Chapter 4: Projective varieties

Projective n-space

We define the projective n-space as the set

equivalence relation

$$\mathbb{P}^n = \mathbb{A}^{n+1} - 0 \Big/ \underset{\lambda \in k^\times}{(x_0, \dots, x_n) \sim (\lambda x_0, \dots, \lambda x_n)}$$

We let $\pi : \mathbb{A}^{n+1} - 0 \longrightarrow \mathbb{P}^n$ denote the quotient map.
$\mathbb{P}^n$ is given the quotient topology: This means that we define

$$V \subseteq \mathbb{P}^n \text{ is closed} \iff \pi^{-1}(V) \text{ is closed in } \mathbb{A}^{n+1} - 0$$

Here we have the Zariski topology

Homogeneous coordinates

For $x = (x_0, .., x_n) \in \mathbb{A}^{n+1} - 0$, we let

$$[x] = (x_0 : \cdots : x_n) \in \mathbb{P}^n$$

denote the equivalence class of x, i.e. $[x] = \pi(x)$.

These are called homogeneous coordinates on $\mathbb{P}^n$.

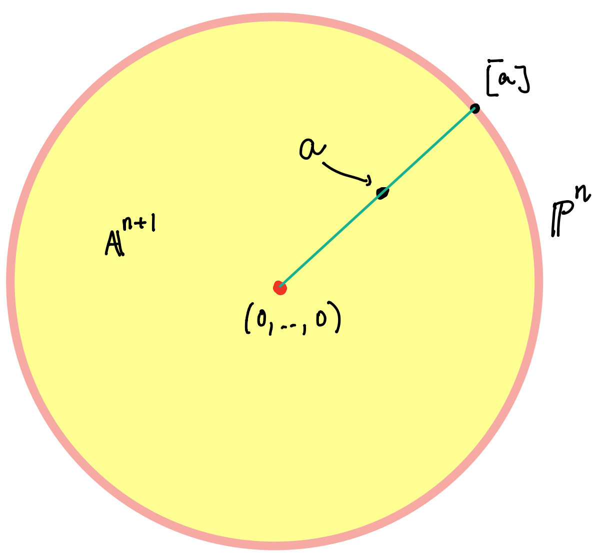
So for instance,

$$(2:2:0) = (1:1:0) \qquad (3:4:5) = (6:8:10) \qquad (1:0) = (4:0)$$

Note that for $a = (a_0, .., a_n) \in \mathbb{A}^{n+1} - 0$

$$\pi^{-1}([a]) = \{ (\lambda a_0, .., \lambda a_n) \mid \lambda \in k^* \}$$

Thus geometrically, the points of $\mathbb{P}^n$ correspond to <u>lines</u> in $\mathbb{A}^{n+1}$ through the origin.



Graded rings and homogeneous polynomials

Defn A graded ring R is a ring of the form

$$R = \bigoplus_{d \geq 0} R_d \quad \text{such that}$$

$$f \in R_d,\ g \in R_e \implies f \cdot g \in R_{d+e}$$

For $f \in R$ we may write $f = f_0 + f_1 + f_2 + \cdots$ where $f_i \in R_i$ are the homogeneous components of f.

ex The ring $R = k[x_0, \ldots, x_n]$ is graded with

$$R_d = \left\{ \sum_{i_0 + \cdots + i_n = d} a_{i_0 \cdots i_n} x_0^{i_0} \cdots x_n^{i_n} \;\middle|\; a_{i_0 \cdots i_n} \in k \right\}$$

Note that a polynomial $F \in k[x_0, \ldots, x_n]$ is homogeneous if

$$F(tx_0, \ldots, tx_n) = t^d F(x_0, \ldots, x_n) \qquad d = \deg F$$

Equivalently, all monomials in F have the same degree.

ex $x_0^2 + x_1^2$, $x_0 x_1 x_2$, and $x_0 x_3 - x_2^2$ are homogeneous

$x_0^2 + x_1^3 - x_1$, $x_0^4 + x_1$ are not homogeneous.

We say that an ideal $I \subset R$ is **homogeneous** if it is generated by homogeneous elements.

If I and J are homogeneous, then so is

$$I \cap J, \quad I+J, \quad IJ, \quad \sqrt{I} \text{ and } \sqrt{J}.$$

Homogenization

If $f \in k[x_0, \dots, x_n]$ is a polynomial of degree d, there is an associated homogenization of f with respect to x_i given by

$$f^h(x_0, \dots, x_n) = x_i^d f\left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i}\right)$$

ex The homogenizations of $x_0^2 + x_1^3 - x_1$ and $x_0^4 + x_1$ with respect to x_0 are: $x_0^3 + x_1^3 - x_1 x_0^2$, $x_0^4 + x_1 x_0^3$ and with respect to x_1: $x_0^2 x_1$, $x_0^4 + x_1^4$.

