a product of fields, or phrased in a different manner, if and only if the algebraic fibre $A(X)/\mathfrak{m}_y A(X)$ is a reduced ring (*i.e.* without nilpotent elements).

EXAMPLE 7.2 Normalization maps have typically fibres with too many points. Our two acquaintances, the standard rational cusp and the standard rational double point, are good examples. The cusp C is the image of the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ defined by $x = t^2$ and $y = t^3$ and is given by the equation $y^2 = x^3$. On the level of rings the map $\phi^*\colon k[x,y] \to k[t]$ is given as $x \mapsto t^2$ and $y \mapsto t^3$. For a point $(a,b) \in C$ with $b^2 = a^2$, one finds when $a \neq 0$, that

$$(x-a, y-b)k[t] = (t^2 - a, t^3 - b) = (at - b) = (t - ba^{-1}).$$

The fibre over (a,b) is therefore just a single simple point, namely the point $b/a \in \mathbb{A}^1$. Over the origin, however, the algebraic fibre is $k[t]/(t^2, t^3) = k[t]/t^2$

which is two-dimensional. That fibre has also just one point, but with multiplicity two. ☆

EXAMPLE 7.3 The double-point D is the image of $\mathbb{A}^1$ under the map $t \mapsto (t^2-1, t(t^2-1))$. The equation of the image is $y^2 = x^2(x+1)$. To determine the fibres, let (a,b) be a point on D that is not the origin.The equalities

$$(x-a, y-b)k[t] = (t^2-1-a, t(t^2-1)-b) = (at-b) = (t-b/a),$$

shows there is a single point in the fibre which is simple. If $(a,b) = (0,0)$ however, on finds

$$k[t]/(t^2-1, t(t^2-1)) = k[t]/(t^2-1).$$

The structure of this ring depends on the characteristic of the ground field k. If it is different from two, the ring splits as the product of two copies of k, and the fibre over the origin consists of two distinct simple points. Is the characteristic two, the ring equals $k[t]/(t-1)^2$ which has just one maximal ideal. The fibre in that case consists of a single point but which has multiplicity two. ☆

EXAMPLE 7.4 Assume that C and D are non-singular curves and that $\phi\colon C \to D$ is a finite morphism. Pick a point $y \in D$ and consider the ibre $A(C)/\mathfrak{m}_y A(C)$. For each point $x_i \in \phi^{-1}(y)$ in the fibre over y the corresponding primary ideal $\mathfrak{q}_i$ is given as $\mathfrak{q}_i = A(D) \cap \mathfrak{m}_y \mathcal{O}_{D,x_i}$ and hence the contribution $A(D)/\mathfrak{q}_i A(D)$ is therefore equal to $\mathcal{O}_{D,x_i}/\mathfrak{m}_y \mathcal{O}_{D,x_i}$.

The rings $\mathcal{O}_{D,y}$ and $\mathcal{O}_{C,x_i}$ are all DVR's. ☆

PROBLEM 7.8 Let $\phi\colon \mathbb{P}^2 \backslash (1:0:0) \to$ be the map that sends $(u:v:w)$ to $(uv : v^2 : w^2 : uw)$. Prove that image is contained in $Z_+(x_0^2 x_2 - x_3^2 x_1)$ and determine all fibres. ★

PROBLEM 7.9 Asume that p and q are two relatively prime numbers. Let $C \subseteq \mathbb{A}^2$ be the image of the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ given as $t \mapsto (t^p, t^q)$. Show that $C = Z(x^q - y^p)$. Prove that ϕ is a finite map and determine all fibres of ϕ. ★

Generic freeness

As alluded to in the previous paragraph, there are examples without reduced fibres. However, even in that case, the dimension $\dim_k A(X)/\mathfrak{m}_y A(X)$ will be constant and equal the $\deg \phi$ for y in a sufficiently small open set (but of course dense). Generic fibres all have $\deg \phi$ points when appropriate multiplicities are taken into account; this ensues straight away from the following proposition. Is the map ϕ is separable, one can say more. In that case, as we shall prove next, the generic fibres will be reduced; that is all their points are simple.

Proposition 7.26 (Generic freeness) *Let $\phi : X \to Y$ be two varieties, and let ϕ be a finite dominating map $\phi\colon X \to Y$. Then there is a dense open and affine set U in Y such that $\phi^{-1}(U)$ is affine and $A(\phi^{-1}(U))$ is a free module over $A(U)$ of rank equal to* $\deg \phi$.

The proof is reduced to a piece of algebra contained in two subsequent lemmas

Lemma 7.27 *Let $A \subseteq B$ be a finite extension of domains and let $K \subseteq L$ be the corresponding extension of fraction fields. Then the $K \subseteq L$ is a finite extension, and if S denotes the multiplicative set $S = A\backslash\{0\}$, it holds true that $L = B_S = B \otimes_A K$*

Proof: The algebra B is a finite module over A and therefore all its elements are integral over A. Any given $f \in A$ thus satisfies an equation of integral dependence

$$f^{\nu} + a_{\nu-1}f^{\nu-1} + \ldots + a_1 f + a_0 = 0$$

with the constant term $a_0 \neq 0$. It follows that

$$f^{-1} = -a_0^{-1}(f^{n-1} + a_{\nu-1}f^{n-2} + \ldots + a_1),$$

and consequently that $f^{-1} \in B_S$. And since this holds for all $f \in B$, it ensues that $L = B_S$. In particular, any generating set of B as an A-module will be a generating set for L over K and L will be finite over K as B is finite over A. ❑

The dimension $\dim_K L$ of L as a vector space over K is called the *degree* of the field extension $K \subseteq L$ and it is common usage to denote it by $[L : K]$.

Degree of field extensions (*graden til kroppsutvidelser*)

Lemma 7.28 *With the setting as in the previous lemma, there is an $g \in A$ such that $B_g = B[1/g] = B \otimes_A A_g$ is a free A_g-module of rank equal to the degree $[L : K]$.*

Proof: This is just a matter of pinning down common denominators. From the previous lemma ensues that there is a basis for L over K of elements of the form $c_i a_i^{-1}$ with $c_i \in B$ and $a_i \in A$. Replacing the a_i's by their product, we may assume that they are all equal and that $c_i = b_i a^{-1}$. The c_i will not *a priori* generated B_a over A_a. They form however a basis for L over K and consequently all elements d_j from a finite generating set for B_a over A_a are sent into A when multiplied by appropriate elements g_j from A_a. Over the localized ring A_g, where g is the product of a and the g_j's, the elements c_i's form a basis for B_g. ❑

Proof of Proposition 7.26: This is just a matter of translating Lemma 7.28 into geomtry. We may assume that X and Y affine; just replaces Y by an open affine whose inverse image is affine and X by that inverse image. After Lemma 7.28 with $A = A(Y)$ and $B = A(X)$ we may find a regular function $g \in A(Y)$ so that the localization $A(X)_g = A(X)_{\phi\circ g}$ is free of rank $[K(X) : K(Y)]$ over $A(Y)_g$. Now, $A(Y)_g$ is the coordinate ring of the distinguished open set $D(g)$ in U, and obviously $\phi^{-1}(D(g)) = D(\phi \circ g)$ whose coordinate ring is $A(X)_{\phi\circ g}$, and that's it. ❑

The separable case

As promised we shall take a closer look at the generic fibres of separable finite morphisms, which we shall prove are reduced. The basic tool will be the Primitive Element Theorem.

7.29 Recall that a field extension $K \subseteq L$ is called primitive if it has one single generator; that is $L = K(f)$ for an element f from K. The generator f is naturally called a *primitive element*. Primitive extensions are a lot easier to handle than general ones, and luckily they are frequent. For instance, any separable extension is primitive—this is the Primitive Element Theorem—and of course, every finite extension is a sequence of primitive ones.

Psrimitive element (*primitive elementer*)

7.30 We are interested in primitive extensions because they makes it easier to unveil the finer generic behaviour of finite morphisms based on the following lemma.

LEMMA 7.31 *Let $\phi\colon X \to Y$ be a finite morphism between two varieties and assume that function field $K(X)$ is a primitive extension of $K(Y)$. Then there is an open affine U of Y such that $V = \phi^{-1}(U)$ is affine and such that*

$$A(V) \simeq A(U)[t]/(F(t))$$

for some polynomial $F(t)$ whose coefficient are regular functions on U.

PROOF: Let f the element in $K(X)$ that generate $K(X)$ over $K(Y)$. The field $K(X)$ is algebraic over $K(Y)$ and f satisfies an algebraic dependence equation

$$a_n f^n + a_{n_1} f^{n-1} + \ldots + a_0 = 0$$

where the a_i are rational functions on Y, and we may suppose that $F(t) = \sum_i a_i t^i$ is an irreducible polynomial over $K(Y)$. Now, we may find an open affine U in Y where the coefficients a_i all are regular and where a_n is without zeros. Additionally we may assume that $V = \phi^{-1}(U)$ is affine after having shrunk U further if necessary. The ring $A(V)$ is a finite module over $A(U)$, and the members of a generating set are linear combinations of powers of f with coefficients in $K(Y)$. Shrinking U further, we may assume that all these coefficients are regular in U. Then clearly

$$A(V) = A(U)(f) \simeq A(U)[t]/(F(t)).$$

❏

7.32 The isomorphism in the lemma is defined by sending the variable t to the function f. One may interpret this in a geometric way as V lying in the product $U \times \mathbb{A}^1$ embedded as the zero locus $Z(F(t))$, and the map ϕ is induced by the projection onto the first factor; remember that the coefficients of F are functions on U, so $F(y,t)$ would be a more precise notation. For a given point

$y_0 \in U$, the fibre over y_0 is formed by the zeros of the polynomial $F(y_0, t)$, obtained by evaluating the coefficients a_i at the point y_0.

The cardinality of a fibre (over a point in U) is clearly bounded above by $\deg \phi$, but in general any smaller cardinality may occur. However, an assumption that ϕ be separable, improves the situation conciderably as the following proposition shows.

Proposition 7.33 *Assume that $\phi\colon X \to Y$ is a finite and separable morphism between two varieties. Then there is an open affine subset U of Y such all fibres over points in U consist of $\deg \phi$ different points.*

Proof: The derivative of $F'(y, t)$ with respect to t is as usual comuputed as $\sum_i ia_i t^{i-1}$. The hypothesis that f be separable ensures that the derivative $F'(t)$ share no common zero with $F(t)$ in $K(X)$. This shows that $Z(F'(y, t)) \cap Z(F(y, t))$ is a proper subset of $Z(F(y, t))$. Hence it does not dominate U, and the image D is a proper closed set of U (finite maps are closed xxx). The points of the fibre over y_0 are the zeros of the $F(y_0, t)$ but when $y_0 \notin D$, the derivative $F'(y_0, t)$ does not vanish in any of the points, and they are all simple zeros. Consequently there are as many as the $\deg \phi$ indicates. ❑

Example 7.5 Consider the map $\mathbb{A}^1$ to $\mathbb{A}^1$ sending x to x^p. On the level of coordinate rings it is given as $f(t) \mapsto f(t^p)$. Ir is finite og degree p, but all its fibres consist of just one point. $k[t]/(t^p - a)k[t]$. But $(t^p - a) = (t - b)^p$ so $(t - a)$ is the only maximal ideal containing $t^p - a$. ☆

Problem 7.10 Given natural numbers n and let m with $m \leqslant n$ Construct a finite map of degree n having exactly m points in one of its fibres. ★

7.4 *Curves over regular curves*

When the target of a morphism is a non-singular curve, more can be said of its fibres. We illustrate this with finite maps, and the result will be useful when we attack the proof of Bezout's theorem in next section. The staging will be as follows. The givens are a regular curve C and a closed subset $Z \subseteq C \times \mathbb{P}^n$ whose components $Z_1, \ldots, Z_r$ are curves that all dominates C.

