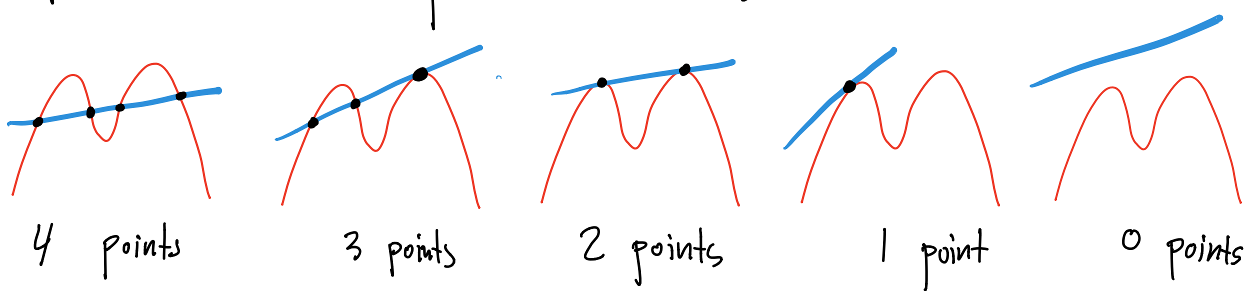
# Chapter 11 - Bezout's theorem

Given two curves $C = Z(F)$ and $D = Z(G)$ of degree $m$ and $n$ respectively, the number of intersection points $C \cap D$ is at most $mn$.




However, if we

- work in $\mathbb{P}^2$
- work over an algebraically closed field
- Count intersection points with multiplicities

then we have <u>exactly</u> $m \cdot n$ points (Bezout's theorem)

**Thm (Classical form of Bezout's theorem)**

Let $Z_1, \ldots, Z_n$ be hypersurfaces in $\mathbb{P}^n$ with only finitely many points in common. Then

$$\deg Z_1 \cdots \deg Z_n = \sum_{p \in \mathbb{P}^n} \mu_p (Z_1, \ldots, Z_n)$$

multiplicity of the intersection at $p \in \mathbb{P}^n$.
(to be defined..)

# Multiplicities of modules

$R$ = noetherian graded $k$-algebra

$M$ = a f.g graded $R$-module

**Defn** The *multiplicity* of $M$ at $\mathfrak{p} \in \operatorname{Spec} R$ is defined by

$$\mu_{\mathfrak{p}} = \operatorname{length}_{R_{\mathfrak{p}}} M_{\mathfrak{p}}$$

$\leftarrow$ Recall: length = supremum of chains of submodules $M = M^0 \supsetneq M^1 \supsetneq \cdots \supsetneq M^d = 0$

If $(R, m)$ is local, then

$$\mu_p(M) < \infty \iff m^d M = 0 \text{ for some } d$$

In this case,

$$\mu_p(M) = \dim_k (M \otimes_R k) \qquad k = R/m$$

## Main example

$X$ a variety

$x \in X$ a point

$I \subset \mathcal{O}_{X,x}$ an ideal with $\sqrt{I} = m \subset \mathcal{O}_{X,x}$

$I \supseteq m^r$ for some $r \gg 0$

$\rightsquigarrow \dim_k \left( \mathcal{O}_{X,x}/I \right) < \infty$ and so we define

the multiplicity of $I$ at $x = \mu_x \left( \mathcal{O}_{X,x}/I \right) = \underset{\substack{\uparrow \\ \text{dimension as a } k\text{-vector sp}}}{\dim_k} \left( \mathcal{O}_{X,x}/I \right)$

**Affine case:** $X \subseteq \mathbb{A}^n$

$$\mathcal{O}_{X,x}/I = \left( A(X)/I \right)_m$$

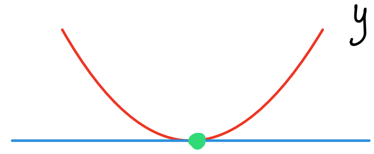
$$\rightsquigarrow \quad \mu_x \left( \mathcal{O}_{X,x}/I \right) = \mu_x \left( A(X)/I \right)$$

**ex** $X = \mathbb{A}^2$

$f = y - x^2$

$g = y$

$p = (0,0)$




$$\rightsquigarrow \quad \mu_p\left(\mathcal{O}_{X,x}/(f,g)\right) = \dim_k\left(\left(\frac{k[x,y]}{(y,\, y-x^2)}\right)_{(x,y)}\right)$$

$$= \dim_k\left(\frac{k[x]}{(x^2)}\right)_{(x)}$$

$$= \dim_k \frac{k[x]}{x^2} = 2.$$

ex $F = (x^2+y^2)z + x^3 + y^3$

$G = x^3 + y^3 - 2xyz$

Consider $I = (F, G)$ and $X = Z_+(I) \subset \mathbb{P}^2$.

We check the affine charts:

$U_0 = D_+(x) \simeq \mathbb{A}^2$ $\quad$ $X \cap U_0 \subset \mathbb{A}^2$ is defined by

$$f = (1+u^2)v + 1 + u^3$$
$$g = 1 + u^3 - 2uv$$

$u = \frac{x_1}{x_0}$, $v = \frac{x_2}{x_0}$

$J = (f, g)$ has primary decomposition

$$J = \underbrace{(3u+3v+3,\ v^3)}_{q_1} \cap \underbrace{(v,\ u^2-u+1)}_{q_2}$$

If $p \in Z(q_1)$ then

$$\mathcal{O}_{X,p} \simeq \left(\frac{k[u,v]}{(3u+3v+3,\ v^3)}\right)_{m_p} \simeq \frac{k[v]}{v^3} \rightsquigarrow \mu_p = \underline{3}$$

one point.