If $f \in k[x_1, \dots, x_n]$, we can recover f from the homogenization f^h wrt x_0 by

$$f = f^h \big|_{x_0 = 1} .$$

The importance of homogeneous polynomials comes from the following observation:

If $I = (F_1, .., F_r) \subset k[x_0, .., x_n]$ is a homogeneous ideal then the zero-locus of I

$$Z_+(I) = \{(x_0 : \dots : x_n) \in \mathbb{P}^n \mid F_i(x_0, .., x_n) = 0 \quad i = 1, .., r\}$$

is a closed subset of $\mathbb{P}^n$.

Indeed, we have

$$\pi^{-1}(Z_+(I)) = Z(I) \cap (\mathbb{A}^{n+1} - 0)$$

which is closed in $\mathbb{A}^{n+1} - 0$.

These satisfy the following identities:

$$Z_+(\mathfrak{a}\mathfrak{b}) = Z_+(\mathfrak{a}) \cup Z_+(\mathfrak{b})$$
$$Z_+(\mathfrak{a} + \mathfrak{b}) = Z_+(\mathfrak{a}) \cap Z_+(\mathfrak{b})$$
$$Z_+(\mathfrak{a}) = Z_+(\sqrt{\mathfrak{a}})$$

Conversely, for a subset $X \subset \mathbb{P}^n$ we define

$$I(X) = \left\langle f \in k[x_0, .., x_n] \;\middle|\; \begin{array}{l} f \text{ homogeneous} \\ f(x) = 0 \quad \forall x \in X \end{array} \right\rangle$$

Cones

- An algebraic set $W \subset \mathbb{A}^{n+1}$ is called a cone if $0 \in W$ and $x \in W \Longrightarrow \lambda x \in W$ for all $\lambda \in k$.

- If W is a cone, then the projectivization of W is

$$\mathbb{P}(W) = \pi(W - 0) = \{ (x_0 : \cdots : x_n) \in \mathbb{P}^n \mid (x_0, \ldots, x_n) \in W \}$$

- For a closed subset $X \subset \mathbb{P}^n$ the cone of X is

$$C(X) = \pi^{-1}(X) \cup \{0\} \hookrightarrow \mathbb{A}^{n+1}$$

Two observations

- If $S \subset k[x_0, \ldots, x_n]$ is a set of homogeneous polynomials then $Z(S) \subset \mathbb{A}^{n+1}$ is a cone. $F(tx) = t^d F(x)$

- Conversely, if $X \subset \mathbb{A}^{n+1}$ is a cone, then $I(X)$ is homogeneous.

$f \in I(X) \Rightarrow$ write $f = f_0 + f_1 + \cdots$

$0 = f(\lambda x) = \sum \lambda^d f_d(x)$ for all $\lambda \in k$ since X is a cone.

$\longrightarrow$ this is the zero polynomial in λ

$\longrightarrow$ $f_i(x) = 0$ for all $i \Longrightarrow f_i \in I(X) \Longrightarrow I(X)$ homogeneous.

Prop There is a bijection

$$\left\{\text{cones in } \mathbb{A}^{n+1}\right\} \longleftrightarrow \left\{\text{projective algebraic sets}\right\}$$

$$C(X) = Z(\mathfrak{a}) \qquad\qquad X = Z_+(\mathfrak{a})$$

For a homogeneous ideal $\mathfrak{a} \subset k[x_0, \dots, x_n]$, we have

$$\mathbb{P}(Z(\mathfrak{a})) = Z_+(\mathfrak{a}) \quad \text{and} \quad C(Z_+(\mathfrak{a})) = Z(\mathfrak{a})$$

$Z(\mathfrak{a})$ is a cone ✓

Any cone is of this form ✓

Any projective algebraic set is of the form $Z_+(\mathfrak{a})$. ✓

$\square$

The irrelevant ideal

Note that the ideal $(x_0, \dots, x_n)$ defines the empty zero set in $\mathbb{P}^n$. We call this the irrelevant ideal.