We let $\pi\colon C \times \mathbb{P}^n \to C$ be the projection and for index each i its restriction to Z_i will be denoted by π_i. By assumption the π_i's are dominating, and by xxx they are all finite morphisms. Finally, π_Z will be the restriction of π to Z. We shall debote by $\deg \pi_Z$ the sum $\deg \pi_Z = \sum_i \deg \pi_i$.

7.34 Except for closed algebraic subsets of affine space, we have developed the general theory for varieties which all have been assumed to be irreducibe.

This has certainly made life agreeable, but we now have come to point where there is a price for this, and admittedly some inelegant wriggling will be necessary. When proving Bezout's theorem in the next section the Z

will certainly not be irreducible. The minimal thing is to stick with an affine situation.

PROPOSITION 7.35 (THE PERMANENCE OF NUMBERS) *Let Z be a closed algebraic set whose irreducible components are $Z_1, \ldots, Z_r$. Let C be a non-singular affine curve and assume that $\pi\colon Z \to C$ is a finite polynomial map and that the restrictions π_{Z_i} all are dominating. Then all fibres of π have the same algebraic cardinality; that is, for $\dim_k A(Z)/\mathfrak{m}_y A(Z)$ is independent of $y \in C$.*

Before proving the Proposition, we give an elegant corollary:

COROLLARY 7.36 *Let X be a projective curve (i.e. a variety of dimension one) and π a dominating morphism to a non-singular curve C. The all fibres of π have the same algebraic cardinality.*

PROOF: By xxx, the morphism π is finite, and we may thus cover C by open affines U_i so that $V_i = \pi^{-1}U_i$ are affine and each $A(V_i)$ is a finite module over $A(U_i)$. By the Proposition the algebraic fibre dimension will be constant within each U_i, but C being irreducible, any two U_i's will meet. ❑

PROOF OF PROPOSITION 7.35: The main point is that $A(Z)$ is a torsion free $A(C)$-module, and since it by assumption is finite, we are through by the little pieces of algebra below since C being non-singular is synonymous to $A(C)$ being a Dedekind ring. To see that $A(Z)$ is torsion free, we consider the inclusion

$$A(Z) \subseteq \prod_i A(Z_i).$$

Since each Z_i is assumed to dominate C, the natural maps $A(C) \to A(Z_i)$ are injective, but the $A(Z_i)$'s being integral domains, they are torsion free. ❑

Two little pieces of algebra

7.37 Two little pieces of algebra The pieces of algebra we need are is contained in the two subsequent lemmas

LEMMA 7.38 *If the ring A is a PID then every finitely genetared and torsion free A-module M is free of rank $\dim_K M \otimes_A K$.*

There is a more general version of this lemma with the hypothesis relaxed to A being a Dedekind ring, but the conclusion is then that the module is a projective A-module. The proof is *mutatis mutandis* the same.

PROOF: Let K be the fraction field of A. The proof goes by induction on the rank of M, that is the dimension $\dim_K K \otimes_A M$. We clearly may assume that M is not the zero module.

The first observation is that $\mathrm{Hom}_A(M, A)$ is non-zero. Indeed, since M is torsion free, the localization $M \otimes_A K$ is non-zero, and there is a non-zero linear functional $\lambda\colon M\otimes_A K \to K$. The values $\lambda(m_i)$ that λ assumes on a finite set of generators $\{m_i\}$ for M, are all of the form $a_i b_i^{-1}$ with a_i and b_i being non-zero elements from A. If b denotes the product $b = \prod_i b_i$, then $b\lambda$ is non-zero and takes M into A.

The next observation is that the image $\lambda(M) \subseteq A$ is isomorphic to A because all ideals in A are principal. The exact sequence

$$0 \longrightarrow \ker\lambda \longrightarrow M \xrightarrow{\lambda} A \longrightarrow 0 \tag{7.1}$$

is split so that $M \simeq \ker\lambda \oplus A$. Hence $\ker\lambda$ is torsion free and of rank one less that M, and it is therefore free by induction. It follows that the sequence (7.1) is split, and M is free as well. ❑

Lemma 7.39 *If A is a Dedekind ring and M is a finitely genertaed torsion free A-module, then $\dim_{A/\mathfrak{m}} M\otimes_A A/\mathfrak{m} = \dim_K M\otimes_A K$ for all maximal ideals $\mathfrak{m}$ in A.*

Proof: The ring A being Dedekind means that all the localizations $A_\mathfrak{m}$ are DVR's; hence they are PID's. Now the localiszations $M_\mathfrak{m}$ persist being torsion free and finitely generated over $A_\mathfrak{m}$, hence by Lemma 7.38 above they are free of the same rank, namley $\dim_K M\otimes_A K$. ❑

Problem 7.11 Assume that C is non-singular projective curve so that all the local rings $\mathcal{O}_{C,x}$ ar DVR's. Denote the normalized valuation of $\mathcal{O}_{C,x}$ by v_x. With any non-zero rational function f on C, one associate the divisor $(f) = \sum_{x\in C} v_x(f)x$. Show that $\deg(f) = 0$. ★

Lecture 8

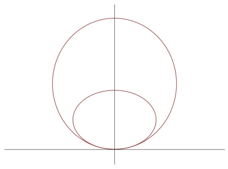
Bézout's theorem

Hot themes in Lecture 8: *Divisors—Local multiplicities—Bézout's theorem—Regular sequences and depth—Cohen–Macaulay rings—The Unmixedness theorem—A little about graded modules*

Étienne Bézout (1730–1783) French Mathematician

Sir Isaac Newton observed in a note dated 30 May 1665 that the number of intersection points of two curves in $\mathbb{P}^2$ equals the product of their degrees. If one starts looking at examples, this pattern emerges almost immediately. Two lines meet in one point and two conics in four—at least if the two conics are in what one calls general position; that is, they are not tangent at the intersection points.

That a line L in $\mathbb{P}^2$ meets a curve X which is the zero locus of a homogeneous form $F(x_0, x_1, x_2)$ of degree n, is a direct consequence of the fundamental theorem of algebra. Choosing appropriate coordinates we can parametrize the line as $(u : v : 0)$. The parameter values of the intersection points will be the roots of the equation $F(u, v, 0) = 0$, of which there are n, unless, of course, x_2 is a factor of F, in which case the line L is a component of X. There is also an issue of multiplicities, roots need not be simple, and to get n intersection points these multiplicities must be taken into account. This issue persists in the general situation and is an inherent part of the problem.

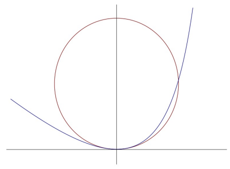


Two ellipses with fourth order contact.

Example 8.1 The local multiplicity is, as this example shows, a quite subtle invariant even for conics. The two conics $zy - x^2$ and $zy - x^2 - y^2$ only intersect at $(0 : 0 : 1)$. Indeed, the difference of the two equations being y^2, it must hold that $y = 0$ at a common zero and then x must vanish there as well. The ellipses have contact oder four at $(0; 0; 1)$: Inserting the parametrization (uv, u^2, v^2) of the first into the equation of the second yields the equation $u^4 = 0$ which has a quadruple root at $u = 0$. ☆

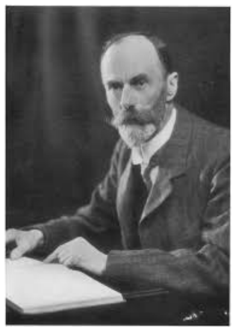
Problem 8.1 Along the lines above, prove that a conic intersects a curve of degree n in $2n$ points multiplicities taken into account unless the conic is a component of the curve. Hint: Parametrize the conic as $(u^2, uv : v^2)$. ★

8.1 What nowadays is called Bézout's theorem in the plane was, as we have indicated in the beginning, known long time before Bézout published his famous paper *Théorie générale des équations algébriques* in 1779. His original contribution is the generalization to projective n-space $\mathbb{P}^n$. He asserted that the number of points n hypersurfaces in $\mathbb{P}^n$ have in common, when finite, is at most the product of the degrees of the hypersurfaces, and there is equality when the hypersurfaces are general; that is, when they meet transversally. As usual there is an issue of multiplicities. Local multiplicities are part of the accounting and with the correct definition of these multiplicities, the number of intersections will always be the product of the degrees. It seems that the first correct proofs of the full Bézout-theorem were given by Georges Henri Halphen and a little later by Hurewitz.



Needless to say, but Bézout's theorem is the seed of intersection theory and has been vastly generalized. To day it is merely a tiny part of a great theoretical body called intersection theory. See Fultons's book xxx.

8.2 To prove Bézout's Theorem there are several lines of reasoning to follow, but at the end they all rely on one of Macaulay's renown results called the "Unmixedness theorem". The classical technique used by Bézout and his contemporaries was build on projections. They projected the intersection of the n hypersurfaces into a line where it was described by the vanishing of a certain polynomial, the so-called *resultant*. We met a specimen of the kind in exercise xxx.



Francis Sowerby Macaulay (1862–1937) British Mathematician

8.3 The proof we shall present naturally belongs to the realm of what are called coherent sheaves on $\mathbb{P}^2$ and their Euler characteristics. However, we do not have all that advanced machinery to our disposal and have to do with an *ad hoc* version of it.

It is based on the notion of regular sequences, the forms $F_1, \ldots, F_n$ defining the hypersurfaces must form a regular sequence (see paragraph 8.4 below). This is *a priori* far stronger assumption than the locus of their common zeros being finite, but then the Macaulay's Unmixedness Theorem enters the scene. It asserts that whenever $Z_+(F_1, \ldots, F_n)$ is finite the F_i's in fact form a regular sequence. Thus the algebraic condition that the hypersurfaces form a regular sequence (hard to check) is reduced to the geometric condition that the intersection be finite (substansially easier to check).

8.1 *Bézout's Theorem*

8.4 In the rest of section we are given n hypersurfaces $Z_1, \ldots, Z_n$ in $\mathbb{P}^n$ defined by the vanishing of the homogeneous polynomials $F_1, \ldots, F_n$. There is a standing hypothesis that they intersect in finitely many points.