If $p \in Z(q_2)$ then

$$\mathcal{O}_{X,p} \simeq \left(\frac{k[u,v]}{(v,\ u^2-u+1)}\right)_{m_p} \simeq \left(\frac{k[u]}{(u-\alpha)(u-\beta)}\right)_{m_p} \rightsquigarrow \mu_p = 1$$

two points.

$\rightsquigarrow$ Checking the remaining points where $x=0$:

In $D_+(x_2)$, $X$ is given by

$$Z(u^2 + 2uv + v^2,\ u^3 + v^3 - 2uv) = Z(u,v)$$

$u = x_0/x_2$

$v = x_1/x_2$

$$\left(\frac{k[u,v]}{((u+v)^2,\ u^3+v^3-2uv)}\right)_m = \left(\frac{k[u,v]}{((u+v)^2,\ uv)}\right)_m = \left(\frac{k[u,v]}{(u^2+v^2,\ uv)}\right)_m \simeq \left(\frac{k[u,v]}{(uv,\ u^2+v^2,\ v^3,\ u^3)}\right)_m$$

$$v^3 = v(u^2+v^2) - u\cdot(uv)$$

$$((u+v)^2,\ u^3+v^3-2uv)$$

$$= ((u+v)^2,\ uv(3u+3v+2))$$

invertible in

$$= k \oplus ku \oplus kv \oplus ku^2$$

$$\rightsquigarrow \mu_p = 4$$

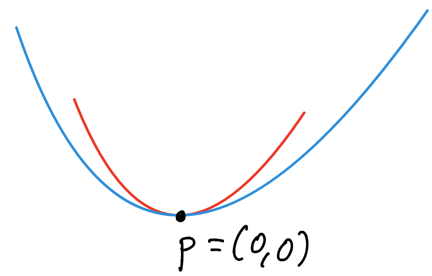
$\therefore$ one point of multiplicity 4

one point of multiplicity 3

two points of multiplicity 1

$$\left(\text{so } \sum \mu_p = 9 = 3\cdot 3\right)$$

**ex** $X = \mathbb{A}^2$




$f = y - x^2$

$g = y - x^2 - xy$

$p = (0,0)$

$$\rightsquigarrow \quad \mathcal{O}_{X,x}/(f,g) = \left( \frac{k[x,y]}{(y-x^2,\ y-x^2-xy)} \right)_{(x,y)}$$

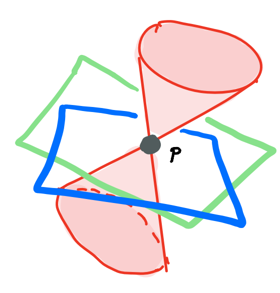
$$= \left( \frac{k[x,y]}{(y-x^2,\ x^3)} \right)_{(x,y)} \cong \left( \frac{k[x]}{x^3} \right)_{(x)} = \frac{k[x]}{x^3}$$

$$\therefore \quad \mu_p(\mathcal{O}/I) = 3$$

**ex** $X = \mathbb{A}^3$ $\quad f_1 = xy - z^2, \quad f_2 = x - z, \quad f_3 = y + x$

$p = (0,0,0)$

$$\rightsquigarrow \quad \mathcal{O}_{X,p}/(f_1,f_2,f_3) = \left( \frac{k[x,y,z]}{(xy-z^2,\ x-z,\ y+x)} \right)_{(x,y,z)}$$




$$= \left( \frac{k[x]}{x(-x) - x^2} \right)_{(x)} = \frac{k[x]}{(x^2)}$$

$\rightsquigarrow$ multiplicity $= 2$

# Transverse intersections

Given $p \in \mathbb{A}^n \longleftrightarrow$ maximal ideal $m \subset k[x_1, .. x_n]$

For $f \in k[x_1, ..., x_n]$ s.t $f(p) = 0$ (so $f \in m_p$)

cotangent space

$\rightsquigarrow$ differential $df \in m/m^2$ (the class of $f$)

Note: $Z(f)$ non-singular at $p \iff df \neq 0$

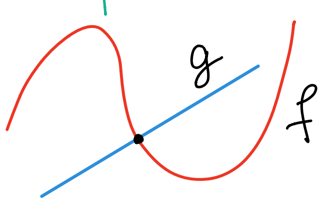
$$df = \sum \frac{\partial f}{\partial x_i}(p) \cdot x_i$$

**Defn** $r$ hypersurfaces with equations $f_1, ..., f_r$ meet *transversely* at $p$ if

(i) they are non-singular at $p$; and

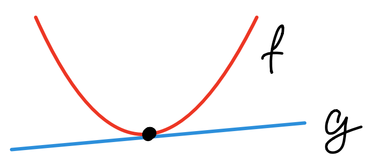
(ii) $df_1, ..., df_r \in m/m^2$ are linearly independent.