More generally, for a graded ring $R = \bigoplus R_d$ the submodule

$$R_{>0} = \bigoplus_{d>0} R_d$$

is an ideal of R: the irrelevant ideal of R

Projective Nullstellensatz

(i) For any projective algebraic set $X \subset \mathbb{P}^n$ we have

$$Z_+(I(X)) = X \quad \leftarrow \text{easy}$$

(ii) For any homogeneous ideal $J \subset k[x_0, \dots, x_n]$ with $\sqrt{J} \neq (x_0, \dots, x_n)$, we have

$$I(Z_+(J)) = \sqrt{J}.$$

This gives an inclusion-reversing bijective correspondence

projective algebraic sets in $\mathbb{P}^n$

homogeneous radical ideals in $k[x_0, \dots, x_n]$ not equal to the irrelevant ideal.

$$X \mapsto I(X)$$

$$Z_p(J) \leftarrow J$$

Proof

(i) $Z_+(I(X)) = X$ follows like in the affine case.

(ii) $I(Z_+(J)) \supseteq \sqrt{J}$ clear.

"$\subseteq$": We have

$$I(Z_+(J)) = \left\langle f \in k[x_0,..,x_n] \;\middle|\; \begin{array}{l} f \text{ homogeneous}, \\ f(x) = 0 \;\; \forall x \in Z_+(J) \end{array} \right\rangle$$

$$= \left\langle f \in k[x_0,..,x_n] \;\middle|\; \begin{array}{l} f \text{ homogeneous}, \\ f(x) = 0 \;\; \forall x \in Z(J) - 0 \end{array} \right\rangle$$

affine zero locus is closed

$$= \left\langle f \in k[x_0,..,x_n] \;\middle|\; \begin{array}{l} f \text{ homogeneous}, \\ f(x) = 0 \;\; \forall x \in \overline{Z(J) - 0} \end{array} \right\rangle$$

$\overline{Z(J) - 0} = Z(J)$

$$= \left\langle f \in k[x_0,..,x_n] \;\middle|\; \begin{array}{l} f \text{ homogeneous} \\ f(x) = 0 \;\; \forall x \in Z(J) \end{array} \right\rangle$$

$Z(J)$ is a cone, and its ideal is automatically homogeneous.

$$= I(Z(J)) = \sqrt{J}$$

usual Nullstellensatz

The bijections above follows from this. $\square$

Distinguished open sets

We define the distinguished open set $D_+(x_i)$ to be

$$D_+(x_i) = \{ x = (x_0 : \cdots : x_n) \in \mathbb{P}^n \mid x_i \neq 0 \}$$

$$= \{ x = (x_0 : \cdots : x_{i-1} : 1 : x_{i+1} : \cdots : x_n) \in \mathbb{P}^n \}$$

If $A_i = Z(x_i - 1) \subset \mathbb{A}^{n+1} - 0$, then the map

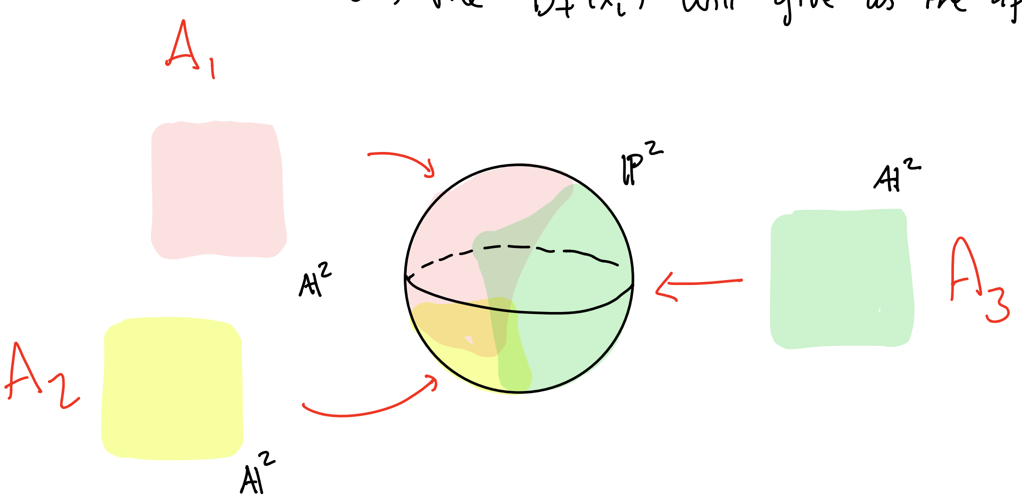
$$\pi|_{A_i} : A_i \longrightarrow D_+(x_i)$$

is bijective : the inverse is given by

$$\alpha : \underbrace{(x_0 : \cdots : x_n)}_{x_i \neq 0} \longmapsto \left(\frac{x_0}{x_i}, \frac{x_1}{x_i}, \ldots, 1, \ldots \frac{x_n}{x_i} \right) \in A_i$$

Main idea: We want to show that $\mathbb{P}^n$ is a (pre) variety

$\rightsquigarrow$ the $D_+(x_i)$ will give us the affine cover.