With every point $p \in \mathbb{P}^n$ one may, and we shall shortly do, associate a multiplicity $\mu_p(Z_1, \ldots, Z_n)$. It is a non-negative integer which is positive if and only if p belongs to the intersection $Z_1 \cap \ldots \cap Z_n$. With this in place, Bézout's theorem reads as follows

THEOREM 8.5 *Let $Z_1, \ldots, Z_n$ be hypersurfaces in $\mathbb{P}^n$ with only finitely many points in common. Then*

$$\deg Z_1 \cdots \deg Z_n = \sum_p \mu_p(Z_1, \ldots, Z_n).$$

Notice that one can only hope for such a result when the number of hypersurfaces is n. If there are less, the intersection cannot be finite; indeed, by Krull's Hauptidalsatz the codimension of the intersection would be less than n and the dimension at least one, and the intersection would have had an infinity of points. And if there are more, we have no control on the number of intersection points, allthough it will be zero for a general choice of hypersurfaces.

PROBLEM 8.2 Give examples of three conics in $\mathbb{P}^2$ that intersect in 0, 1, 2, 3 and 4 points. ★

8.6 It is quit natural to extend the scope of Bezout's theorem slightly to also encompass intersections of *effective divisor*[1]. Recall that such an animal is a *formal* linear combination $\sum_i m_i Z_i$ of irreducible hypersurfaces Z_i with non-negative integral coefficients. This might look enigmatic at the first encounter, but it is merely a convenient and geometrically suggestive way to keep track of the irreducible components of a homogeneous form. Indeed, if F is a homogeneous form of degree n that splits as $F = \prod_i F_i^{m_i}$ into a product of irreducible forms, the associate divisor is $\sum_i m_i Z_+(F_i)$.

Effective divisors (*effektiv divisorer*)

[1] The significance of the attribute *effective* is that the coefficients n_i are non-negative. A *divisor* is a linear combination $\sum_i n_i Z_i$ with integral coefficients

The *degree* of an effective divisor is defined to be the degree of the corresponding homogeneous form. Divisors can be added just by adding coefficients, and one clearly has $\deg(Z + Z)' = \deg Z + \deg Z'$. The definition of the local multiplicities extends, and in terms of effective divisors Bézout's Theorem takes the form

Degree of divisors (*graden til divisorer*)

THEOREM 8.7 *Let $Z_1, \ldots, Z_n$ be effective divors in $\mathbb{P}^n$ with only finitely many points in common. Then*

$$\deg Z_1 \cdots \deg Z_n = \sum_p \mu_p(Z_1, \ldots, Z_n).$$

8.2 *The local multiplicity*

The natural starting point is to define the local multiplicities. So we continue working with the given hypersurfaces $Z_1, \ldots, Z_n$, or one should rather say effective divisors, and t their homogeneous equations are syill $F_1, \ldots, F_n$. The F_i's need not be irreducible and can even have factors with exponents higher than one, but we the standing assumption that the intersection $Z_1 \cap \ldots \cap Z_n$ is finite, is in force.

8.8 To set the stage, we chose coordinates $(x_0 : \ldots : x_n)$ on $\mathbb{P}^n$ so that the intersection is contained in the distinguished open set $U = D_+(x_0)$. One may in fact use any open affine containing the intersection, but for the presentation it is convenient to use a standard affine.

The coordinates we shall use in U will be $t_i = x_i/x_0$, and the equations of the subvarieties $Z_i \cap U$ of U are the dehomogenized polynomials $f_i(t_1, \ldots, t_n) = F_i(1, x_1/x_0, \ldots, x_n/x_0)$ which live in the coordinate ring $A(U) = k[t_1, \ldots, t_n]$.

In this setting, the intersection $Z_1 \cap \ldots \cap Z_n$ equals the closed algebraic subset $Z(f_1, \ldots, f_n)$ of $D_+(x_0)$, and by the standing hypothesis this is a finite set. Consequently the ring $\mathcal{O}_{Z_1 \cap \ldots \cap Z_n} = k[t_1, \ldots, t_n]/(f_1, \ldots, f_n)$ is Artinian. As any Artinian rings, it is isomorphic to the direct product of its localizations; that is, one has a natural isomorphism

$$\mathcal{O}_{Z_1 \cap \ldots \cap Z_n} \simeq \prod_p \mathcal{O}_{Z_1 \cap \ldots \cap Z_n, p}$$

where the product extends over points p from the intersection $Z_1 \cap \ldots \cap Z_n$.

We are ready for the definition of the *intersection multiplicity* at p also called the *local intersection number* at p. The local rings $\mathcal{O}_{Z_1 \cap \ldots \cap Z_n, p}$ are all finite dimensional over the ground field k, and we define

The intersection multiplicity (snittmultiplisitet)
Local intersection numbers (lokale snitt-tall)

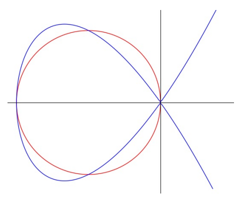
$$\mu_p(Z_1, \ldots, Z_n) = \dim_k \mathcal{O}_{Z_1 \cap \ldots \cap Z_n, p}.$$

If p does not belong to the intersection, we let $\mu_p(Z_1, \ldots, Z_n) = 0$.

Examples

The higher multiplicities are caused by two different phenomena, tangency and singularity—the hypersurfaces involved can be tangent at the intersection point, or one or more of them can have a singularity; thas is, the defining form vanishes to the second order at the point. In both cases there will be a higher multiplicity assigned to the point. If neither of the two phenomena occur, the contribution of the point is just one, and we say that intersection is *transversal* at the point.

Transversal intersections (transversale snitt)

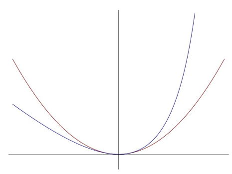


EXAMPLE 8.2 In the margin we have depicted two curves, the circle $(x+1)^2 + y^2 = 1$ and the cubic $y^2 = x^2(x+2)$. They intersect in four points. From the left there is a tangency where the multiplicity is two, then come two points where the intersection is transversal each contributing one to the total, and finally in the last point the cubic aquires a double point, and the local multiplicity is two. ☆

EXAMPLE 8.3 Let $f(x, y) = y - x^2$ and $g(x, y) = y - x^2 - xy$. Then the two conics $X = Z(f)$ and $Y = Z(g)$ have triple contact at the origin; that is, $\mu_p(X, Y) = 3$. One finds

$$(f, g) = (y - x^2, y - x^2 - xy) = (y - x^2, x^3),$$

and hence $\mathcal{O}_{X \cap Y, p} = k[x, y]/(f, g) \simeq k[x]/x^3$ ☆



Two parabolas with triple contact at the origin.

PROBLEM 8.3 Find the intersection and the local multiplicities of the three surfaces in $\mathbb{P}^3$ given by $xy - zw$, $xz - yw$ and $xw - yz$. ★

PROBLEM 8.4 Prove that $xy - zw$ and $x^2y - z^2x$ intersect along five lines. Find the intersection of $y - x$, $xy - zw$ and $x^2y - z^2x$. ★

Transversal intersections

8.9 A hypersurface X in $\mathbb{A}^n$ given by the polynomial f which passes through the point p, is said to be *regular* or *non-singular* at p if f does not vanish to the second order there. If $\mathfrak{m}$ denotes the maximal ideal at p, the polynomial f is required not to belong to the square $\mathfrak{m}^2$; that is, $f \notin \mathfrak{m}^2$.

Non-singular or regular points (ikkesingulære eller regulære punkter)

One may write $f = \lambda + g$ where $g \in \mathfrak{m}^2$ and λ is an element whose class in $\mathfrak{m}/\mathfrak{m}^2$ is non-zero; this class $\bar{\lambda}$ defines a *normal* to the hypersurface X at p. Its orthogonal complement in the dual space $\mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k)$ is the *tangent space* to X at p.

Normals of hypersurfaces (normalen til en hyperflate)

Tangent spaces (tangentrom)

One says that r hypersurfaces with equations $f_1, \ldots, f_r$ meet *transversally* at the point p if they all are regular at p and their normals $\bar{\lambda}_i$ are linearly independent in k-vector space $\mathfrak{m}/\mathfrak{m}^2$.

Transversal hypersurfaces (transversale hyperflater)

8.10 The following lemma is almost tautological, and tells us that the Z_i's meet transversally at p if and only if $\mu_p(Z_1, \ldots, Z_n) = 1$.

LEMMA 8.11 *Let A denote the localisation of the polynomial ring $k[t_1, \ldots, t_n]$ in the maximal ideal $\mathfrak{m} = (t_1, \ldots, t_n)$. Then n elements $f_1, \ldots, f_n$ from $\mathfrak{m}$ meet transversally at the origin if and only if $(f_1, \ldots, f_n) = \mathfrak{m}$.*

PROOF: This is just Nakayama's lemma. The images of each f_i in $\mathfrak{m}/\mathfrak{m}^2$ is the normal vector $\bar{\lambda}_i$, hence the f_i generate $\mathfrak{m}$ if and only if the $\bar{\lambda}_i$'s generate $\mathfrak{m}/\mathfrak{m}^2$. But there are many $\bar{\lambda}_i$'s as the dimesnion of $\mathfrak{m}/\mathfrak{m}^2$, hence they generate if and only if they are linearly independent. ❑

Additivity

The local intersection number is additive in the sense that if one of the divisors splits into as sum, say that the last one decomposes as $Z_n = Z_n' + Z_n''$, the following addition formula holds true

$$\mu_p(Z_1, \ldots, Z_n' + Z_n'') = \mu_p(Z_1, .., Z_n') + \mu_p(Z_1, \ldots, Z_n'').$$

However it is astonishingly subtle to prove and hinges on the "Unmixedness theorem" of Macaulay.

Let f_n' and f_n'' be the equations of Z_n' and Z_n'', and let A be the local ring $A = \big(k[t_1, \ldots, t_n]/(f_1, \ldots, f_{n-1})\big)_{\mathfrak{m}_p}$. One has the exact sequence

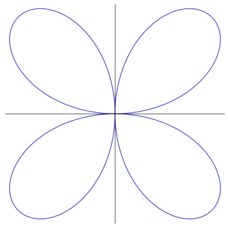
$$A/(f_n'')A \xrightarrow{\alpha} A/(f_n' f_n'')A \longrightarrow A/(f_n')A \longrightarrow 0$$

where $\alpha(a) = f_n' a$ and the rightmost map is the natural surjection. The sequence is exact—that is, α is injective— precisely when the vector space dimensions over k of the involved rings add up; in other words, when the local intersection numbers add up. However, this requires that f_n' be a non-zero divisor in A so that $f_n' a = b f_n' f_n''$ implies that $a = b f_n''$.

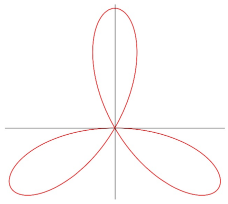
This is not generally true for one dimensional local rings even if neither of f_n' and f_n'' lies in any of the minimal primes of A. Denizens from the deep algebraic waters—the notorious embedded components threaten emerge—but thank's to Macaulay the threat is not realized in our situation. There are no embedded components and additivity holds true.