Intuitive picture:



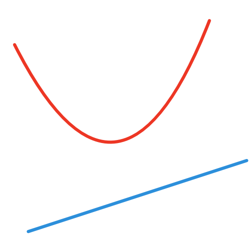

transversal

$\langle df, dg \rangle = k^2$




not transversal

$\langle df, dg \rangle = k$



Also transversal

**Lemma** Let $A = k[t_1, .., t_n]_{(t_1,..,t_n)} = \mathcal{O}_{\mathbb{A}^n, 0}$
$f_1, ..., f_n \in m$

TFAE :

(1) $f_1, ..., f_n$ meet transversally at $0$

(2) $(f_1, .., f_n) = m$

(3) $\mu_p(Z_1, .., Z_n) = 1$ where $Z_i = Z(f_i)$ $i = 1..n$

$$df_1, .., df_n \text{ generate } m/m^2 \overset{\text{Nakayama}}{\Longleftrightarrow} f_1, .., f_n \text{ generates } m$$

$$\Updownarrow$$

$df_1, .., df_n$ are linearly independent

So $(1) \Leftrightarrow (2)$.

$(2) \Leftrightarrow \dim_k \left( A/(f_1, .., f_n) \right) = 1 \Leftrightarrow \mu_p(Z_1, .., Z_n) = 1$ by definition.

**Rmk** If, say, $f_1 \in m^2 \setminus m$, then $f_1, .. f_n$ cannot generate the maximal ideal $m$

$\therefore$ If $Z(f_1)$ is singular at $p \rightarrow \mu_p \geq 2$.

# Numerical polynomials

**Defn** A numerical polynomial is a polynomial $P \in \mathbb{Q}[z]$ s.t $P(m) \in \mathbb{Z}$ for each $m \in \mathbb{Z}$.

**ex**

$$\binom{z}{n} := \frac{z(z-1)\cdots(z-n+1)}{n!} \in \mathbb{Q}[z]$$

is a numerical polynomial which has non-integer coefficients.

If $P$ is a numerical polynomial, then so is $\Delta P(z)$ where

$$\Delta P(z) = P(z+1) - P(z)$$

is the (forward) difference operator.

## Lemma

(1) If $P \in \mathbb{Q}[z]$ and $P(m) \in \mathbb{Z}$ for all $m \gg 0$

$\rightsquigarrow$ $P$ is a numerical polynomial

(2) $\binom{z}{n}$ form a $\underline{\mathbb{Z}\text{-basis}}$ for the group of numerical polynomials

i.e. $$P(z) = c_0 \binom{z}{n} + c_1 \binom{z}{n-1} + \cdots + c_n$$

for $c_0, \ldots, c_n \in \mathbb{Z}$

(3) If $f : \mathbb{Z} \to \mathbb{Z}$ is a function s.t

$\Delta f(m)$ is a polynomial for $m \gg 0$

then $\exists$ numerical polynomial $P(z) \in \mathbb{Q}[z]$ s.t

$$f(m) = P(m) \quad \text{for } m \gg 0.$$

# Hilbert functions and Hilbert polynomials

For the rest of the section $R = k[x_0, \dots, x_n]$

All modules $M$ are f.g. and graded.

**Defn** The Hilbert function of $M$ is given by

$$h_M(i) = \dim_k M_i$$

dimension of the $i$-th graded part of $M$.

**ex** For $M = R$, we have

$$h_M(d) = \binom{n+d}{n}$$

**ex** $R = k[x_0, x_1]$ $\qquad M = R/(x_0^3, x_0x_1, x_1^5)$

$\rightsquigarrow$ $h_M$ has values

| $i$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|---|---|---|---|---|---|---|---|---|---|
| $h_M$ | 1 | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 |
| | | $x_0, x_1$ | $x_0^2, x_1^2$ | $x_1^3$ | $x_1^4$ | | | | |

$\rightsquigarrow$ polynomial growth for $i \geq 5$.

## Theorem (Hilbert-Serre)

$I \subset R = k[x_0, \dots, x_n]$ homogeneous ideal.

$\rightsquigarrow$ unique polynomial $P_I(z) \in \mathbb{Q}[z]$ s.t

$$h_I(m) = P_I(m) \quad \text{for all } m \gg 0$$

Furthermore,

we define $\deg 0 = -1 = \dim \emptyset$

a) $\deg P_I(m) = \dim Z_+(I) \subseteq \mathbb{P}^n$

b) If $Z_+(I) \neq \emptyset$, then the leading coefficient of $P_{R/I}(z)$ is of the form $\frac{1}{(\deg P_I)!}$ (integer)

See Ellingsrud's "Lectures on Commutative Algebra".

**Defn** $P_I(z)$ is called the Hilbert polynomial of $I$.

# The Degree of a variety

**Defn**

If $d = \dim Z_+(\operatorname{ann} M)$, we define the degree of $M$ as

$$\deg M = d!\,(\text{leading coefficient of } P_M)$$

If $X \subseteq \mathbb{P}^n$ is a closed subset, we define

this is an integer.

$$\deg X = \deg\left(R/I(X)\right).$$

Note: If $X \neq \emptyset \Rightarrow (I(X))_d \neq R_d \quad \forall d \geq 0$

$\Rightarrow P_X \neq 0$ with positive leading coefficient

$\Rightarrow \deg X > 0.$

( If $X = \emptyset$, then $\exists\, d$ s.t. $x_i^d \in I(X)$ $\forall i$ by Nullstellensatz

$\Rightarrow \left(R/I(X)\right)_i = 0$ for $i \geq d\cdot(n+1)+1$

$\Rightarrow P_X = 0.$ )