In fact:

Prop $\pi|_{A_i} : A_i \to D_+(x_i)$ is a homeomorphism

$\pi|_{A_i}$ is clearly continuous and a bijection.
To prove that it is a homeomorphism, we need to show that $\pi|_{A_i}$ is closed.
Any closed subset $Z \subseteq A_i$ is an intersection of sets of the form

$$Z = Z(f) \cap A_i \qquad f \in k[x_0, \dots, x_n]$$

$\rightsquigarrow$ suffices to show $\pi(Z(f) \cap A_i)$ is closed in $D_+(x_i)$

$$\text{But } \pi(Z(f) \cap A_i) = \left\{ (x_0 : \dots : x_n) \in \mathbb{P}^n \;\middle|\; \begin{array}{l} x_i = 1 \text{ and} \\ f(x_0, \dots, x_n) = 0 \end{array} \right\}$$

$$= \left\{ (x_0 : \dots : x_n) \in \mathbb{P}^n \;\middle|\; \begin{array}{l} x_i = 1 \\ \underset{\text{wrt } x_i}{f^h(x_0, \dots, x_n)} = 0 \end{array} \right\}$$

$$= Z(f^h) \cap D_+(x_i)$$

which is closed in $D_+(x_i)$ $\square$

Note that the complement of $D_+(x_i)$ is the closed subset $Z_+(x_i)$ which is a projective space $\mathbb{P}^{n-1}$. We may write

$$\mathbb{P}^n = \mathbb{A}^n \cup \mathbb{P}^{n-1}$$

Regular functions on projective varieties

$g, h \in k[x_0 .. x_n]_d$ homogeneous of the same degree d.

$$\frac{g(tx_0, .., tx_n)}{h(tx_0, .., tx_n)} = \frac{t^d g(x_0, .., x_n)}{t^d h(x_0, .., x_n)} = \frac{g(x_0, .., x_n)}{h(x_0, .., x_n)} \quad \forall t \in k^\times$$

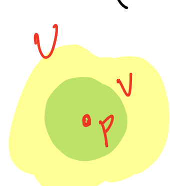
$\rightsquigarrow$ $f = \dfrac{g(x_0, .., x_n)}{h(x_0, .., x_n)}$ gives a <u>rational</u> function on $\mathbb{P}^n$

This is defined on $D_+(h) \subseteq \mathbb{P}^n$.

Defn Let $X \subseteq \mathbb{P}^n$ be a projective algebraic set.

The structure sheaf $\mathcal{O}_X$ is defined by

$$\mathcal{O}_X(U) = \left\{ f : U \longrightarrow k \;\middle|\; \begin{array}{l} f \text{ is continuous and} \\ \text{locally of the form} \\ \quad \dfrac{g(x_0, .., x_n)}{h(x_0, .., x_n)} \\ \text{where } g, h \text{ are homogeneous} \\ \text{of the same degree.} \end{array} \right\}$$



This means: For any $p \in U$, there is an open nbh $V \ni p$ so that $f = g/h$ on V and $h(x) \neq 0$ for all $x \in V$.

Elements of $\mathcal{O}_X(U)$ are called regular functions on U.

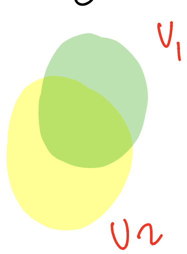
$\mathcal{O}_X$ is a presheaf, with ρ_{UV} = restriction of functions

Lemma $\mathcal{O}_X$ is a sheaf (of k-algebras)

Note that $\mathcal{O}_X(U)$ is a k-algebra: Given $f_1, f_2 \in \mathcal{O}_X(U)$ we have $f_1 + f_2$, $f_1 \cdot f_2$, $k \cdot f_1 \in \mathcal{O}_X(U)$, since we may for every $p \in X$ find an open set $U \subseteq X$ such that both f_1 and f_2 are of the form g/h.

Locality: $\mathcal{O}_X$ is a subpresheaf of the sheaf R_X of continuous k-valued functions $f: U \to k$.
$\therefore$ Locality holds for $\mathcal{O}_X$ since it holds for R_X.

Gluing: Given U and U_i as above, and elements $f_i \in \mathcal{O}_X(U_i)$ such that $f_i = f_j$ on $U_i \cap U_j$



$\rightsquigarrow$ f_i glue to a continuous map

$$f: U \longrightarrow k$$

By construction $f|_{U_i}$ is locally $\frac{g}{h}$, so f is an element of $\mathcal{O}_X(U)$.

$\therefore$ Any projective algebraic set $X = Z_+(I) \subseteq \mathbb{P}^n$ gives rise to a ringed space $(X, \mathcal{O}_X)$.