Problem 8.5 In case $n = 2$ the unmixedness theorem is almost trivial, and additivity comes for free. Prove additivity for local intersections multiplicities for two effective divisors in $\mathbb{P}^2$. ★

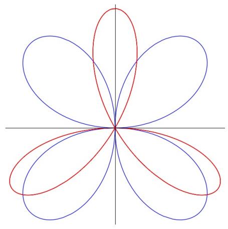
Problem 8.6 The relative behaviour of two intersecting curve can be rather complicated. The following example is taken from William Fulton's book[2] Show that the local intersection number at the origin of the two curves $(x^2 + y^2)^2 + 3x^2y - y^3 = 0$ and $(x^2 + y^2)^3 - x^2y^2 = 0$ equals 14. Where else do they intersect? Hint: Additivity can be useful. ★



$(x^2 + y^2)^2 + 3x^2y - y^3 = 0$



$(x^2 + y^2)^3 - x^2y^2 = 0$



[2]

8.3 *Proof of Bezout's theorem*

With the Unmixedness Theorem in mind, we proceed with proving Bézout's theorem under the assumption that the polynomials $F_1, \ldots, F_n$ form a regular sequence. And there will two parts, first we shall identifying the product of the degrees of the n hypersurfaces as the Hilbert polynomial of the graded ring $S = k[x_0, \ldots, x_n]/(F_1, \ldots, F_n)$ (which is constant), and subsequently show that this constant equals the sum of the local multiplicities.

The Hilbert polynomial

So the first step is thus to establish the following formula

Lemma 8.12 $h_S(t) = \deg Z_1 \ldots \deg Z_n$.

It follows as a special case of lemma 8.14 below

8.13 To prepare for lemma 8.14 we introduce the quotient ring $S_r = R/(F_1, \ldots, F_r)$ for each r with $1 \leqslant r \leqslant n$, and for convenience we let $S_0 = R$; moreover we put $d_i = \deg Z_i$ and $d_0 = 1$. The sequence $F_1, \ldots, F_n$ being regular means by definition there are short exact sequences

$$0 \longrightarrow S_r[-d_{r+1}] \xrightarrow{F_{r+1}} S_r \longrightarrow S_{r+1} \longrightarrow 0$$

for each $0 \leqslant r \leqslant n-1$, where the indicated map is just multiplication by F_{r+1}. Bboth maps in the sequences are homogeneous of degee 0, and the following identity between Hilbert polynomials ensues

$$h_{S_{r+1}}(t) = h_{S_r}(t) - h_{S_r}(t - d_{r+1}).$$

Now, it is an elementary fact that for any polynomial $P(t)$ of degree r with leading coefficient a, the difference $P(t) - P(t-d)$ is of degree $r-1$ with leading coefficient rda; indeed, for any natural number m the Binomial Theorem yields the equality

$$t^m - (t-d)^m = md \cdot t^{m-1} + o(m-2),$$

where $o(m-2)$ stands for a polynomial term of degree less than $m-2$. Using this, a straightforward induction gives the following (remember that $d_0 = 1$):

Lemma 8.14 *For any r with $0 \leqslant r \leqslant n$ it holds true that*

$$h_{S_r}(t) = d_0 \cdot \ldots \cdot d_r \frac{t^{n-r}}{(n-r)!} + o(n-r-1)$$

where the term $o(n-r-1)$ is a polynomial of degree $n-r-1$. In particular for $r = n$, we have $h_S(t) = d_1 \ldots d_n$.

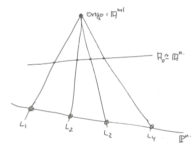
Proof: Induction on r ❑

The local intersection numbers

The graded ring $S = k[x_1, \ldots, x_n]/(F_1, \ldots, F_n)$ is the coordinate ring of the cone $Z(F_1, \ldots, F_n)$ over the intersection $Z_1 \cap \ldots \cap Z_n$, and this cone is just a bunch of lines through the origin. Hence S is a one dimensional ring and its Hilbert polynomial is therefore constant. The crucial equality we are to establish is the following

$$h_S(t) = \dim_k \mathcal{O}_{Z_1 \cap \ldots \cap Z_n}. \tag{8.1}$$

8.15 It is convenient to continue working with the notation established in pargraph 8.8 on page 145. Recall the affine hyperplane A_0 of $\mathbb{A}^{n+1}$ where $x_0 = 1$ which was introduced in paragraph 4.5 on page 60. The projection $\mathbb{A}^{n+1} \backslash \{0\} \to \mathbb{P}^n$ sends A_0 isomorphically onto $U = D_+(x_0)$, and the intersection $A_0 \cap Z(F_1, \ldots, F_n)$ corresponds to $Z_+(F_1, \ldots, F_n) = Z_1 \cap \ldots \cap Z_n$. On the algebraic level this is just the interplay between homogenizing and dehomgenizing polynmials, and there is an isomorphism



$$k[x_0, \ldots, x_n]/(F_1, \ldots, F_n, x_0 - 1) \simeq k[t_1, \ldots, t_n]/(f_1, \ldots, f_n),$$

or in other terms

$$S/(x_0 - 1)S \simeq \mathcal{O}_{Z_1 \cap \ldots \cap Z_n}.$$

The next step, which in view of the equality (8.1) above will prove Bézout's theorem, is the following is lemma

LEMMA 8.16 $h_S(t) = \dim_k S/(x_0 - 1)S$

PROOF: Let $(0) = \mathfrak{q}_0 \cap \mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_s$ be an irredundant primary decomposition of 0 in S. The corresponding prime ideals $\mathfrak{p}_i$ are all homogeneous, and we chose the indices so that $\mathfrak{p}_0$ is the irrelevant ideal $\mathfrak{m}_+$ (in fact, it ensues from the Unmixedness Theorem that there is no such component, but we do not need that heavy artillery here). The other $\mathfrak{p}_i$'s define the lines in the cone $Z(F_1, \ldots, F_n)$. By the Chinese Remainder Theorem there is an exact sequence

$$0 \longrightarrow C \longrightarrow S \longrightarrow \textstyle\prod_i S/\mathfrak{p}_i \longrightarrow D \longrightarrow 0$$

where C and D are supported at the origin. They are therefore Artinian and have vanishing Hilbert functions, so that $h_S(t) = \sum_i h_{S/\mathfrak{p}_i}(t)$. It also follows that there is an isomorphism

$$S/(x_0 - 1)S \simeq \prod_i S/(\mathfrak{p}_i + (x_0 - 1)S),$$

and in view of this, we are through by Lemma 8.17 below. Ideed, we find

$$h_S(t) = \sum_i h_{S/\mathfrak{p}_i}(t) = \sum_i \dim_k h_{S/(\mathfrak{p}_i + (x_0-1)S)}(t) = \dim_k S/(x_0 - 1)S$$

❑

8.3.1 *A general lemma*

The second part of the proof is to apply from commutative algebra, and prove general lemma about graded modules supported along a line. Chosing appropriate coordinates $(x_0, \ldots, x_n)$ for $\mathbb{A}^{n+1}$, we may assume that line L is the zero-locus of the ideal $\mathfrak{p} = (x_1, \ldots, x_n)$.

LEMMA 8.17 *Let M be a finitely generated and graded R-module. Assume that the support of M is the line $L = Z(\mathfrak{p}) \subseteq \mathbb{A}^{n+1}$. Then*

$$h_M(t) = \ell_{R_\mathfrak{p}}(M_\mathfrak{p}) = \dim_k M/xM$$

where x is any element in R such that $\mathfrak{p} + (x)$ is a maximal ideal distinct from $\mathfrak{m}_+$.

PROOF: A fundamental result from commutative algebra asserts that there is a descending chain $\{M_i\}$ of submodules of M beginning with $M_0 = M$ and whose subquotients all are of the form $A/\mathfrak{q}_i$ with the $\mathfrak{q}_i$'s amongst the prime ideals associated to M. The module M being graded ensures that all its associated ideals are homogeneous, and since the support is a line, there can at most be two of them, the irrelevant ideal $\mathfrak{m}_+$ and the prime ideal $\mathfrak{p} = (x_1, \ldots, x_n)$ that defines the line. Indeed, any associated prime different from $\mathfrak{p}$ is homogeneous and maximal, and the only one of that kind is $\mathfrak{m}_+$.

Moreover, the chain $\{M_i\}$ may be chosen to respect the grading, so that for suitable integers e_i we have exact sequences

$$0 \longrightarrow M_{i+1} \longrightarrow M_i \longrightarrow R/\mathfrak{q}_i[e_i] \longrightarrow 0 \tag{8.2}$$

where the maps are homogeneous of degree zero, and the ideals $\mathfrak{q}_i$ are either $\mathfrak{m}_+$ or the prime ideal $\mathfrak{p}$ of the line.

Let N be the number of times the ideal $\mathfrak{p}$ occurs as a subquotient of the chain. The rest of the proof consists of interpreting N in three different ways, as each of the three numbers in the lemma.

Localising at $\mathfrak{p}$ makes all the subquotients that are equal to $R/\mathfrak{m}_+$ disappear, and since $(R/\mathfrak{p})_\mathfrak{p}$ of course is of length one over $R_\mathfrak{p}$, we may conclude that $\ell_{R_\mathfrak{p}}(M_\mathfrak{p})$ equals the number of subquotients of type $R/\mathfrak{p}[e_i]$; that is, it is equal to N.

Secondly, the element x lies neither in $\mathfrak{p}$ nor in $\mathfrak{m}_+$ and is therefore not a zero-divisor in M nor in any of the M_i's. A snake argument gives exact sequences

$$0 \longrightarrow M_{i+1}/xM_{i+1} \longrightarrow M_i/xM_i \longrightarrow R/(\mathfrak{q}_i + (x)R) \longrightarrow 0$$

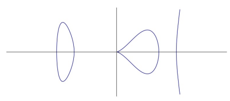
of vector spaces over $R/(\mathfrak{p} + (x)R) = k$. Moreover, a subquotient $R/(\mathfrak{q}_i + (x)R)$ vanishes if $\mathfrak{q}_i = \mathfrak{m}_+$ and equals k if $\mathfrak{p}_i = \mathfrak{p}$, and we can conclude that $N = \dim_k M/xM$.