**Lemma** $F \in k[x_0, \dots, x_n]_d \Rightarrow X = Z_+(F) \subset \mathbb{P}^n$ has degree $d$.

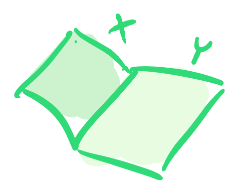
$$0 \to R(-d) \xrightarrow{\cdot F} R \to R/F \to 0$$ is exact

$$\leadsto h_M(z) = h_R(z) - h_R(z-d)$$

$$= \binom{z+n}{n} - \binom{z-d+n}{n} = \frac{d}{(n-1)!} z^{n-1} + \dots$$

$$\Rightarrow \deg X = d \quad ✓$$

**Lemma** $X, Y \subseteq \mathbb{P}^n$ of the same dimension $m$ and with no common component.




$$\leadsto \deg(X \cup Y) = \deg X + \deg Y$$

$$I_{X \cup Y} = I_X \cap I_Y$$

$\leadsto$ s.e.s $$0 \to R/I_{X\cup Y} \to R/I_X \oplus R/I_Y \to R/I_X + I_Y \to 0$$

$$\leadsto P_{X \cup Y} = P_X + P_Y - P_{X \cap Y}$$

$X \cap Y$ has $\dim < m$
$\Rightarrow$ no contribution to leading term

$$= \frac{\deg X}{(m-1)!} z^m + \frac{\deg Y}{(m-1)!} z^m + \text{terms of lower order}$$

$\Rightarrow$ ok.

ex Recall the twisted cubic $C \subset \mathbb{P}^3$ defined by the $2\times 2$-minors of the matrix

$$A = \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

$\rightsquigarrow$ s.e.s

$$0 \to R(-3)^2 \xrightarrow{\begin{pmatrix} x_2 & -x_1 & x_0 \\ x_3 & -x_2 & x_1 \end{pmatrix}} R(-2)^3 \longrightarrow I \longrightarrow 0$$

$$e_1 \to x_0x_2 - x_1^2$$
$$e_2 \to x_0x_3 - x_1x_2$$
$$e_3 \to x_1x_3 - x_2^2$$

$$\therefore P_I = 3\cdot P_R(z-2) - 2P_R(z-3)$$

$$= 3\binom{z-2+3}{3} - 2\binom{z-3+3}{3} = 3\binom{z+1}{3} - 2\binom{z}{3}$$

$$\rightsquigarrow P_{R/I} = P_R - P_I = \binom{z+3}{3} - 3\binom{z+1}{3} + 2\binom{z}{3}$$

$$= \underline{3z+1}$$

$$0 \to I \to R \to R/I \to 0$$

$\dim X = 1$

$\Downarrow$

$\therefore \deg X = \underline{3}$

# Regular sequences and Unmixedness

**Defn** $f_1, \dots, f_k \in R$ forms a regular sequence if $f_{r+1}$ is not a zero-divisor modulo $(f_1 \dots f_r)$ $\forall r = 1 \dots k-1$

**Note** Krull's principal ideal theorem $\Rightarrow$ $\dim Z(f_1, \dots f_r) = n - r$ $\forall r = 1 \dots k-1$.

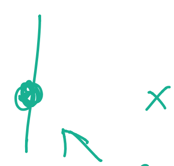
**ex** $x^2, yz$ $\quad x^2, y^2, z^2$ are regular,

$x^2, xy$ , $\quad x^2, x+y, zx+y^2$ are not.

$Z(x)$ is a component $\quad Z(x,y)$ is a component $\rightsquigarrow \dim Z(f_1, f_2, f_3) = 2$

Note:

$(x^2, xy) = (x) \cap (x^2, y)$



embedded prime

$\to$ problem: there are associated prime ideals which aren't minimal.

The following result is fundamental:

## The Unmixedness Theorem (Macaulay)

The polynomial ring $R = k[x_0 \ldots x_n]$ is unmixed.

In other words, any ideal $I \subset R$ generated by $r = \operatorname{ht} I$ elements does not have any embedded primes.

$\Longleftrightarrow$ every associated prime is a minimal prime.

"I is unmixed"

**Prop** If $f_1, \ldots, f_n \in R$ have $Z(f_1, \ldots, f_n)$ finite, then $f_1, \ldots, f_n$ form a regular sequence:

$f_{r+1}$ is not a zero-divisor modulo $(f_1 \ldots f_r)$ $\forall r$.

**Lemma** If $Z(f_1, \ldots f_n)$ is finite, then any irreducible component $Z$ of $Z(f_1, \ldots f_r)$ has dimension $n-r$ for $r = 1 \ldots n$.

First, $\dim Z \geq n-r$ by Krull's principal ideal theorem

Similarly, $\dim(Z \cap Z(f_{r+1}, \ldots, f_n)) \geq \dim Z - (n-r)$

By assumption, LHS $= 0$, so $\dim Z \leq n-r$. ✓

**Cor** The ideal $(f_1 \ldots f_r)$ has ht $= r$ and is therefore unmixed.

Now, let $M = R/(f_1, \ldots f_r)$.

All associated primes of $M$ have ht $r$ and $\dim \frac{M}{f_{r+1}} < \dim M$.

$\Rightarrow$ $f_{r+1}$ is not contained in any associated prime of $M$

$\Rightarrow$ $f_{r+1}$ is not a zerodivisor.