We want to show that this is a variety.

First $X = \mathbb{P}^n$:

Affine cover: $\mathbb{P}^n = D_+(x_0) \cup \dots \cup D(x_n)$

Let $A_i = Z(x_i - 1) \subset \mathbb{A}^{n+1}$ (so $A_i \cong \mathbb{A}^n$)

The quotient map $\pi : \mathbb{A}^{n+1} - 0 \longrightarrow \mathbb{P}^n$ gives the diagram

$$\begin{tikzcd} A_i \arrow[r, hook] \arrow[dr, "\pi|_{A_i}"'] & \pi^{-1} D_+(x_i) \arrow[r, hook] \arrow[d, "x_i \neq 0"] & \mathbb{A}^{n+1} - 0 \arrow[d, "\pi"] \\ & D_+(x_i) \arrow[r, hook] & \mathbb{P}^n \end{tikzcd}$$

Prop The projection $\pi|_{A_i}$ is an isomorphism (of ringed spaces)

The inverse is given by $\alpha : D_+(x_i) \longrightarrow A_i$

$$\alpha(x_0 : \dots : x_n) = \left(\frac{x_0}{x_i}, \dots, \frac{x_n}{x_i} \right)$$

We have already shown that $\pi_i = \pi|_{A_i}$ is a homeomorphism.

$\longrightarrow$ need to check that $\pi_i^* f \in \mathcal{O}_{A_i}(\pi_i^{-1} U)$ for every $f \in \mathcal{O}_{D_+(x_i)}(U)$.

Pick $V \subseteq U$ such that $f|_V = g/h$ with g, h homogeneous polynomials of the same degree.

$\longrightarrow$ On $\pi_i^{-1} V$ we have

$$\pi_i^* f = f \circ \pi = \frac{g(x_0, .., 1, -- x_n)}{h(x_0, --, 1, --, x_n)}$$

$$= \frac{g(x_0 -- x_n)}{h(x_0 -- x_n)} \Big|_{A_i}.$$

This is regular.

Since $\pi^{-1} U$ is covered by such $\pi^{-1} V \rightsquigarrow$ OK.

We also need to check that $\alpha : D_+(x_i) \longrightarrow A_i$

$$\alpha^* f \in \mathcal{O}_{D_+(x_i)}(\alpha^{-1} U) \quad \text{for all } f \in \mathcal{O}_{A_i}(U)$$

Let $V \subseteq U$ be an open so that $f = g/h$ there.

$$\rightsquigarrow \quad \alpha^* f = f \circ \alpha = \frac{g\left(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}\right)}{h\left(\frac{x_0}{x_i}, ..., \frac{x_n}{x_i}\right)}$$

x_0/x_i are regular on $D_+(x_i)$ and $h(x) \neq 0$ for all $x \in V$ $\longrightarrow$ $\alpha^* f$ also regular. $\square$

Since closed irreducible subsets of affine varieties are affine varieties, we get also

Prop $X \subseteq \mathbb{P}^n$ irreducible closed projective set

$\longrightarrow$ $V_i = D_+(x_i) \cap X$ (with $\mathcal{O}_{V_i} = \mathcal{O}_X|_{V_i}$)

is an affine variety.

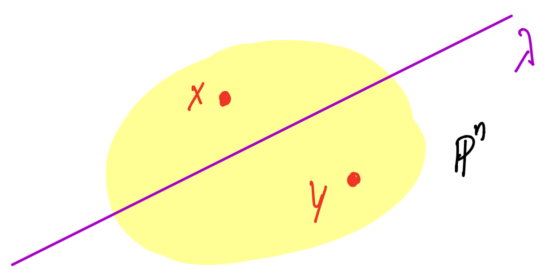
Theorem $X \subseteq \mathbb{P}^n$ irreducible, closed projective set.
$\rightsquigarrow$ $(X, \mathcal{O}_X)$ is a variety.

$\Leftarrow$ we call such varieties projective varieties

Ringed space ✓
Affine covering: $X = \bigcup_{i=0}^{n} D_+(x_i) \cap X$ ✓

Just need to check that X satisfies the Hausdorff axiom

Lemma The Hausdorff axiom holds provided the following condition holds:
For any two points $x, y \in X$ there is an affine open $U \subseteq X$ containing both x and y.



Now, given $x, y \in X$ $\rightsquigarrow$ $\exists$ linear form on $\mathbb{A}^{n+1}$
$\lambda = a_0 x_0 + \dots + a_n x_n$
such that $\lambda(x) \neq 0$
and $\lambda(y) \neq 0$

$\rightsquigarrow$ $x, y \in D_+(\lambda)$ which is affine

$\rightsquigarrow$ OK.