Finally, by a straightforward induction using the additivity of the Hilbert polynomial one shows that

$$h_M(t) = \sum_i h_{R/\mathfrak{q}_i[e_i]}(t) = \sum_i h_{R/\mathfrak{q}_i}(t - e_i) = \sum_{\mathfrak{q}_i = \mathfrak{p}} 1 = N,$$

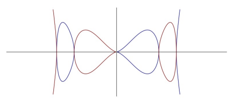
observing that the Hilbert polynomial of $R/\mathfrak{p} = k[x_0]$ is the constant 1 and that of $R/\mathfrak{m}_+ = k$ the constant 0. ❑

PROBLEM 8.7 Let $n > m$ be two natural numbers and let $\alpha(x)$ and $\beta(x)$ be two polynomials which do not vanish at $x = 0$. Determine the local intersection multiplicity at the origin of the two curves defined respectively by $y - \alpha(x)x^n$ and $y - \beta(x)x^m$. If $m = n$, show by exhibiting an example that the local multiplicity can take any integral value larger than n. ★



The affine pieces in $D_+(z)$ of one the two curves in problem 8.9

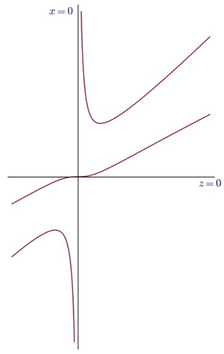
PROBLEM 8.8 Find all intersection points of the two cubic curves defined by the forms $zy^2 - x^3$ and $zy^2 + x^3$ (we assume the characteristic of the ground field to be different from two). Determine all the local intersection multiplicities of the two curves. ★



The affine pieces in $D_+(z)$ of the two curves in problem 8.9

PROBLEM 8.9 Let X and Y be two curves in $\mathbb{P}^2$ being the zero loci of the polynomials $z^5y^2 - x^3(z^2 - x^2)(2z^2 - x^2)$ and $z^5y^2 + x^3(z^2 - x^2)(2z^2 - x^2)$. Determine all intersection points and the local multiplicities in all the intersection points of X and Y ★

PROBLEM 8.10 Let C be the curve given as $zy^2 - x(x-z)(x-2z)$. Determine the intersection points and the local multiplicities that X has withe the line $z = 0$. Same task, but with the line $x - z = 0$. ★



8.4 Appendix: Depth, regular sequences and unmixedness

An important ingredient in the full proof of Bézout's theorem is the concept of so-called unmixed rings. These are Noetherian rings all whose associated prime ideals are of the same height, or what amounts to the same in our context of algeras of finite type over a field, that $\dim A/\mathfrak{p}$ is the same for all associated primes $\mathfrak{p}$. In particual A has no embedded components, the height of an embedded prime would of course be larger than the height of at least on of the others. In geometric terms, if $A = k[x_1, \ldots, x_n]/\mathfrak{a}$, all the components of the closed algtebraic subset $X = Z(\mathfrak{a})$ are of the same dimension and A has no embedded component.

Macaulay showed that if $(F_1, \ldots, F_r)$ is of height r, then $k[x_1, \ldots, _x]/(f_1, \ldots, f_r)$ is unmixed. That the irreducible components of the closed algebraic set $Z(f_1, \ldots, f_r)$ all are of codimension r is clear—the height being the smallest codimension of a components, and Krull's Hauptidealsatz tells us that every component is of codimension most r— so the subtle content is that there are no embedded components. This has consequence that if F_{r+1} is a new polynomial not vanishing along any of the components, then F_{r+1} is a non-zero divisor in $k[x_1, \ldots, x_n]/(f_1, \ldots, f_r)$. So we see that $(F_1, \ldots, F_r)$ being of height r is equivalent to $F_1, \ldots, F_r$ being a regular sequence.

Regular sequences

The theory of Cohen–Macaulay rings and more generally of the Cohen–Macaualy modules, is based on the concept of *regular sequences* which was introduced by Jean Pierre Serre in 1955. Their basic properties are described in this paragraph.

8.18 The stage is set as follows. We are given a ring A together with a proper ideal $\mathfrak{a}$ in A and an A-module M. Most of the time An will be local and Noetherian and M will be finitely generated over A.

A sequence $x_1, \ldots, x_r$ of elements belonging to the ideal $\mathfrak{a}$ is said to be *regular for M*, or *M-regular* for short, if the following condition is fulfilled where for notational convenience we let $x_0 = 0$.

Regular sequences (*regulære følger*)

- For any i with $1 \leqslant i \leqslant r$ the multiplication-by-x_i map

$$M/(x_1, \ldots, x_{i-1}) \longrightarrow M/(x_1, \ldots, x_{i-1})$$

is injective.

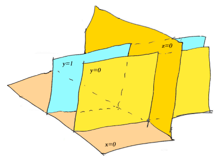
In other words, x_i is a not a zero-divisor in $M/(x_1, \ldots, x_{i-1})$. In particular, x_1 is not a zero-divisor in M, and this has lead to the usage that x_1 being regular in M is synonymous with x_1 being a non-zero divisor in M.

Maximal regular sequences (*maksimale regulære følger*)

8.19 A regular sequence $x_1, \ldots, x_r$ is said to be *maximal* if it is no longer regular when an element is added to it. When M is a Noetherian module, this is equivalent to $\mathfrak{a}$ being contained in one of the associated primes of $M/(x_1, \ldots, x_r)$; indeed, the union of the associated primes of $M/(x_1, \ldots, x_r)$ is precisely the set of zero-divisors in $M/(x_1, \ldots, x_r)$.

PROBLEM 8.11 Show that maximal regular sequences for Noetherian modules are finite. Exihibit a counterexample when M is not Noetherian. HINT: Consider the ascending chain $(x_1, \ldots, x_i)$ of ideals. ★

8.20 Be aware that in general the order of the x_i's is important, permute them and the sequence may no more be regular. However, regular sequences for modules finitely generated over local Noetherian rings remain regular after an arbitrary permutation, and the same holds true for graded rings with appropriate finiteness conditions. Henceforth, we shall work with rings that are Noetherian and local and with modules finitely generated over A (but with a sideways glimpse into the graded case, so important for projective geometry).



EXAMPLE 8.4 The simplest example of a sequence that ceases being regular when permuted is as follows. Start with the three coordinate planes in $\mathbb{A}^3$; they are given as the zero loci of x, y, z. Add a plane disjoint from one of them to the two others; *e.g.* consider the zero loci of the three polynomials $x(y-1)$, y and $z(y-1)$.

Clearly $x(y-1), z(y-1), y$ is *not* a regular sequence in $k[x, y, x]$. The point is that $z(y-1)$ kills any function on $Z(x(y-1))$ that vanishes on the component $Z(x)$ (for example x) and is thus not a zero-divisor in $k[x, y, z]/(x(y-1))$.

On the other hand, the sequence $x(y-1), y, z(y-1)$ is regular. Indeed, it holds that $k[x, y, z]/(x(y-1), y) = k[z]$, and in that ring $z(y-1)$ is congruent to z and thus not a zero-divisor. Geometrically, capping $Z(x(y-1))$ with $Z(y)$ makes the villain component $Z(y-1)$ go away.

This example is in fact arche-typical. The troubles occur when two of the involved closed algebraic sets have a common component disjoint from one of the components of a third. If all components of all the closed algebraic subsets involved have a point in common, one is basically in a local situation, and permutations are permitted. ☆

Permutation permitted

8.21 As mention in the previous example, in local Noetherian rings a sequence being regular is a property insensitive to order. The same holds true in a graded setting, and in both cases Nakayama's lemma is the tool that makes it work.

LEMMA 8.22 *Assume that A is a local Noetherian ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module. If x_1, x_2 is a regular sequence in $\mathfrak{m}$ for M, then x_2, x_1 is one as well.*

PROOF: There are two things to be checked. Firstly, that x_2 is a non-zero divisor in M. The annihilator $(0 : x_2)_M = \{ a \in M \mid x_2 a = 0 \}$ must map to zero in M/x_1M because multiplication by x_2 in M/x_1M is injective. Hence $(0 : x_2)_M + x_1M = x_1M$, and since $x_1 \in \mathfrak{m}$ and M is finitely generated, Nakayama's lemma applies and $(0 : x_2)_M = 0$.

Secondly, we are to see that multiplication by x_1 is injective on M/x_2M, so assume that $x_1a = x_2b$. But multiplication by x_2 is injective on M/x_1M, and it follows that $b = cx_1$ for some c; that is, $x_1a = x_1x_2c$. Cancelling x_1, which is legal since x_1 is a non-zero divisor in M, we obtain $a = cx_2$. ❑

PROPOSITION 8.23 *Let A be a local Noetherian ring with maximal ideal $\mathfrak{m}$ and M a finitely generated A-module. Assume that $x_1, \ldots, x_r$ is a regular sequence in $\mathfrak{m}$ for M. Then for any permutation σ the sequence $x_{\sigma(1)}, \ldots, x_{\sigma(n)}$ is regular.*

PROOF: It suffices to say that any permutation can be achieved by successively swapping neighbours . ❑

8.24 The graded version reads as follows:

PROPOSITION 8.25 *Let A be a graded ring satisfying $A_i = 0$ when $i < 0$, and let M be a finitely generated graded A-module. If $x_1, \ldots, x_r$ is a sequence of elements from A, homogeneous of positive degree, that form a regular sequence in M, then for any permutation σ the sequence $x_{\sigma(1)}, \ldots, x_{\sigma(r)}$ is also a regular in M.*

PROOF: As above, one may assume that $r = 2$. The proof of Lemma 8.22 goes through *mutatis mutandis*; the sub module $(0 : x_2)_M$ will be a graded submodule because x_2 is homogeneous, and a version of Nakayma's lemma for graded things is available. ❑

PROBLEM 8.12 With assumptions as in 8.23 or 8.25, prove that if $x_1, \ldots, x_r$ is a regular sequence for M and $\nu_1, \ldots, \nu_r$ is a sequence of natural numbers, then $x_1^{\nu_1}, \ldots, x_r^{\nu_r}$ will be a regular sequence as well. HINT: Reduce to the case of $x_1, \ldots, x_{r-1}, x_r^{\nu}$. ★

PROBLEM 8.13 Jean Dieudonné gave the following example of a regular sequence x_1, x_2 in local non-Noetherian ring such that x_2, x_1 is not regular. Consider the ring B of germs of C^∞-functions near 0 in $\mathbb{R}$. It is a local ring whose maximal ideal $\mathfrak{m}$ constists of the functions vanishing at zero. Let $\mathfrak{a}$ be the ideal $\mathfrak{a} = \bigcap_i \mathfrak{m}^i$ of functions all whose derivatives vanish at the origin. Let $A = B[T]/\mathfrak{a}TB[T]$. Let I be the function $I(x) = x$. Show that the sequence I, T is a regular sequence in A whereas T, I is not. ★

Enters homolocigal algebra—the depth

8.26 One of the first appearances of homological methods in commutative algebra was in the circle of ideas round of regular sequences and Cohen-Macaulay modules. These methods give a caracterization of the maximal length of M-regular sequences in terms of certain homologically defined modules. The criterion has the virtue of not explicitly refering to any sequence, and has the consequence that all maximal sequence are of the same length.

The homological modules in question are modules $\operatorname{Ext}^i_A(M,N)$ associated with a pair of A-modules M and N. In the lingo of homological algebra they appear as derived functors of the functor $\operatorname{Hom}_A(-,-)$. Students not already acquainted with these useful creatures should consult a textbook about homological algebra for the few of their very basic properties we shall need.