$\Rightarrow$ $f_1 \ldots f_n$ is a regular sequence

**Prop** Let $f_1 \dots f_n \in R$ be a regular sequence of homogeneous elements of degrees $d_1, \dots, d_n$ respectively. Then

$$h_{R/I}(I) = d_1 \cdots d_n \qquad I = (f_1, \dots, f_n)$$

Write $S_r = R/(f_1 \dots f_r)$. We have sequences $\quad h_M(i) = \dim_k M_i$

$$0 \longrightarrow S_r(-d_{r+1}) \xrightarrow{\cdot f_{r+1}} S_r \longrightarrow S_{r+1} = S_r/f_{r+1} \longrightarrow 0$$

$f_1 \dots f_n$ regular sequence $\Longrightarrow$ this is exact. (injective on left)

$\underset{\text{induction on } r}{\leadsto}$
$$h_{S_{r+1}}(z) = h_{S_r}(z) - h_{S_r}(z - d_{r+1})$$
$$= \left( \frac{d_1 \cdots d_r}{(n-r)!} z^{n-r} + a z^{n-r-1} + \dots \right)$$
$$- \left( \frac{d_1 \cdots d_r}{(n-r)!} (z - d_{r+1})^{n-r} + a(z - d_{r+1})^{n-r-1} + \dots \right)$$

$$= \frac{d_1 \cdots d_r}{(n-r)!} \cdot (n-r) \cdot d_{r+1} z^{n-r-1} + \dots$$

$$= \frac{d_1 \cdots d_{r+1}}{(n-r-1)!} z^{n-r-1} + \dots \qquad \square$$

## Alternative proof for $n=2$  $R = k[x_0, x_1, x_2]$

If $f_1, f_2$ have no common factors (so $f_1, f_2$ is a regular sequence), then the following sequence is exact:

"Koszul complex"

$$0 \to R(-d-e) \xrightarrow{\alpha} \begin{array}{c} R(-d) \\ \oplus \\ R(-e) \end{array} \xrightarrow{\beta} R \to R/(f_1, f_2) \to 0$$

Here $\alpha(w) = (f_2 \cdot w, -f_1 \cdot w)$  $d = \deg f_1$

$e = \deg f_2$

$\beta(a, b) = a f_1 + b f_2$

This gives

$$h_{R/(f_1,f_2)}(z) = h_R(z) - h_R(z-d) - h_R(z-e) + h_R(z-d-e)$$

$$= \binom{z+2}{2} - \binom{z-d+2}{2} - \binom{z-e+2}{2} + \binom{z-d-e+2}{2}$$

$$= \underline{d \cdot e}$$

**Rmk** There is a Koszul complex in $\mathbb{P}^n$ $n \geq 3$ as well:

$$\cdots \to \bigoplus_{i<j} R(-d_i - d_j) \to \bigoplus_i R(-d_i)^n \to R \to R/(f_1, \ldots, f_n) \to 0$$

But showing exactness here is trickier.

# From Hilbert polynomials to local multiplicities

Now, $I = (f_1, \dots, f_n) \subset R$ has $\dim Z_{+}(I) = 0$

$\rightsquigarrow$ after a change of coordinates, we may assume

$$Z(I) \subset D_+(x_0) \subset \mathbb{P}^n$$

**Lemma** Let $f_1, \dots, f_n \in k[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}]$ denote the dehomogenizations wrt $x_0$, and set

$$\mathcal{O}_{Z_1 \cap \dots \cap Z_n} = k\left[\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right] \Big/ (f_1, \dots, f_n) \overset{\text{Artinian}}{\cong} \prod_p \mathcal{O}_{Z_1 \cap \dots \cap Z_n, p}$$

Then there is a decomposition

$$S_{x_0} = \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_{Z_1 \cap \dots \cap Z_n} \cdot x_0^i$$

Note: $(F_1, \dots, F_n) = (f_1, \dots, f_n)$ in $S_{x_0}$ $\quad (x_i^{d_i} f_i = F_i)$

The degree 0 part: Consider

$$R \to R/(F_1, \dots, F_n) \rightsquigarrow R_{x_0} \xrightarrow{\theta} R_{x_0}/(F_1, \dots, F_n)R_{x_0} = S_{x_0}$$

$$(\ker \theta)_0 = \left((F_1, \dots, F_n)_{x_0}\right)_0 = \left((f_1, \dots, f_n)_{x_0}\right)_0$$

$$\rightsquigarrow \quad \frac{(R_{x_0})_0}{((f_1, \dots, f_n)_{x_0})_0} \simeq (S_{x_0})_0 \quad \Rightarrow \quad (S_{x_0})_0 \simeq \mathcal{O}_{Z_1 \cap \dots \cap Z_n}$$

Now, $x_0$ is invertible in these localizations, so

$$(S_{x_0}) \cdot x_0^i \subseteq (S_{x_0})_i$$

Conversely, any $w \in (S_{x_0})_i$ is of the form $w = a x_0^s$

where $s \in \mathbb{Z}$, $a \in (S_{x_0})_0$. Hence

$$(S_{x_0})_i = (S_{x_0})_0 \cdot x_i^i$$

$$\rightsquigarrow \quad S_{x_0} = \bigoplus_{i \in \mathbb{Z}} (S_{x_0})_0 \, x_0^i$$

$$= \bigoplus_{i \in \mathbb{Z}} \mathcal{O}_{Z_1 \cap \dots \cap Z_n} \cdot x_0^i \qquad \square$$

**Lemma** For $d \gg 0$, the localization map $S \longrightarrow S_{x_0}$ induces an isomorphism

$$\rho : S_d \longrightarrow (S_{x_0})_d$$

$\underline{\rho \text{ injective}}$:

**Claim** $M = \ker \rho$ has support at the origin. ← as an $R$-module

$Z_+(F_1, \dots, F_n, x_0) = \emptyset \implies (F_1, \dots, F_n, x_0)$ is $\underbrace{(x_0, \dots, x_n)}_{=: m_+}$-primary

$\therefore$ $Z(x_0)$ and $\mathrm{Supp}(S)$ have only the origin in common in $\mathbb{A}^{n+1}$. ✓

$w \in \ker \rho \implies x_0^N w = 0$ for some $N > 0$

Also, $w \cdot F_1 = \dots = w \cdot F_n = 0$

$\implies M$ is annihilated by some power $m_+^N$

$\Rightarrow$ $M$ is finite-dimensional as a $k$-vector space

$\Rightarrow$ $M_i = 0$ for $i \gg 0$

$\Rightarrow$ $\rho$ is injective in large degrees.