Global regular functions on projective varieties

For $X \subseteq \mathbb{P}^n$ a projective variety $\rightsquigarrow$ what are the global regular functions $f : X \longrightarrow k$?

Recall: If $X \subseteq \mathbb{A}^n$ is an affine variety, then the affine coordinate ring $A(X)$ gives us all such functions. $A(X)$ is big enough so that it in fact recovers X.

On the other hand, we will prove

Theorem Let $X \subseteq \mathbb{P}^n$ be a projective variety. Then

$$\mathcal{O}_X(X) = k$$

$\therefore$ The only global regular functions $X \to k$ are the constants!

Compare this to Liouville's theorem in complex analysis:

If $f : \mathbb{CP}^1 \longrightarrow \mathbb{C}$ is a holomorphic function

$\rightsquigarrow$ $f|_{\mathbb{C}} : \mathbb{C} \to \mathbb{C}$ is a bounded holomorphic function

$\rightsquigarrow$ f is constant.

Some notation:

$$S(X) = A(C(X))$$

the homogeneous coordinate ring

$C(X) \subseteq \mathbb{A}^{n+1}$

cone over X

(recall: If $X = Z_+(I)$ then $C(X) = Z(I) \subseteq \mathbb{A}^{n+1}$!)

This is a graded ring because X is defined by a homogeneous ideal.

ex $S(\mathbb{P}^n) = k[x_0, \dots, x_n]$

ex $C \subseteq \mathbb{P}^3$ twisted cubic $\Rightarrow$ $S(C) = \dfrac{k[x_0, x_1, x_2, x_3]}{(x_1^2 - x_0x_2,\ x_2^2 - x_1x_3,\ x_0x_3 - x_1x_2)}$

Regular functions on $\mathbb{P}^n$

$f : \mathbb{P}^n \longrightarrow k$

$$\begin{tikzcd} \mathbb{A}^{n+1} - 0 \arrow[d, "\pi"] \arrow[dr, bend left, "\pi^* f = f \circ \pi"] & \\ \mathbb{P}^n \arrow[r, "f"'] & k \end{tikzcd}$$

$\rightsquigarrow$ regular function $\pi^* f = f \circ \pi$ on $\mathbb{A}^{n+1} - 0$

$\rightsquigarrow$ $\pi^* f$ is a polynomial in $x_0, \dots, x_n$

But $\pi^* f$ is constant on lines through the origin

$\Rightarrow$ it must be constant!

Prop $\mathcal{O}_{\mathbb{P}^n}(\mathbb{P}^n) = k$

Now to prove the theorem in general:

Let $f : X \to k$ be a global regular function

$\rightsquigarrow$
$$\begin{tikzcd} C(X) - 0 \arrow[d, "\pi"] \arrow[dr, bend left, "\pi^* f =: F"] & \\ X \arrow[r, "f"] & k \end{tikzcd}$$

Suppose for simplicity that $X \subseteq \mathbb{P}^n$, so $C(X) \subseteq \mathbb{A}^{n+1}$.

We let $D_i = C(X) \cap D(x_i)$.

Each D_i is an affine variety, with affine coordinate ring
$$A(D_i) = S(X)_{x_i} \qquad \leftarrow \text{localization at } x_i$$

For each i, we may write the restriction $F|_{D_i}$ as
$$F|_{D_i} = \frac{g_i}{x_i^{r_i}}$$
where g_i is a homogeneous polynomial of degree r_i

F has degree 0!

because F is constant on lines through 0!

$\rightsquigarrow \quad x_i^{r_i} F = g_i$

$g_i \in S(X)_{r_i}$ (the r_i-th graded piece)

$\rightsquigarrow \quad h x_i^{r_i} F \in S(X)_{r_i + j}$ for all $h \in S(X)_j$.

Let $r > \sum r_i$:

$\rightsquigarrow$ any monomial $x_0^{a_0} \cdots x_n^{a_n}$ of degree r must contain one of the variables, say x_i, with exponent $a_i > r_i$.

$\longrightarrow$ $x_0^{a_0} \cdots x_n^{a_n} \cdot F \in S(X)_r$ $\forall$ monomial $x_0^{a_0} \cdots x_n^{a_n}$

$\rightsquigarrow$ multiplication by F $S(X) \xrightarrow{\cdot F} S(X)$ leaves the subspace $S(X)_r$ invariant.

Cayley-Hamilton

$\rightsquigarrow$ F satisfies a monic relation

$$F^m + a_{m-1} F^{m-1} + \cdots + a_1 F + a_0 = 0$$

where $a_i \in k$.