8.27 It is natural to introduce the number $\operatorname{depth}_{\mathfrak{a}} M$ as the length of the longest (maximal) regular M-sequence in $\mathfrak{a}$. It is called the *depth* of M in $\mathfrak{a}$. In the end, it turns out that all maximal M-sequences in $\mathfrak{a}$ have the same length, but for the moment we do not know that, and *a priori* the number is not even bounded. However, we have:

The depth of a module (dybden til en modul)

LEMMA 8.28 *If A is a local Noetherian ring, $\mathfrak{a}$ a proper ideal and M a finitely generated A-module, then $\operatorname{depth}_{\mathfrak{a}} M \leqslant \dim M$. In particular, $\operatorname{depth}_{\mathfrak{a}} M$ is finite.*

PROOF: Induction on $\dim M$ (which is finite!). If $\dim M = 0$, the maximal ideal $\mathfrak{m}$ is the only associated prime of M. Therefore every element in $\mathfrak{m}$ is a zero divisor and $\operatorname{depth}_{\mathfrak{a}} M = 0$.

Next, observe that if x is a non-zero divisor in M, it holds true that $\dim M/xM < \dim M$, and by induction one may infer that

$$\operatorname{depth}_{\mathfrak{a}} M/xM \leqslant \dim M/xM < \dim M. \tag{8.3}$$

So if $x_1, \ldots, x_r$ is a maximal regular sequence in M (they are all finite after Problem 8.11), the sequence $x_2, \ldots, x_r$ will be one for M/x_1M, and by (8.3) $r - 1 < \dim M$; that is $r \leqslant \dim M$. ❑

8.29 We have comes to the homological characterization. It is notable since it determines the depth of a module without referring to any regular sequence. We introduce a number $p(M)$ which is the smallest integer i such that $\operatorname{Ext}^i_A(A/\mathfrak{a}, M) \neq 0$.

PROPOSITION 8.30 *Let A be a local Noetherian ring, $\mathfrak{a}$ a proper ideal and M a finitely generated A-module. Then $\operatorname{depth}_{\mathfrak{a}} M = p(M)$.*

PROOF: The proof goes by induction on the depth of M (which is finite by Lemma 8.28 above). That $\operatorname{depth}_{\mathfrak{a}} M = 0$, means that there no element in $\mathfrak{a}$ is regular in M. In other words, $\mathfrak{a}$ is contained in one of the associated primes of

M, say $\mathfrak{p}$. There is then an inclusion $A/\mathfrak{p} \to M$ and a surjection $A/\mathfrak{m} \to A/\mathfrak{p}$. Consequently $\operatorname{Hom}_A(A/\mathfrak{a}, M) \neq 0$, and $p(M) = 0$.

Assume next that $\operatorname{depth}_\mathfrak{a} M > 0$. If x is the first member of an M-regular sequence of maximal length, the quotient M/xM satisfies $\operatorname{depth}_\mathfrak{a} M/xM = \operatorname{depth}_\mathfrak{a} M - 1$. Moreover, since x is regular on M, one has the short exact sequences

$$0 \longrightarrow M \xrightarrow{x} M \longrightarrow M/xM \longrightarrow 0,$$

from which one derives a long exact sequence the relevant part for us being

$$\operatorname{Ext}^i_A(A/\mathfrak{a}, M) \longrightarrow \operatorname{Ext}^i_A(A/\mathfrak{a}, M/xM) \longrightarrow \operatorname{Ext}^{i+1}_A(A/\mathfrak{a}, M) \xrightarrow{x=0} \operatorname{Ext}^{i+1}_A(A/\mathfrak{a}, M).$$

Since $x \in \mathfrak{a}$ the multiplication by x on the ext-modules is the zero map. Now, if $i+1 < p(M)$ it ensues that $\operatorname{Ext}^i_A(A/\mathfrak{a}, M/xM) = 0$, and we may conclude that $p(M/xM) + 1 \leqslant p(M)$. And if $i+1 = p(M)$ it follows that $\operatorname{Ext}^i_A(A/\mathfrak{a}, M/xM) \simeq \operatorname{Ext}^{i+1}_A(A/\mathfrak{a}, M) \neq 0$ so equality holds.

So both the quantities $\operatorname{depth}_\mathfrak{a} M$ and $p(M)$ drops by one when we mod out by x, and thence they are equal by induction. ❑

The proposition has an important corollary, which in fact is the main target of this paragraph:

Theorem 8.31 *Let A be a local Noetherian ring, $\mathfrak{a}$ an ideal in A and M a finitely generated A-module. Then all maximal regular M-sequences in $\mathfrak{a}$ have the same length.*

Example 8.5 A Noetherian zero-dimensional local ring has of course depth zero. A Noetherian one-dimensional local ring A has depth one if and only if the maximal ideal is not associated; that is, A has no embedded component. ☆

As usual, we also give a graded version:

Theorem 8.32 *Let A be a graded ring satisfying $A_i = 0$ when $i < 0$, and let M be a finitely generated graded A-module. Then all homogeneous maximal regular M-sequences have the same length.*

The bound

In geometry the dimension of a close algebraic set is the maximum dimension of the irreducible components, and the algebraic counterpart is that the dimension of a ring is the maximum of the dimensions $\dim A/\mathfrak{p}$ for $\mathfrak{p}$ running through the associated prime ideals of A. This maximum is never assumed at an embedded prime since these by definition strictly contain another associated prime. For a module M, the same holds true as $\dim M = \dim / \operatorname{Ann} M$.

The word *depth* has the flavour of something down, and indeed, $\operatorname{depth}_\mathfrak{m} M$ is smaller then all the dimensions $\dim A/\mathfrak{p}$ where this time $\mathfrak{p}$ runs through *all* of the associated primes, including the embedded ones. And this is the crucial point.

Proposition 8.33 *As usual, let A be local Noetherian ring with maximal ideal $\mathfrak{m}$ and let M be a finitely generated A-module. It then holds true that*

$$\operatorname{depth}_{\mathfrak{m}} M \leqslant \dim A/\mathfrak{p}$$

for all prime ideals associated to M.

Proof: The proof goes by induction on the depth of M. If $\operatorname{depth}_{\mathfrak{m}} M = 0$ there is nothing to prove. So assume that $\operatorname{depth}_{\mathfrak{m}} M \geqslant 1$. Then there is a short exact sequence

$$0 \longrightarrow M \xrightarrow{\ x\ } M \longrightarrow M/xM \longrightarrow 0. \tag{8.4}$$

Let $\mathfrak{p}$ be prime ideal associated to M. The sequence (8.4) above induces an exact sequence

$$0 \longrightarrow \operatorname{Hom}_A(A/\mathfrak{p}, M) \xrightarrow{\ x\ } \operatorname{Hom}_A(A/\mathfrak{p}, M) \longrightarrow \operatorname{Hom}_A(A/\mathfrak{p}, M/xM),$$

and by Nakayama's lemma the cokernel of the multiplication-by-x map is a non-zero submodule of $\operatorname{Hom}_A(A/\mathfrak{p}, M/xM)$. Hence $\operatorname{Hom}_A(A/\mathfrak{p}, M/xM) \neq 0$, and the ideal $\mathfrak{p} + (x)$ is contained in an associated ideal prime ideal $\mathfrak{q}$ of M/xM. Since x is a non-zero divisor in M and $\mathfrak{p}$ is associated to M, we may infer that $x \notin \mathfrak{p}$, and therefore $\mathfrak{p}$ is strictly contained in $\mathfrak{q}$. It follows that $\dim A/\mathfrak{p} > \dim A/\mathfrak{q}$. Now, $\operatorname{depth}_m M/xM = \operatorname{depth}_{\mathfrak{m}} M - 1$, and by induction

$$\operatorname{depth}_{\mathfrak{m}} M - 1 \leqslant \dim A/\mathfrak{q} < \dim A/\mathfrak{p}.$$

❏

8.34 The A-module M is said to be a *Cohen-Macaulay module* if $\operatorname{depth}_{\mathfrak{m}} M = \dim M$, in particular, the ring A itself is *Cohen-Macaulay* if $\operatorname{depth}_{\mathfrak{m}} A = \dim A$. If x is a non-zero divisor in A, both the depth and the dimension of A/xA are one less than of A, and hence A is Cohen–Macaulay if and only if A/xA is.

Cohen-Macaulay modules (*Cohen-Macaulay moduler*)

Theorem 8.35 *Assume that A is a local Noetherian Cohen-Macaulay ring. Then A is unmixed. That is $\dim a/\mathfrak{p} = \dim A$ for all associated primes $\mathfrak{p}$ of A; in particular, A has no embedded components.*

Proof: In view of Proposition 8.33 this is almost a tautology. The lower and the upper bound of the dimensions $\dim A/\mathfrak{p}$ for $\mathfrak{p}$ associated with A coincide, hence these dimensions all coincide. ❏

8.36 To check that a ring is Cohen–Macaulay, it suffices to exhibit one regular sequence of length the dimension of the ring. For instance, the local rings $A_n = k[x_1, \ldots, x_n]_{\mathfrak{m}_n}$ where $\mathfrak{m}_n = (x_1, \ldots, x_n)$ are Cohen–Macaulay since the sequence $x_1, \ldots, x_n$ is regular. This follows easily by induction because there are natural isomorphisms $A_n/x_nA_n \simeq A_{n-1}$ induced by the maps $k[x_1, \ldots, x_n] \to k[x_1, \ldots, x_{n-1}]$ that send x_n to zero.

8.37 Recall that a *system of parameters* in a local ring A of dimension n is a sequence $x_1, \ldots, x_n$ so that the quotient $A/(x_1, \ldots, x_n)$ is of dimension zero. In our geometric terms, a a sequence of functions $f_1, \ldots, f_n$ in a coordinate ring $A(X)$ is a system of parameters at a point p if in a neighbourhood of p their only common zero is p. What was need in the proof of Bézout's theorem, was that such a sequence of polynomials in fact form a regular sequence in $k[t_1, \ldots, t_n]$, and this is generally a property of Cohen–Macaulay rings.

Theorem 8.38 *In a local Noetherian Cohen–Maccaulay ring A every system of parameters is a regular sequence.*

Proof: The approach is by induction on $\dim A$. If $x_1, \ldots, x_r$ is a system of parameters, it must hold $\dim A/x_1A < \dim A$. Indeed, by Krull's Hauptidealsatz the dimension can drop by at most one each time we mod out by an x_i, and to reach zero after n steps, it must drop every time.