<u>$\rho$ surjective</u>

Let $\omega = a x_0^r \in S_{x_0}$ with $a \in (S_{x_0})_0$.

Modulo $(F_1, \dots, F_n)$ we may write $\omega$ as

$$\omega = a x_0^r = H x_0^{r-d} \quad \text{where} \quad H \in R_d.$$

$\rightsquigarrow$ If $r > d$, $\omega$ lies in the image of $\rho$.

Now, take a basis $a_1, \dots, a_v \in (S_{x_0})_0$ (as a $k$-vector space) and write these as $a_i = H_i x_0^{-d_j}$ where $H_i \in R_{d_i}$

$d > d_1 \cdots d_j \Rightarrow$ all products $a_j \cdot x_0^{d_j} \in \operatorname{im} \rho$

$$\begin{tikzcd}
S_d \arrow[r, "\rho"] & (S_{x_0})_d \\
S_0 \arrow[u, "\cdot x_0^d"] \arrow[r] & (S_{x_0})_0 \arrow[u, "x_0^d"', "\simeq"]
\end{tikzcd}$$

$\Rightarrow$ $\rho$ is surjective ✓ $\square$

**Prop** For $d \gg 0$, the localization map $S \longrightarrow S_{x_0}$ induces an isomorphism

$$S_d \simeq \mathcal{O}_{Z_1 \cap \cdots \cap Z_n} \cdot x_0^d$$

In particular,

$$\dim_k S_d = \dim_k \mathcal{O}_{Z_1 \cap \cdots \cap Z_n}$$

## Proof of Bezout's theorem

For $d \gg 0$, we have

$$\begin{aligned}
d_1 \cdots d_n &= h_S(d) && \text{(by Hilbert polynomial computations)} \\
&= \dim S_d && \text{(def)} \\
&= \dim \mathcal{O}_{Z_1 \cap \cdots \cap Z_n} && \text{(by proposition)} \\
&= \sum_P \dim_k \mathcal{O}_{Z_1 \cap \cdots \cap Z_n, P} && \Big(\mathcal{O}_{Z_1 \cap \cdots \cap Z_n} = \prod \mathcal{O}_{Z_1 \cap \cdots \cap Z_n, P}\Big) \\
&= \sum_P \mu_P(Z_1, \ldots, Z_n)
\end{aligned}$$

# Basic examples

<u>Prop</u> Given $p_1, \dots, p_5 \in \mathbb{P}^5$ no 4 lying on a line

$\Rightarrow \exists$ unique conic through $p_1 \dots p_5$.

Uniqueness: Suppose $C_1, C_2$ are two such conics

$\Rightarrow \quad C_1 \cap C_2 \supseteq \{p_1, \dots, p_5\}$

$\Rightarrow \quad C_1$ and $C_2$ share a component by Bezout

$\Rightarrow \quad C_1$ and $C_2$ are both reducible, say

$C_1 = Z_+(L_1 L_2) \qquad C_2 = Z_+(L_1 L_3)$

$\Rightarrow \quad C_1 \cap C_2 = L_1 \cup (L_2 \cap L_3)$

$\Rightarrow \quad \{p_1, \dots, p_5\} \subseteq L_1 \cup \{\text{point}\}$

$\Rightarrow \quad$ some 4 of $p_1 \dots p_5$ lie on $L_1$.

Existence: Let $Q = a_{00} x_0^2 + \dots + a_{22} x_2^2$

$Q(p_1) = 0 \rightsquigarrow$ linear condition on $a_{00}, a_{01}, \dots, a_{22}$

6 equations $\Longrightarrow$ at least one non-zero solution $a_{00}, a_{01}, \dots, a_{22}$.

# Application: Automorphisms of $\mathbb{P}^n$

**Thm** Any automorphism $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ is a linear transformation. That is,

$$\mathrm{Aut}\,\mathbb{P}^n = PGL_{n+1}(k) = GL_{n+1}(k)/k^*$$

We have a map

$$\rho : PGL_{n+1}(k) \longrightarrow \mathrm{Aut}\,\mathbb{P}^n$$

$\rho$ injective:

If $M \in GL_n(k)$ induces the identity, then $M = c \cdot \mathrm{Id}_{n+1}$, $c \neq 0$. ✓

$\rho$ surjective:

Let $\varphi : \mathbb{P}^n \longrightarrow \mathbb{P}^n$ be an automorphism.

Consider $H = Z(x_0)$.