$\rightsquigarrow$ $F \in K(S(X))$ is algebraic over k $\rightsquigarrow$ $F \in k$ since $\overline{k} = k$. $\square$

Cor If $f : X \longrightarrow Y$ is a morphism
If X is projective and Y is affine
$\longrightarrow$ f is constant.

Morphisms $X \to Y$ with Y affine
$\longleftrightarrow$ k-alg homomorphisms $\mathcal{O}(Y) \longrightarrow \mathcal{O}_X(X) = k$

So all we get is the constant maps.

Cor If X is a variety which is both projective and affine, then X is a point.

X affine $\rightsquigarrow$ $\exists$ embedding $X \hookrightarrow \mathbb{A}^n$

X projective $\rightsquigarrow$ that embedding is constant.

$\rightsquigarrow$ X is a point.

quasi projective = open subvariety of a projective variety

Morphisms from quasiprojective varieties

Suppose we have a continuous map $\phi : X \longrightarrow Y$ which fits into a diagram

punctured cones

$$\begin{tikzcd} C_0(X) \arrow[r, "\Phi"] \arrow[d, "\pi_X"'] & C_0(Y) \arrow[d, "\pi_Y"] \\ X \arrow[r, "\phi"] & Y \end{tikzcd}$$

quotient map

$\leftarrow$ $C_0(X) = C(X) - 0$
$C_0(Y) = C(Y) - 0$

Lemma If Φ is a morphism (of quasiaffine varieties) then so is ϕ.

Being a morphism is a local property $\rightsquigarrow$ suffices to check that $\phi\big|_{D_+(x_i)}$ is a morphism.

Let $A_i \subset C_0(X)$ be the subvariety given by $x_i = 1$.

$\rightsquigarrow$ $\pi_X : A_i \longrightarrow D_+(x_i) \cap X$ is an isomorphism

Let β_i denote the inverse:

$$\begin{tikzcd} A_i \cap C_0(X) \arrow[r, hook, "\text{morphism}"] & C_0(X) \arrow[r, "\Phi"', "\text{morphism}"] \arrow[d] & C_0(Y) \arrow[d, "\text{morphism}"] \\ D_+(x_i) \cap X \arrow[u, "\beta_i", "\cong\ \text{Morphism}"'] \arrow[r] \arrow[rr, bend left] & X \arrow[r, "\phi"] & Y \end{tikzcd}$$

$\rightsquigarrow$ $D_+(x_i) \longrightarrow Y$ is also a morphism ✓

ex $X \hookrightarrow \mathbb{P}^n$

$f_0, \dots, f_m$ homogeneous of degree d.

s.t $Z(f_0, \dots, f_m) \cap X = \emptyset$

prototype example

$\rightsquigarrow$ morphism $X \xrightarrow{\phi} \mathbb{P}^m$

$x \longmapsto (f_0(x) : \dots : f_m(x))$

ex A projection is a rational map $\mathbb{P}^n \overset{p}{\dashrightarrow} \mathbb{P}^m$ $n > m$

induced from $\mathbb{A}^{n+1} \longrightarrow \mathbb{A}^{m+1}$

$(x_0, \dots x_n) \mapsto (x_0, \dots, x_m)$

p is a morphism outside the linear $\mathbb{P}^{n-m-1} \subset \mathbb{P}^n$ given by $x_0 = \dots = x_m = 0$.

Special case: projection from the point $(0 : \dots : 0 : 1) \in \mathbb{P}^n$

$p : \mathbb{P}^n \dashrightarrow \mathbb{P}^{n-1}$

ex $\mathbb{P}^1 \xrightarrow{\phi} \mathbb{P}^3$

$(u:v) \longmapsto (u^3 : u^2v : uv^2 : v^3)$

$C =$ the image of ϕ $= Z_+(I)$ where $I =$ 2×2-minors of $\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$

Consider the projection from $(0:0:0:1)$:

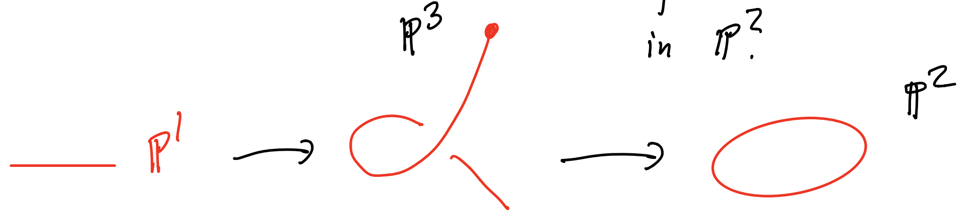
$\mathbb{P}^1 \to C \hookrightarrow \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$ not defined at $u=0$!

