Problem 1. Assume X and Y are two affine varieties and that $\phi : X \to Y$ is a morphism. Show that ϕ is a closed embedding if and only if the map $\phi^* : A(Y) \to A(X)$ between the coordinate rings is surjective.

Recall: φ is a closed embedding $\Longleftrightarrow$ $W = \varphi(X)$ is closed, and $\varphi : X \longrightarrow W$ is an isomorphism.

If φ is a closed embedding, then $W = Z(I) \subset Y$ for some ideal $I \subset A(Y)$, and $A(W) = A(Y)/I$.

We have

$$\begin{tikzcd} A(Y) \arrow[r, "\phi^*"] \arrow[dr] & A(X) \arrow[d, "\simeq"] \\ & A(W) = A(Y)/I \end{tikzcd}$$

$\Rightarrow$ ϕ^* is surjective.

Conversely, if ϕ^* is surjective, let $I = \ker \phi^*$, so that

$A(X) \cong A(Y)/I$ via ϕ^*

Let $W = Z(I) \subseteq Y$. Then ϕ^* induces an isomorphism

$\phi^* : A(W) \longrightarrow A(X)$

$\Rightarrow$ $\varphi : X \longrightarrow W$ is an isomorphism, by the correspondence between ring maps and morphisms.

$\Rightarrow$ φ is a closed embedding.

Problem 2. Consider the algebraic set $X = Z(I) \subset \mathbb{A}^3$ given by the ideal

$$I = (y - x^2, yz^2, xz^2) \subset \mathbb{C}[x, y, z]$$

Find a decomposition of X into irreducible components and compute its dimension.

Solution 1 (Geometry) $X = Z(I) = Z(\sqrt{I}) = Z(y-x^2, yz, xz)$

either: $z=0$ $\rightsquigarrow$ component $X_1 = Z(y-x^2, z)$

or $x=y=0$ $\rightsquigarrow$ component $X_2 = Z(x,y)$

$\therefore\ X = X_1 \cup X_2$

$\dim X = \max(\dim X_1, \dim X_2) = 1$

Solution 2 (algebra) claim: $I = (y-x^2, z^2) \cap (x,y)$

$\subseteq$ OK

$\supseteq$: Take $f = A(y-x^2) + B z^2$

If $f \in (x,y)$, then $B \in (x,y) \Rightarrow B = xC + yD$

$\Rightarrow\ f = A(y-x^2) + Cxz^2 + Dyz^2$

$\in$ LHS ✓

$\dim X = \dim A(X) = \max\left(\dim \frac{k[x,y,z]}{(y-x^2, z^2)}, \dim \frac{k[x,y,z]}{(x,y)}\right)$

$= \underline{1}$

Problem 3. Find all the singular points on the curve
$$C = Z(x^4 + y^3 z - x^2 yz) \subset \mathbb{P}^2$$
and show that C is rational (i.e., birational to $\mathbb{P}^1$).

(I) $\frac{\partial f}{\partial x} = 4x^3 - 2xyz = 2x(x^2 - yz)$

(II) $\frac{\partial f}{\partial y} = 3y^2 z - x^2 z = z(3y^2 - x^2)$

(III) $\frac{\partial f}{\partial z} = y^3 - x^2 y = y(y^2 - x^2)$

Assume first that $xyz \neq 0$.

(III) gives $y = ax$. $a = \pm 1$

(II) gives $3y^2 - x^2 = 3x^2 - x^2 = 2x^2 \rightsquigarrow x = 0 \rightsquigarrow$ contradiction

$\rightsquigarrow xyz = 0$.

If $x = 0$: $\overset{(III)}{\Longrightarrow} y = 0 \Rightarrow$ the point $(0:0:1)$, which lies on C.

If $y = 0$ (and $x \neq 0$) $\overset{(II)}{\Longrightarrow} z = 0$

$\Rightarrow$ the point $(1:0:0)$, but this does not lie on C.

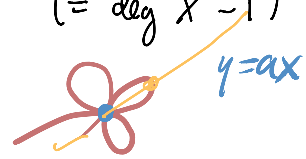
If $z = 0$ (and $xy \neq 0$): (I) $\Rightarrow x = 0 \Rightarrow$ contradiction.

$\therefore$ $(0:0:1)$ is the only singular point.

Consider the open set $D_+(z) \simeq \mathbb{A}^2$

$\rightsquigarrow$ C is given by $x^4 + y^3 - x^2 y$

The point $(0,0)$ has multiplicity 3 $(= \deg X - 1)$
so we expect that C is rational.



Write $y = ax$

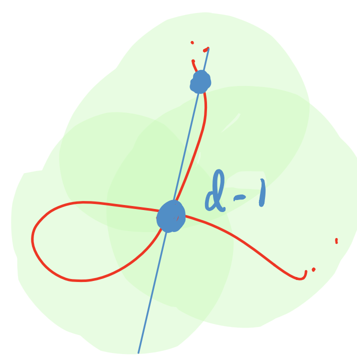
$$\rightsquigarrow \quad x^4 + a^3x^3 - x^2ax = x^3(x + a^3 - a)$$

$\rightsquigarrow$ we get the parameterization

$$\begin{aligned} \mathbb{A}^1 &\xrightarrow{\ \phi\ } C \\ a &\longmapsto (a - a^3,\ a^2 - a^4) \end{aligned}$$

This is 1-1 on an open set by construction, so C is rational.