Since A is a Cohen–Macaulay ring, all its associated primes $\mathfrak{p}$ are of dimension $\dim A/\mathfrak{p} = \dim A$, hence x_1 can not belong to any of the $\mathfrak{p}$'s, and as the zero-divisors in A constitute the union $\bigcup \mathfrak{p}$, x_1 is non-zero divisor. Thence A/x_1A is Cohen–Macaulay, the induction hypothesis ensures that $x_2, \ldots, x_r$ is a regular sequence in A/x_1A and we are through. ❏

8.5 *Appendix: Some graded algebra*

Graded modules

Recall that a graded k-algebra is a k-algebra S with a decomposition $S = \bigoplus_d S_d$ into a direct sum of k-vector spaces. The summands are called the *homogeneous parts* of S, and the elements of S_d are said to be homogeneous of degree d. The decomposition is subjected to the requirement

$$S_d \cdot S_{d'} \subseteq S_{d+d'},$$

which can be considered a compatibility relation between the grading and the multiplicative structure of S. The part S_0 of elements of degree zero acts on each of the parts S_d making them S_0-modules. The field k is contained in S_0.

Example 8.6 The archetype of a graded ring is the polynomial ring $R = k[x_0, \ldots, x_n]$ with the homogenous part of degree consisting of the homogenous forms of degree d. ☆

Example 8.7 If one localizes R in x_n, the resulting algebra R_{x_n} is graded. The homogeneous elements of R_{x_n} are the ones of the form $z = H(x_0, \ldots, x_n)x_n^{-r}$ for some homogeneous polynomial H and some non-zero integer r. The degree of the element z equals $\deg H - r$. When $\deg z = 0$, it holds true that $z = H(x_0x_n^{-1}, \ldots, x_{n-1}x_n^{-1})$; that is the dehomogenization of H. This implies that the degree zero piece of R_{x_n} is given as the polynomial ring $(R_{x_2})_0 =$

$k[x_0x_n^{-1},\dots,x_{n-1}x_n^{-1}]$. Hence the decomposition of R_{x_n} into homogeneous pieces is shaped like

$$R_{x_2} = \bigoplus_{i\in\mathbb{Z}} k[x_0x_n^{-1},\dots,x_{n-1}x_n^{-1}]\cdot x_n^i.$$

☆

Graded S-modules

A graded S-module is an S module M with a decomposition $M = \bigoplus_d M_d$ into a direct sum of k-vector space such that

$$S_{d'} \cdot M_d \subseteq M_{d+d'}$$

Notice that all the summands M_d are modules over the degree zero piece S_0.

EXAMPLE 8.8 Every homogenous ideal $\mathfrak{a}$ in R is a graded R_n-module. It satisfies the equality $\mathfrak{a} = \bigoplus_d \mathfrak{a}\cap R_d$ so that the homogeneous part $\mathfrak{a}_d$ of degree d is given as the intersection $\mathfrak{a}_d = \mathfrak{a}\cap R_d$.

The quotient $R/\mathfrak{a}$ is a graded module over R_n as well as a graded k-algebra. It holds true that $R/\mathfrak{a} = \bigoplus_d R_d/\mathfrak{a}_d$. ☆

The introduction of a new concept in mathematics is almost always followed by the introduction of corresponding "morphism's'; that is, "maps" preserving the new structure. In the present case a "morphism" between two graded S-modules M and M' is an S-homomorphism $\phi\colon M\to M'$ that respects the grading; that is $\phi(M_d)\subseteq M'_d$. One says that ϕ is a homogeneous homomorphism of degree zero, or *homomorphism of graded modules*. Two graded modules are *isomorphic* if there is a homomorphism of graded modules $\phi\colon M\to M'$ having an inverse.

Homomorphism of graded modules
Isomorphic graded modules

One easily checks that the kernel and the cokernel of a homomorphism of degree zero are graded in a natural way. Students initiated in the categorical language would say that the graded modules form an *abelian category*.

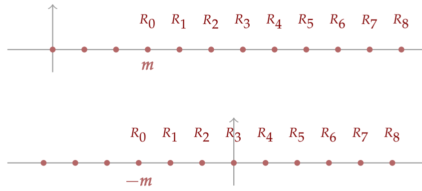
PROBLEM 8.14 Show that if $\phi\colon M\to M'$ is invertible and homogeneous of degree zero, the inverse is automatically is homogeneous of degree zero. ★

There is a collection of shift operators acting on the category of graded S-modules. For each graded module M and each integer $m\in\mathbb{Z}$ there is fresh graded module $M(m)$ associated to a graded module M. The shift operators do not alter the module structure of M, not even the set of homogeneous elements is affected, but they give new degrees to the homogeneous elements. The new degrees are defined by setting

$$M(m)_d = M_{m+d}.$$

In other words, one declares the degree of elements in M_m to be $d-m$.

EXAMPLE 8.9 For instance, when $m>0$, the shifted polynomial ring $R(-m)$ has no elements of degree d when $d<f$, indeed, $R(m)_d = R_{d-m}$, and the ground field k sits as the graded piece of degree m. Whereas the twisted algebra $R(m)$ has non-zero homogeneous elements of degree down to $-m$ with the ground field sitting as the piece of degree $-m$. ☆



The graded modules $R(-m)$ and $R(m)$

EXAMPLE 8.10 One simple reason for introducing the shift operators, is to keep track of the degrees of generators. For instance, consider the principle ideal $\mathfrak{a} = (F)$ in the polynomial ring R generated by a homogeneous form F of degree m. As every principal ideal in R is, $\mathfrak{a}$ is isomorphic to R as an R-module—multiplication by F gives an isomorphism. However, this not an isomorphism of *graded* modules since it alters the degrees; a homogeneous element a is mapped to the product aF which is of degree $\deg a + m$. But multiplication by F induces a graded isomorphism between $R(-m)$ and $\mathfrak{a}$, since for elements $a \in R(-m)_d$ it holds that $\deg a = d - m$ and consequently $\deg aF = d$.

The classical short exact sequence is therefore an exact sequence of graded modules:

$$0 \longrightarrow R(-m) \xrightarrow{\mu} R \longrightarrow R/F \longrightarrow 0,$$

where thew map μ is multiplication by F. ☆

All graded modules we shall meet in this course are finitely generated over the polynomial ring R. Their generators may be taken to be homogeneous, but they can of course be of different degrees. If the degrees of generators are $d_1, \ldots, d_r$, then M is quotient of a module shaped like a finite direct sum $\bigoplus_{1 \leqslant i \leqslant r} R(-d_i)$; the factor $R(-d_i)$ is sent to the generator of degree d_i. The twists make the quotient map homogenous of degree zero.

LEMMA 8.39 *If M is a graded module finitely generated over the polynomial ring R, then all the graded pieces M_d are finite dimensional vector spaces over k.*

PROOF: This is more or less obvious. It is true for R itself, hence for all twists $R(m)$, hence for direct sums $\bigoplus_i R(-d_i)$. And if M is a quotient of $\bigoplus_i R(-d_i)_d$, the graded piece M_d of M of degree d is a quotient of the graded piece $\bigoplus_i R(-d_i)_d$. ❑

Hilbert functions and Hilbert polynomials

There are some numerical invariants attached to a graded module M finitely generated over a polynomial ring R, which makes working with graded modules much easier. These functions, or their alter egos, are ubiquitous in algebraic geometry and they play an extremely important role. One is the so called *Hilbert function* $h_M(d)$ of M defined as $h_M(d) = \dim_k M_d$. It turns out that $h_M(d)$ behaves like a polynomial for d sufficiently large; that is, there is a unique polynomial $P_M(d)$ coinciding with $h_M(d)$ when $d >> 0$. This is the *Hilbert polynomial* of M.

Hilbert functions

Hilbert polynomials

A fundamental property of the Hilbert functions that makes it possible to calculate at least of them, is that, just like the vector space dimension, they are additive over short exact sequences. If

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

is an exact sequence of graded module, it holds true that $h_M = h_{M'} + h_{M''}$. Indeed, for each degree d the graded pieces of degree d fit into an exact sequence

$$0 \longrightarrow M'_d \longrightarrow M_d \longrightarrow M''_d \longrightarrow 0$$

of vector spaces, and the assertion follows since vector space dimension is additive.

For functions $h\colon \mathbb{Z} \to \mathbb{Z}$; *i.e.* functions taking integral values on the integers, one introduces a *difference operator* Δ. It is some sort of *discrete derivative* and it is defined as

The difference operator or the discrete derivative

$$\Delta h(d) = h(d) - h(d-1).$$

Just like a derivative, if $\Delta h(d) = 0$ for all d, then h is constant. And so two functions h and h' having the same discrete derivative are equal up to a constant.

Multiplication by an element $x \in R$ of degree one which is not a zero-divisor in the graded module M, induces an exact sequence

$$0 \longrightarrow M(-1) \longrightarrow M \longrightarrow M/xM \longrightarrow 0,$$

from which we infer the equality

$$h_{M/xM}(d) = h_M(d) - h_M(d-1) = \Delta h_M(d). \tag{8.5}$$

A polynomial $P(t)$ with rational coefficient is called a *numerical polynomial* if it assumes integral values for integral arguments; that is, if $P(t) \in \mathbb{Z}$ whenever $t \in \mathbb{Z}$.

Numerical polynomials

EXAMPLE 8.11 The binomial coefficients are archetypes of numerical polynomials. Recall that they are defined for any t by the identity

$$\binom{t+n}{n} = (t+n)(t+n-1)\cdot\ldots\cdot(t+1)/n!,$$

and it is well known they are numerical polynomials. A straightforward calculation shows that

$$\Delta\binom{t+n}{n} = \binom{t+n-1}{n-1}. \tag{8.6}$$

☆

EXAMPLE 8.12 The Hilbert function of the polynomial[3] ring ${}^nR$ vanishes for negative arguments and is given as the binomial coefficient

[3] Recall that ${}^nR = k[x_0,\ldots,x_n]$

$$h_{{}^nR}(d) = \binom{n+d}{n}$$

when $d \geqslant 0$. Indeed, multiplication by x_n induces the exact sequence

$$0 \longrightarrow {}^nR(-1) \longrightarrow {}^nR \longrightarrow {}^{n-1}R \longrightarrow 0$$

of graded modules, and hence $\Delta h_{^nR} = h_{^{n-1}R}$. By induction on n and the identity (8.6) above the assertion follows. Because $^0R = k[x_0]$ it obviously holds that $h_{^0R}(d) = 1$ for $d \geqslant 0$ and $h_{^0R} = 0$ when $d < 0$, so that the induction can start. ☆

Example 8.13 For a graded R-modules M of finite support, the Hilbert polynomial P_M vanishes identically. Indeed, the module M is finite dimensional as a vector space over k, and there is only room for finitely many non-zero graded pieces. But of course, the Hilbert *function* of M is not identically zero. ☆

We shall mostly be concerned with the leading term of numerical polynomials; they are of a special form as described in the following lemma:

Lemma 8.40 *Assume that $P(t)$ is a numerical polynomial of degree m. Then*

$$P(t) = c_m/m!t^m + \dots$$

where c_m is an integer[4]*. The discrete derivative $\Delta P(t)$ is of degree $m-1$ and its leading coefficient equals $c_m/(m-1)!$*

[4] As is customary, the dots stand for terms of lower degree than m.