$(u:v) \to (u^3 : u^2v : uv^2 : v^3) \mapsto (u^3 : u^2v : uv^2)$

$(u^2 : uv : v^2)$

However, the rational map extends over $(0:1)\in\mathbb{P}^1$ by defining

$(u:v) \mapsto (u^2:uv:v^2)$ $\quad\therefore$ The projection of the twisted cubic equals the conic $Z(x_1^2-x_0x_2)$ in $\mathbb{P}^2$.



If we instead project from the point $(0:0:1:0)$:

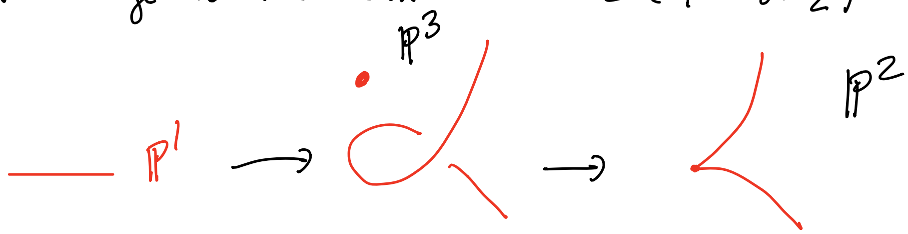
this gives

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^3 \dashrightarrow \mathbb{P}^2$$

$$(u:v) \longmapsto (u^3:u^2v:v^3)$$

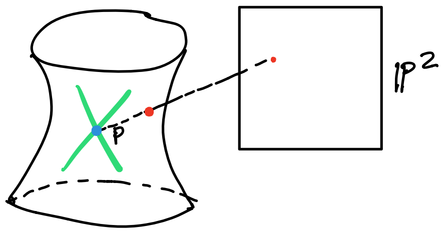
This is in fact a morphism $\phi:\mathbb{P}^1 \to \mathbb{P}^2$

The image is the curve $C = Z(x_1^3 - x_0^2x_2) \subset \mathbb{P}^2$



ex $Q = Z(x_0x_3 - x_1x_2) \subset \mathbb{P}^3$

projection from $p = (0:0:0:1) \in Q$



$$\begin{aligned} \mathbb{P}^3 &\dashrightarrow \mathbb{P}^2 \\ (x_0:x_1:x_2:x_3) &\mapsto (x_0:x_1:x_2) \end{aligned}$$

Restrict to $U = D_+(x_3)$: Then $U \cap Q \subset \mathbb{A}^3_{u,v,w}$ is given by $u = vw$.

and $Q \cap U \simeq \mathbb{A}^2$ via $\begin{aligned} \mathbb{A}^2 &\to Q \subset \mathbb{A}^3 \\ (v,w) &\mapsto (vw, v, w) \end{aligned}$

The projection is given by

$$\begin{aligned} \mathbb{A}^2 \hookrightarrow Q \hookrightarrow \mathbb{P}^3 &\dashrightarrow \mathbb{P}^2 \\ (v,w) &\longmapsto (vw : v : w) \end{aligned}$$

this is defined on $\mathbb{A}^2 - \{(0,0)\}$.

$\ell_1 = Z(v) \subset \mathbb{A}^2 - 0$ get mapped to $(0:0:1)$ respectively.
$\ell_2 = Z(w) \subset \mathbb{A}^2 - 0$ get mapped to $(0:1:0)$

and the projection is an isomorphism outside $\ell_1 \cup \ell_2$.

The remaining points : $(x_3 = 0)$

$$\begin{aligned} Q \cap Z(x_3) &= Z(x_3, x_0x_3 - x_1x_2) = Z(x_3, x_1x_2) \\ &= Z(x_1, x_3) \cup Z(x_2, x_3) \end{aligned}$$

$Z(x_1, x_3)$ $\mathbb{P}^3 \dashrightarrow \mathbb{P}^2$

$(x_0 : 0 : x_2 : 0) \mapsto (x_0 : 0 : x_2)$ is an isomorphism

$Z(x_1, x_2)$: $(x_0 : x_1 : 0 : 0) \mapsto (x_0 : x_1 : 0)$ —— 11 ——

$\therefore$ The projection $p : Q \dashrightarrow \mathbb{P}^2$
collapses the two lines $l_1 = Z(x_0, x_1)$ to points
$l_2 = Z(x_0, x_2)$
and is an isomorphism outside $l_1 \cup l_2$.

Q and $\mathbb{P}^2$ are "birational" (there are mutually inverse rational maps in both directions – more on this later!)