$\rightsquigarrow$ $\varphi(H) \subset \mathbb{P}^n$ is a subvariety of codim 1

$\overset{\text{Krull}}{\rightsquigarrow}$ $\varphi(H) = Z_+(F_0)$ for some $F_0 \in k[y_0, \dots, y_n]$

Consider $\ell = Z(x_1) \cap \dots \cap Z(x_{n-1}) = Z(x_1, \dots, x_{n-1})$,

so that $H \cap \ell = (0 : \cdots : 0 : 1)$

Then since $\varphi$ is an automorphism,

$$
\begin{aligned}
\varphi((0:\cdots:0:1)) &= \varphi(H \cap \ell) \\
&= \varphi(H) \cap \varphi(\ell) \\
&= Z_+(F_0) \cap \varphi(Z(x_1)) \cap \cdots \cap \varphi(Z(x_{n-1}))
\end{aligned}
$$

Hence

$$1 = (\deg F_0) \cdot (\deg \varphi(Z(x_1))) \cdot \cdots \cdot (\deg \varphi(Z(x_{n-1})))$$

This means that $F_0 = a_{00} y_0 + \cdots + a_{0n} y_n$ is a linear form.

Also, we get that $\varphi$ induces an isomorphism

$$\varphi|_{D_+(x_0)} : \overset{\simeq \mathbb{A}^n}{D_+(x_0)} \longrightarrow D_+(F_0) \simeq \mathbb{A}^n$$

Repeating the argument for $H = Z(x_i)$ for $i = 1 \ldots n$ shows that $\varphi$ induces isomorphisms

$$\varphi|_{D_+(x_i)} : \overset{\simeq \mathbb{A}^n}{D_+(x_i)} \longrightarrow D_+(F_i) \simeq \mathbb{A}^n$$

$\varphi|_{D_+(x_0)} : \mathbb{A}^n \longrightarrow \mathbb{A}^n$ is given by

$$\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right) \mapsto \left( f_1\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right), \dots, f_n\left(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0}\right)\right)$$

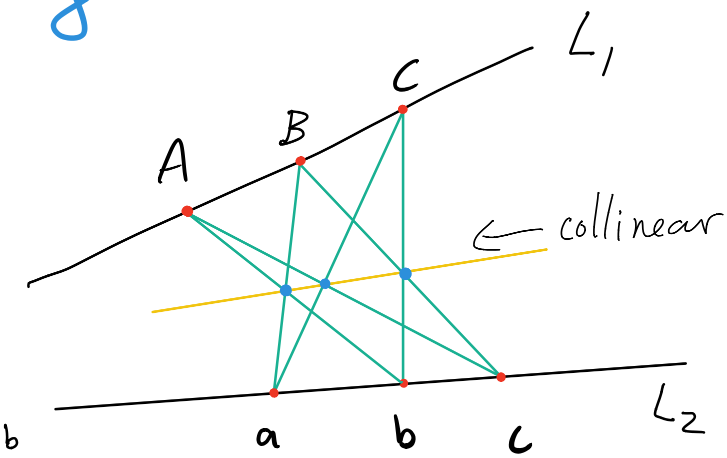
Since $\varphi$ sends hyperplanes to hyperplanes, we see that the $f_i$ must be <u>linear</u> polynomials, and thus $\varphi$ must be induced by a linear map $\mathbb{P}^n \to \mathbb{P}^n$.

# Applications to geometry

## Pappus' theorem

$L_1, L_2$ two lines

Pick $A, B, C \in L_1$,
$a, b, c \in L_2$




⇝ draw lines connecting A to b
A to c
B to a etc

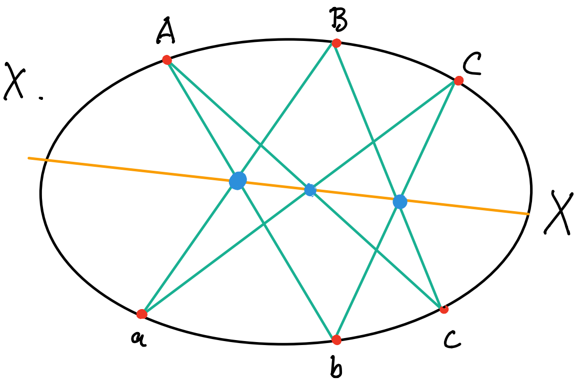
⟿ the intersection points of these 6 lines lie on a line.

## Pascal's theorem

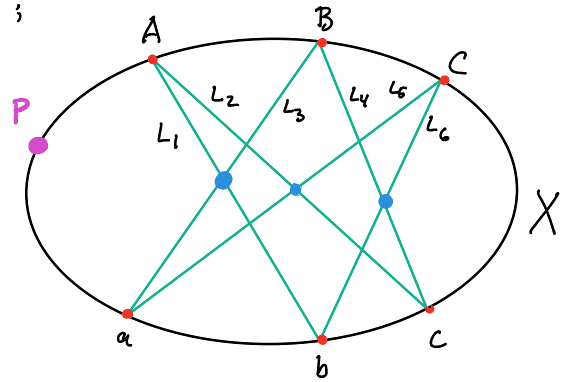
Pick $A, B, C, a, b, c$ on a conic $X$.

Draw 6 lines as before.

⇝ the 3 intersection points are colinear.



Label the lines as follows:




Consider the curves

$C_\lambda = Z(L_1 L_2 L_3 - \lambda L_4 L_5 L_6)$

Let $p \in X$ be some other point, and choose $\lambda$ so that $p \in C_\lambda$.

Consider $C_\lambda \cap X$. This contains $A, B, C, a, b, c$ and $p$

$\rightsquigarrow \quad \#(C_\lambda \cap X) \geq 7$

However, $\deg C_\lambda = 3$ and $\deg X = 2$

Bezout $\Longrightarrow$ $C_\lambda$ and $X$ must have a common component

$\therefore \; C_\lambda = Z(F), \; X = Z(Q) \quad \rightsquigarrow \quad F = Q \cdot L$

where $L$ is a linear form.

