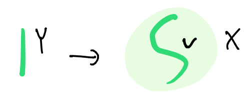
Chapter 5 : Segre and Veronese varieties



Defn $\phi : Y \to X$ is a closed embedding if the image $V = \phi(Y)$ is closed in X and ϕ induces an isomorphism $\phi : Y \xrightarrow{\sim} V \subset X$.

V has a canonical variety structure, being a closed subset of Y.

ex The morphism
$$\begin{aligned} \mathbb{A}^1 &\longrightarrow \mathbb{A}^3 \\ t &\longmapsto (t, t^2, t^3) \end{aligned}$$
is a closed embedding onto the twisted cubic $C \subset \mathbb{A}^3$.

Being an embedding is a local property on the target X:

Lemma If $\phi : Y \to X$ is a morphism, and $\mathcal{U} = \{U_i\}$ is a collection of opens covering $\phi(Y)$. Then

$$\phi \text{ is an embedding} \iff \phi\big|_{\phi^{-1}(U_i)} : \phi^{-1}(U_i) \to U_i$$
is an embedding for all i.

"$\Rightarrow$" ϕ an embedding $\Rightarrow$ $\phi\big|_{\phi^{-1}(U_i)}$ embedding
(image is closed, and restriction of isomorphism is isomorphism)

"$\Leftarrow$" Given ϕ such that each restriction is an embedding:

ϕ injective ✓

ϕ closed $\left(\phi(Z) \text{ is closed iff } \underset{\substack{\| \\ \phi(\phi^{-1}U_i \cap Z)}}{\phi(Z) \cap U_i} \text{ closed} \right)$

$\rightsquigarrow$ ϕ is a homeomorphism onto $\phi(Y)$

Need to show $\left(\phi \big|_{\phi(Y)}\right)^{-1}$ is a morphism.

Let $U \subseteq Y$ be open and $f \in \mathcal{O}_Y(U)$.

Need to check that $f \circ \phi^{-1}\big|_{\phi(U)}$ is regular on $\phi(U)$.

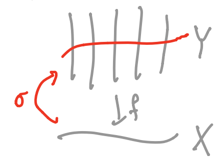
Being regular is a local property $\rightsquigarrow$ wlog $U = \phi^{-1}(U_i) \cap U$

In that case, $f \circ \phi^{-1}\big|_{\phi(U)}$ is regular because

$\phi\big|_{\phi^{-1}(U_i)}$ is an isomorphism so that $\phi^{-1}\big|_{\phi(U)}$ is a morphism

$\square$

Lemma Assume that $\phi : \mathbb{A}^n \longrightarrow \mathbb{A}^{n+m}$ has a left section which is a projection. Then ϕ is a closed embedding.



$\sigma : X \to Y$ is a left-section of $f : Y \to X$ if

$$X \underset{\sigma}{\overset{f}{\rightleftarrows}} Y \qquad \sigma \circ f = id_X$$

Suppose $\sigma : \mathbb{A}^{m+n} \longrightarrow \mathbb{A}^n$ is the section

$$(z_1, \dots, z_{m+n}) \mapsto (z_1, \dots, z_n)$$

The composition $\mathbb{A}^n \xrightarrow{\phi} \mathbb{A}^{m+n} \xrightarrow{\sigma} \mathbb{A}^n$ is the identity

$\Rightarrow$ ϕ is bijective onto its image $V = \phi(\mathbb{A}^n)$.

$\sigma|_V : V \to \mathbb{A}^n$ is an inverse to ϕ $\rightsquigarrow$ ϕ is an embedding.

very useful

Rational normal curves

Affine version

$$\begin{aligned} \phi : \mathbb{A}^1 &\longrightarrow \mathbb{A}^d \\ t &\longmapsto (t, t^2, \dots, t^d) \end{aligned}$$

The image C is given by the equations $z_i = z_1^i \qquad i = 2 .. d$

A left section of ϕ is given by $\mathbb{A}^d \longrightarrow \mathbb{A}^1 \quad (z_1 .. z_d) \mapsto z_1$

$\leadsto$ ϕ is an embedding.

Projective version

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\ \phi\ } \mathbb{P}^d \\ (x_0 : x_1) &\longmapsto (x_0^d : x_0^{d-1} x_1 : \dots : x_1^d) \end{aligned}$$

Prop ϕ is a closed embedding.

Let $U_0 = D_+(u_0)$. Then

$$\phi^{-1}(U_0) = D_+(x_0) \subseteq \mathbb{P}^1$$

The restriction $\phi|_{\phi^{-1}(U_0)}$ is given by

$$\begin{tikzcd} \mathbb{A}^1 \arrow[r, "\phi|_{D_+(x_0)}"] \arrow[d, "D_+(x_0)"] & \mathbb{A}^d \arrow[d, "D_+(u_0)"] \\ \mathbb{P}^1 \arrow[r, "\phi"] & \mathbb{P}^d \end{tikzcd}$$

$t \longrightarrow (t, t^2, \dots, t^d)$

This is the affine RNC

$\Longrightarrow$ $\phi|_{D_+(x_0)}$ is an embedding

A symmetric argument shows that $\phi|_{D_+(x_1)}$ is an embedding.