Inverse:
$$\begin{aligned} C &\dashrightarrow \mathbb{A}^1 \\ (x,y) &\longmapsto y/x \end{aligned}$$

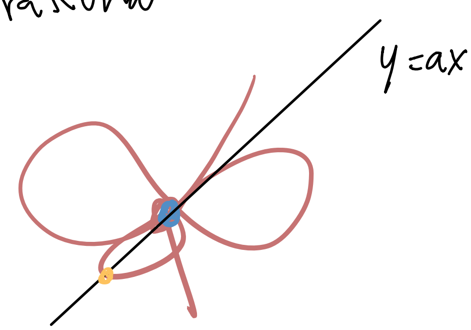


Bezout
$$\sum \mu = d$$

$C \subset \mathbb{A}^2$ degree d

$(0,0) \in C$ multiplicity $d-1$

$\Rightarrow C$ is rational (same argument)



Problem 4. Consider $V \subset \mathbb{A}^2 \times \mathbb{P}^1$ given by the equation

$$u_0 x^2 - u_1 y = 0$$

where $(u_0 : u_1)$ are homogeneous coordinates on $\mathbb{P}^1$ and x, y are affine coordinates on $\mathbb{A}^2$.

(i) Show that V is irreducible and compute its dimension.

(ii) Describe the fibers of the morphism $\pi = p_1 : V \to \mathbb{A}^2$ and show that V is rational.

(iii) Describe the fibers of the morphism $p = p_2 : V \to \mathbb{P}^1$. Which fibers are singular?

(iv)* Find all sections of p, i.e., morphisms $\sigma : \mathbb{P}^1 \to V$ so that $p \circ \sigma = \mathrm{id}_{\mathbb{P}^1}$.

(i) Over the open set $D(u_0)$, $V \cap D(u_0)$ is the hypersurface

$x^2 - u_1 y = 0$ in $\mathbb{A}^3$ $\rightsquigarrow$ irreducible

Over $D(u_1)$ it is given by $u_0 x^2 - y = 0$ in $\mathbb{A}^3$ $\rightsquigarrow$ irred.

Also, $V \cap D(u_0) \cap D(u_1)$ is irreducible $\Rightarrow$ V is irreducible

The dimension of $Z(x^2 - u_1 y)$ is 2 (by Krull) $\Rightarrow$ $\dim V = 2$.

(ii) The fibers of $\pi = p_1 : V \longrightarrow \mathbb{A}^2$

Given $(a,b) \in \mathbb{A}^2$, we have

$(a,b) \times (u_0 : u_1) \in \pi^{-1}(a,b) \Leftrightarrow u_0 a^2 - u_1 b^2 = 0 \quad (*)$

If $(a,b) \neq (0,0)$, the point $(u_0 : u_1)$ is uniquely determined from $(*)$

If $(a,b) = (0,0)$, $\pi^{-1}(0,0) = (0,0) \times \mathbb{P}^1$

X variety $\Rightarrow$ $\dim X = \mathrm{tr.deg}\, k(X)$

$\Rightarrow \dim V = \mathrm{tr.deg}\, k(V)$

$\therefore$ $\pi : V \longrightarrow \mathbb{A}^2$ is generically 1-1 $\longrightarrow$ V is rational.

$= \mathrm{trdeg}\, k(\mathbb{A}^2)$

(birational to $\mathbb{A}^2$) $= 2$.

(iii) Given $(a:b) \in \mathbb{P}^1$, we have

$$p^{-1}(a:b) \simeq \{(x,y) \times (a:b) \mid ax^2 - by = 0\}$$

$$= \text{the parabola } ax^2 - by = 0 \text{ in } \mathbb{A}^2.$$

$\therefore$ The fibers of p are conic curves in $\mathbb{A}^2$.

$p^{-1}(a:b)$ singular $\Leftrightarrow$ $ax^2 - by$ is a singular conic

$b \neq 0$: $\rightsquigarrow$ $y = b^{-1} a x^2$ $\rightsquigarrow$ the conic is a non-sing. parabola

$b = 0$: $\rightsquigarrow$ $p^{-1}(1:0) = \{x = 0\}$ is a line $\Rightarrow$ also non-sing.

Rmk In MAT4215, the "scheme-theoretic fiber" is $x^2 = 0$
$\rightsquigarrow$ singular

(iv) If $\sigma : \mathbb{P}^1 \longrightarrow V$ is a section of p

we get a morphism

$$\begin{tikzcd} \mathbb{P}^1 \arrow[r, "\sigma"] \arrow[rr, bend right] & V \arrow[r, "\pi"] & \mathbb{A}^2 \end{tikzcd}$$

$$\begin{tikzcd} V \arrow[d] & \subset \mathbb{A}^2 \times \mathbb{P}^1 \arrow[r] & \mathbb{A}^2 \\ \mathbb{P}^1 \arrow[u, bend left=60, "\sigma"] \end{tikzcd}$$

This is a morphism from $\mathbb{P}^1$ to an affine variety

$\rightsquigarrow$ $\pi \circ \sigma$ must be constant

b) $\rightsquigarrow$ $\sigma(\mathbb{P}^1)$ must be the curve $(0,0) \times \mathbb{P}^1$

(since π is an isomorphism outside $(0,0) \times \mathbb{P}^1$)

$\rightsquigarrow$ there is exactly one section of p, namely

$$\sigma : \mathbb{P}^1 \longrightarrow V$$
$$(a:b) \longmapsto (0,0) \times (a:b).$$