$\therefore$ We find that the intersection points lie on $Z(L)$ $\square$

# Real plane curves

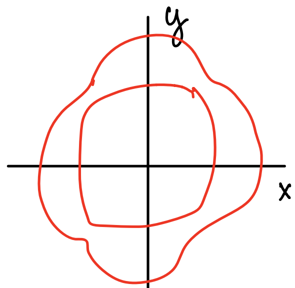
Let $X \subset \mathbb{P}^2$ be a curve defined by a quartic

$$F \in \mathbb{R}[x_0, x_1, x_2]$$

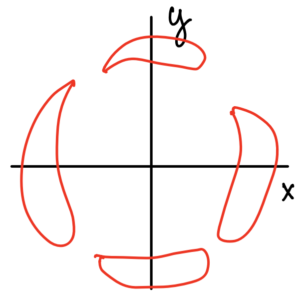
Q: What sort of topology can $Z_{\mathbb{R}}(F) \subset \mathbb{P}^2_{\mathbb{R}}$ have?

real points in $\mathbb{P}^2_{\mathbb{C}}$

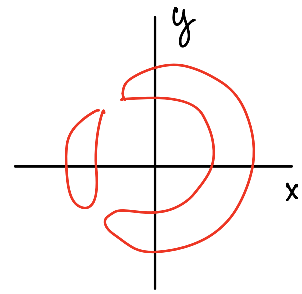
**ex**



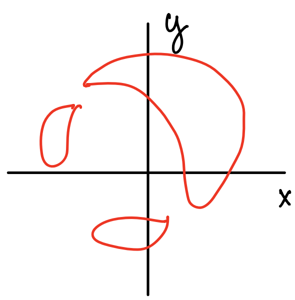

2 nested ovals



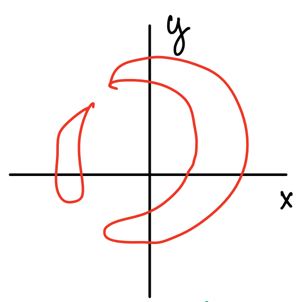

4 ovals



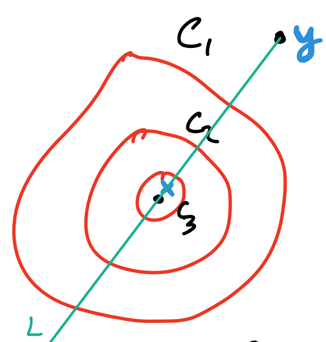

2 ovals




3 ovals




2 ovals




Q: Can $X$ have the topology of 3 nested ovals?

A: No. Label the ovals as in the figure. Pick

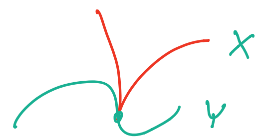
$x \in$ interior of $C_3$
$y \in$ exterior of $C_1$ $\rightsquigarrow$ let $L$ = line through $x$ and $y$.

$L$ meets each oval twice $\Rightarrow$ $L \cap X \geq 6$ $\Rightarrow$ contradicting Bezout.

# Bounds for the number of singular points

Let $X, Y \subset \mathbb{P}^2$ be plane curves given by $F, G$ respectively.

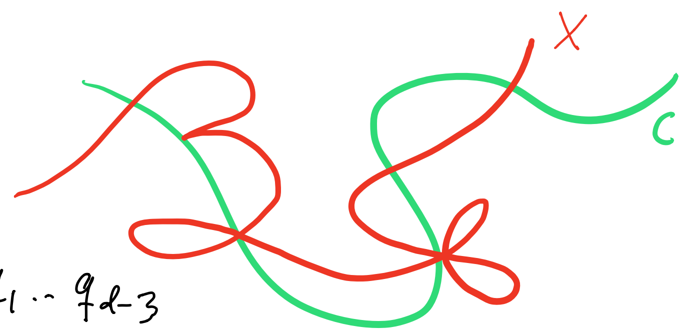
If $p \in X \cap Y$ is a singular point of $X$, then the intersection $X \cap Y$ can not be transversal there, so $\mu_p \geq 2$.




**Prop** A curve $X \subset \mathbb{P}^2$ of degree $d$ can not have more than $\binom{d-1}{2}$ singular points.

Suppose $X$ is singular at

$$p_1, \dots, p_{\binom{d-1}{2}+1}$$




pick $d-3$ extra points $q_1 \dots q_{d-3}$

Let $C$ be a curve of degree $d-2$ which passes through

$$p_1, \dots, p_{\binom{d-1}{2}+1}, q_1, \dots, q_{d-3}$$

$\binom{d-1}{2}+1+d-3$

$=\binom{d}{2}-1$ pts

$\rightsquigarrow$ a dimension count shows that such a curve

$\rightsquigarrow \deg C \cdot \deg X = d(d-2) = d^2 - 2d$

However, $\mu_p \geq 2$ for each singular intersection point exists.

$$\rightsquigarrow \quad \sum_{p \in C \cap X} \mu_p \geq \sum_{p_i} 2 + \sum_{q_i} 1 = 2 \cdot \left( \binom{d-2}{2} + 1 \right) + (d-3)$$

$$= d^2 - 2d + 1$$

Bezout $\rightsquigarrow$ $C$ and $X$ must share a component

$\rightsquigarrow$ contradicting the assumption that $X$ is irreducible.

$\square$