Summary of the curriculum for MAT4210

A word about preparing for the exam: The exam will be oral, so you should focus on concepts, definitions, examples and counterexamples in your preparation. You may also be asked for some basic computations, but nothing very complicated. Think about how the topics fit together, and why terms are defined in the way they are.

Example questions:

- *"What is a sheaf? Why do we introduce sheaves? Can you give an example? A non-example?"*
- *"What is the Hausdorff axiom? Why do we include it?"*

Some of the main examples: Affine space, projective space, conics, plane curves, hypersurfaces, cuspidal curve, nodal cubic, twisted cubic, quadratic cone $Z(xy-z^2)$, quadric surface $Z(xw-yz)$, rational normal curves, Veronese varieties, the Veronese surface, Segre embedding, $\mathbb{A}^n - 0$, blow-ups.

Here is a checklist of the topics covered through the course. The points marked with * are especially important.

1 Chapter 1

- Definition of the algebraic set associated to an ideal.
- Ideal associated to an algebraic set and properties.
- Definition of coordinate ring associated to an algebraic set,
- * Hilbert's Nullstellensatz
- Conics
- the affine twisted cubic

2 Chapter 2

- Zariski topology*
- Bijection between algebraic sets (resp. affine varieties), radical ideals (resp. radical prime ideals) and finitely generated k-algebras (resp. finitely generated k-algebras that are integral domains).*
- Distinguished open sets
- Irreducible topological spaces. Characterisation and consequences (e.g. not Hausdorff, any non-empty open subset is dense, etc...).
- Properties of $A(X)$ vs X (e.g., an algebraic set is irreducible iff its ideal is prime).*
- Definition of an affine variety
- Noetherian topological spaces
- Decomposition in irreducible components of a Noetherian space.
- Relation with primary decomposition of an ideal.
- Dimension of a (irreducible) topological space. Examples.*
- Dimension of a closed subset and of an open subset.
- Definition of morphism of affine varieties as a polynomial map.
- Polynomial maps are continuous
- The coordinate ring of an affine variety
- Equivalence of categories between algebraic sets and reduced finitely generated k-algebras*
- The quadratic cone

3 Chapter 3

- Presheaves
- Sheaves*
- Examples and non-examples of sheaves
- Pullback of functions
- Ringed spaces and morphisms of ringed spaces*

- Regular and rational functions on affine varieties*
- The function field*
- The local ring at a point*
- The structure sheaf of regular functions for an affine variety (proof that this is a sheaf)*
- Definition of an affine variety*
- Prevarieties*
- Theorem on morphisms to $\mathbb{A}^n$*
- Morphisms of varieties vs morphisms of coordinate rings: The Main theorem of affine varieties*
- Example of a non-affine variety
- The Hausdorff axiom; definition and intuition
- The affine line with two origins (is a prevariety)
- Products of varieties, especially the affine case.

4 Chapter 4

- Projective space*
- Homogeneous cordinates
- Projective algebraic sets, Zariski topology*
- Projective Nullstellensatz
- Distinguished open subsets
- All conics are equivalent
- Regular functions on projective varieties*
- The structure sheaf on projective varieties*
- Global regular functions on projective varieties*
- Projective varieties are varieties
- Maps from projective varieties to affine varieties*
- Cones
- Morphisms of cones vs morphisms of projective varieties*
- Linear projections; examples projecting curves and surfaces

5 Chapter 5

- Closed embeddings
- Criterion for being a closed embedding (left section)
- Rational normal curves
- Segre embeddings
- Veronese embeddings
- Consequences: Products of projective varieties, complements of hypersurfaces are affine,..
- Subvarieties of $\mathbb{P}^n \times \mathbb{P}^m$.
- The Veronese surface and conics

6 Chapter 6

- Dimension
- Images and fibers of morphisms
- Dominant maps
- Finite maps
- Properties of finite and domiant maps*
- Examples
- Noether normalization*
- Dimension = transcendence degree of function field
- Birational varieties have the same dimension
- Chains of subvarieties have the same length
- Codimension and height
- Dimension of a product
- Krull's principal ideal theorem*
- Geometric applications of the PIT*
- Systems of parameters, and their existence, examples and non-examples
- Dimension of fibres of morphisms (inequality)*
- Dimension of intersections (inequality), both affine and projective cases*

7 Chapter 7

- Definition of tangent spaces of affine varieties*
- Computations of T_pX, e.g., using the jacobian*
- T_pX vs m/m^2 – the general definition of T_pX.*
- Singular points
- Regular local rings
- The singular set is closed
- Normal varieties
- Examples: Plane cubics, quadric surfaces.

8 Chapter 8

- Rational maps and maximal sets of definition
- Main theorem of rational maps (link to maps between function fields)*
- Birational maps, examples
- Blow-ups
- Explicit computations using blow-ups*

9 Chapter 9

- Curves
- Local rings are DVRs
- Extension properties of rational maps between curves and (projective) varieties*
- Normalization*
- Non-singular curves vs function fields.*
- Cubic curves are irrational

10 Chapter 9

- The generic structure theorem for dominant morphisms*
- Consequences of the GST: fibers of morphisms, semicontinuity,
- Chevalley's theorem*
- Morphisms of projective varieties are closed

11 Chapter 10

- Bezout's theorem*
- Intersection multiplicity
- Computing multiplicities*
- Transversal intersections
- Hilbert functions and Hilbert polynomials
- Computations using Bezout (curves, surfaces,..)*
- Automorphism of projective space