Proof: We proceed by induction on m. The lemma holds for $m = 0$ because a numerical polynomial of degree zero is an integral constant. For $m > 0$ we write $P(t) = a_m t^m + Q(t)$ with Q of degree at most $m-1$. Appealing to the binomial theorem, one finds

$$\begin{aligned}\Delta P(t) &= a_m t^m - a_m(t-1)^m + \Delta Q(t) = \\ &= a_m t^m - a_m t^m + m a_m t^{m-1} + \Delta Q(t) = m a_m t^{m-1} + \Delta Q(t),\end{aligned}$$

by induction $\Delta Q(t)$ is of degree at most $m-2$, the leading coefficient of $\Delta P(t)$ is $m a_m$, and again by induction, it is shaped like $m a_m = c_{m-1}/(m-1)!$ where c_{m-1} is an integer. The lemma follows. ❏

Theorem 8.41 *Let $\mathfrak{a}$ be a homogenous ideal in R. Then $P_{R/\mathfrak{a}}$ is of degree $\dim R/\mathfrak{a}$.*

Proof: We proceed by induction on $\dim R/\mathfrak{a}$. Let $\mathfrak{p}_i$ be the associated prime ideals to $\mathfrak{a}$. Then there is an element $x \in \mathfrak{m}_+$ of degree one not contained in any of the minimal primes of $\mathfrak{a}$, and $\dim R/\mathfrak{a} + (x) = \dim R/\mathfrak{a} - 1$. Hence there is an exact sequence

$$0 \longrightarrow K \longrightarrow S(-1) \xrightarrow{x} S \longrightarrow S/xS \longrightarrow 0$$

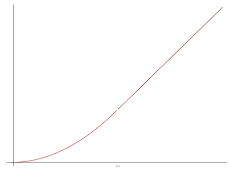
where K is of finite support. By example xxx above, the Hilbert polynomial of K vanishes identically, and hence $\Delta P_S = P_{S/xS}$. By induction we are through. ❏

Example 8.14 Let $F \in R = k[x_0, x_1, x_2]$ be a homogeneous polynomial of degree m. Then there is a short exact sequence of graded modules

$$0 \longrightarrow R(-m) \xrightarrow{\alpha} R \longrightarrow R/F \longrightarrow 0$$

where the map α is multiplication by F. Additivity of the Hilbert functions yields, when $d \geqslant m$, that

$$h_{R/F}(d) = h_R(d) - h_R(d-m) = md + m^2 - 3m/2,$$



The graph of $h_{R/(F)}$.

while

$$h_{R/F}(d) = h_R(d) = d^2/2 + 3d/2 + 1$$

when $0 \leqslant d < m$ since then $h_{R(d-m)} = 0$. For $d < 0$ it obviously holds true that $h_{R/F}(d) = 0$. So the Hilbert function is constant and equal to zero for negative values of the argument, it grows quadratically for d between 0 and m and settles with a linear grows for $d \geqslant m$. The Hilbert polynomial is linear and has leading term md. The geometric interpretation of the algebra $R/(F)R$ is as the homogeneous coordinate ring $S(X)$ of of the curve $X = Z_+(F) \subseteq \mathbb{P}^2$. Notice that the degree of the Hilbert polynomial $P_{S(X)}(d)$ equals the dimension of X (both are one) and that the leading coefficient equals the degree of F. ☆

Lecture 9

Non-singular varieties

Hot themes in Notes :
Super-Preliminary version 0.0 as of 13th February 2019 at 9:52am—Well, still not really a version at all, but better. Improvements will follow!

Geir Ellingsrud — ellingsr@math.uio.no

Manifolds are important,

In algebraic geormetry the corresponding concept is that of non-singular varietes. There som differences. We do not have a invers function theorem tooir disposal; for tha Zarisk opens are to large.

9.1 Regular local rings

In differential topology a maniold is space which locallyisdiffeomorphico opensets in some $\mathbb{R}^n$.

Zariski tangent space. In moder geometry which sttrives after intrinsic invariants. The classical way of itroducing tanget soaces uses and embedding of X and the tanfent space is the lienear space tha fits best fits X; meaning something like if you follow a line into p, the distamce to X decreases more rapidly then t

If you have a function defined on X, you can compute derivatives of f along curves approshing p. So one usually defines thetagent space as point derivation at the point p: That is, the maps $\mathcal{O}_{X,p} \to k$ being k-linear and complyinig with Leibnitz' rule

$$Dfg = f(p)Dg + g(p)Dg$$

Such a derivation vanishes on the constants; indeed if k is not of characteristic two, this folows from $1 \cdot 1 = 1$; if charcteristic is two a slifgtly more elaborate argument is needed. Such derivation vanishe on the square $\mathfrak{m}^2$. Hence

Lemma 9.1 *The $T_pX \simeq \mathrm{Hom}_k(\mathfrak{m}/\mathfrak{m}^2, k,)$*

Proof: Given f and g and consider the product $(f - f(p))(g - g(p))$. It belongs to the squaer $\mathfrak{m}^2$, and hence

$$D((f - f(p))(g - g(p))) = 0$$

Multiplyinig out one finds

$$(f - f(p))(g - g(p)) = fg - f(p)g - gpf - f(p)g(p)$$

and the Leibnitz' rule follows since D kills contants. ❑

Lemma 9.2 *If A is a Noetherian local ring with maximal ideal $\mathfrak{m}$,it holds true that* $\dim_k \mathfrak{m}/\mathfrak{m}^2 \geqslant \dim A$

Proof: By Nakayamas lemma any basis $v_1, \ldots, v_s$ of $\mathfrak{m}/\mathfrak{m}^2$ lifts to a set generators $x_1, \ldots, x_s$ of the maximal ideal $\mathfrak{m}$. Krull's Hauptidealsatz implies that $\mathfrak{m} \leqslant s$, but $\mathfrak{m} = \dim A$. ❑

9.2 *The Jacobian criterion*

Following Hartshorne we turn the Jacobian criterion into a definition.

Let $X \subseteq \mathbb{A}^n$ be a closed subvairaiety and let $f_1, \ldots, f_s$ generators for the ideal $I(Y)$ of polkynomials vanishing along X. The *Jacobian matrix* of the polynomials $f_1, \ldots, f_s$ is well known from calculus courses. It is shaped like

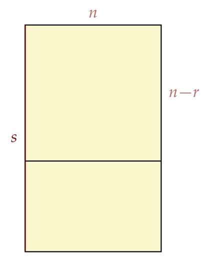
Jacobian matrix

$$J(f_\bullet) = \left(\partial f_i / \partial x_j\right)_{ij}$$

where the row index runs from 1 to s and the row inde from 1 to n. The Jacobian is a matrix function on X, and can be evaluated at points.

We say that the varity X is *non-singular* at the point $p \in X$ if the rank of $J(f_\bullet)(p)$ is equal to $n - \dim X$.

The polynomials fi have Taylor expansions about p shaped like



non-singular

$$f_i = \sum_j \partial_j f_i(p)(x_j - a_j) + H_i$$

where H_i vanishes to the second order at p. In worder words $H_j \in \mathfrak{m}_p^2$. The Jacobian at p is therfoer nothing but the matrix of the liner system consisting of the s of equations

$$\sum_j \partial_j f_i(p) x_j = 0 \quad 1 \leqslant j \leqslant s$$

and the condition that rank of J is $n - r$, says exactly that these linear equation defin a linear subsaåce of dimension r, that is a linear subspace of the same dimension as X. That linear subspace is what one calls the tangent space at p.

The definiton we have gven seems to have some flaws. Appearntly it depends on the given embeddin of X in an affine space, and even worse on the generators.

Proposition 9.3 *The varity X is non-singular at p is and only if* $\dim_k \mathfrak{n}/\mathfrak{n}^2 = \dim X$.

PROOF: Consider the map $k[x_1, \dots, x_n] \to k^n$ sending a polynomial f two the row vector $(\partial_j f)_{1 \leqslant j \leqslant n}$. It is obviously surjective, sending x_i to the i-th standard basic vector of k^n. The ideal $\mathfrak{m}^2$ lies in kernel sins Leibnitz has taught us that $\partial fg = f\partial g + g\partial f$ which vanishs at p when both f and g do. Hence $\theta \colon \mathfrak{m}/\mathfrak{m}^2 \simeq k^n$.

The ideal I lies in $\mathfrak{m}$ and the inage og I under the map θ is vector subspace generated by the rows of the Jacobian matrix evaluated at p. Hence the dimension of the image equals the rank of J. Onte other hand $\theta(I) = I + \mathfrak{m}^2/\mathfrak{m}^2$

On the other hand, one has $\mathfrak{n}/\mathfrak{n}^2 = \mathfrak{m}/(\mathfrak{m}^2 + I)$. Hence tye chain

$$\mathfrak{m}^2 \subseteq I + \mathfrak{m}^2 \subseteq \mathfrak{m}$$

the first subquotient being $\mathfrak{n}/\mathfrak{n}^2$ and the seconf to $\theta(I)$; hence

$$\dim_k \mathfrak{n}/\mathfrak{n}^2 = \dim_k \mathfrak{m}/\mathfrak{m}^2 - \dim(\mathfrak{m}^2 + I)/\mathfrak{m}^2$$

❑

COROLLARY 9.4 *p is a non-singular point on X if and only if the local ring $\mathcal{O}_{X,p}$ is a regular local ring.*

COROLLARY 9.5 *Being non-singular is an intrinsic property.*

9.2.1 *The projective cse*

The Jacobian criterion works well in $\mathbb{P}^n$ as well.

PROPOSITION 9.6 *Let $J = \left(\partial_j F_i(p)\right)$ Then the the rank of J does not depend on the choice of p. Moreover X is non-singular at p if and only if $\mathrm{rk} J = n - r$.*

1) The Euler formula: If F is homogeneous f degree d it holds true that

$$dF = \sum_j x_j \partial_j F$$

Both sides are linear in F, so it sufficies to establish the equality for monials. If F is a monomial, the frmula is clear; say $F = x_1^{d_0} \cdot \ldots \cdot x_n^{d_n}$: Then $x_j \partial_j F = d_j F$, and summing over j closes the case since $d = \sum_j d_j$.

2) If p lies in the basic open subset $D_+(x_k)$, if follow from Euler that

$$\partial_k F_i = x_k^{-1} (\sum_{j \neq k} \partial_j F_i)$$

so after an invertible column operation the whole k-th column of the Jacobian $J(p)$ becomes zero, and can be removed from the matrix without altering the rank.

3) The eauation of $X \cap D$ in $D_+(x_k)$ are the polynomials f_i obtained by dehomogenizingthe F_i's with rspect to x_k. it holds true that $x_k^{d_k} f_i = F_i$. Hence

when $j \neq k$, one has $\partial_j f_i = x_k^{-d_i} \partial_j F_i$, and this means that the i-row have the factor $x_k^{-d_i}$ which can taken out any minor involving that row. So up to factor that are powesr of x_k, the miors of

$$\left(\partial_j F_i\right)$$

and

$$\left(\partial_j f_i\right)$$

are the same.