These two open sets cover $\mathbb{P}^1$ $\Longrightarrow$ ϕ is an embedding. $\blacksquare$

ex For $d=2$, this is $\mathbb{P}^1 \to \mathbb{P}^2$ $(x_0 : x_1) \mapsto (x_0^2 : x_0x_1 : x_1^2)$.
The image is the conic $C = Z(u_1^2 - u_0u_2)$ this is irreducible

ex For $d=3$, this is $\mathbb{P}^1 \to \mathbb{P}^3$ $(x_0 : x_1) \mapsto (x_0^3 : x_0^2x_1 : x_0x_1^2 : x_1^3)$
The image is the twisted cubic $Z_+(I) \subset \mathbb{P}^3$ where
$I = (u_1^2 - u_0u_2,\ u_0u_3 - u_1u_2,\ u_2^2 - u_1u_3)$.

Using the substitutions
$u_1^2 \mapsto u_0u_2$
$u_1u_2 \mapsto u_0u_3$ $\rightsquigarrow$ may write any $f \in k[u_0,..,u_3]$ modulo I as
$u_2^2 \mapsto u_1u_3$

$$f(u_0,u_1,u_2,u_3) = \underset{d}{a}(u_0,u_3) + \underset{d-1}{b}(u_0,u_3)\,u_1 + \underset{d-1}{c}(u_0,u_3)\,u_2$$

If $f \in I(C)$, then modulo I:
$$f(s^3, s^2t, st^2, t^3) = a(s^3,t^3) + b(s^3,t^3)\,s^2t + c(s^3,t^3)\,st^2 \equiv 0$$

exponents of t mod 3 : 0 1 2

$\rightsquigarrow$ if this is $\equiv 0$, then a, b, c are 0.
$\rightsquigarrow$ $f \in I$. $\square$

Prop

on $\mathbb{P}^d$

If $u_0, .., u_d$ are projective coordinates, the ideal of the image $C := \phi(\mathbb{P}^1)$ is the ideal I generated by the 2×2 minors of the matrix

2x2 determinants

$$\begin{bmatrix} u_0 & u_1 & u_2 & \cdots & u_{d-1} \\ u_1 & u_2 & u_3 & \cdots & u_d \end{bmatrix}$$

Note that all the 2×2 minors of

$$\begin{bmatrix} x_0^d & x_0^{d-1}x_1 & x_0^{d-2}x_1^2 & \cdots & x_0x_1^{d-1} \\ x_0^{d-1}x_1 & x_0^{d-2}x_1^2 & x_0^{d-3}x_1^3 & \cdots & x_1^d \end{bmatrix}$$

are zero. Hence $I \subseteq I(C)$.

$\supseteq$: Requires an algebraic computation similar to the above. See the notes.

The Segre embedding

$m, n \geq 1$

$$\sigma_{m,n} : \mathbb{P}^m \times \mathbb{P}^n \longrightarrow \mathbb{P}^{(m+1)(n+1)-1}$$

$$(x_0 : \cdots : x_m) \times (y_0 : \cdots : y_n) \longmapsto (x_0 y_0 : x_0 y_1 : \cdots : x_i y_j : \cdots : x_m y_n)$$

$\uparrow$ lexicographic ordering

Write u_{ij} $i = 0 \ldots m$, $j = 0 \ldots n$ on $\mathbb{P}^{mn+m+n}$

$$M = \begin{pmatrix} u_{00} & u_{01} & \cdots & u_{0n} \\ \vdots & & & \vdots \\ u_{m0} & u_{m1} & \cdots & u_{mn} \end{pmatrix}.$$

rank 1

The matrix $(x_0, \ldots, x_m)^t (y_0, \ldots, y_n) = \begin{pmatrix} x_0 y_0 & x_0 y_1 & \cdots & x_0 y_n \\ \vdots & \ddots & & \vdots \\ x_m y_0 & \cdots & & x_m y_n \end{pmatrix}$

has rank ≤ 1 for all $x_i, y_j \in k$

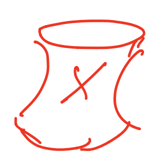
$\rightsquigarrow$ all the 2×2 minors vanish

$\rightsquigarrow$ the image of $\sigma_{m,n}$ is contained in $Z_+(I)$ where I is the ideal generated by the 2×2 - minors of M.

ex $\quad \mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{\sigma_{1,1}} \mathbb{P}^3$

$$(x_0 : x_1) \times (y_0 : y_1) \longmapsto (x_0 y_0 : x_0 y_1 : x_1 y_0 : x_1 y_1) =: (u_0 : u_1 : u_2 : u_3)$$

$\rightsquigarrow$ image $= Z_+(u_0 u_3 - u_1 u_2)$



Prop The Segre map $\sigma_{m,n}$ is a closed embedding. The image equals $Z_+(I)$ where I is the ideal generated by the 2×2-minors of M.

For $0 \leq s \leq m$, $0 \leq t \leq n$ consider the open sets

$$U = D_+(x_s) \times D_+(x_t) \subset \mathbb{P}^m \times \mathbb{P}^n$$

$$D = D_+(u_{st}) \quad \subset \mathbb{P}^{mn+m+n}$$

Note that $\sigma^{-1}(D) = U$

$\rightsquigarrow$ so suffices to show that $\phi|_D$ is a closed embedding.

We have $U \simeq \mathbb{A}^m \times \mathbb{A}^n$ $\left(\text{coordinates } \frac{x_i}{x_s}, \frac{y_j}{y_t}\right)$ and $U \cong \mathbb{A}^{mn+m+n}$ $\left(\text{coordinates } \frac{u_{ij}}{u_{st}}\right)$

The corresponding morphism

$$\mathbb{A}^m \times \mathbb{A}^n \longrightarrow \mathbb{A}^{mn+m+n}$$

$$\left(\frac{x_i}{x_s}\right)\left(\frac{y_j}{y_t}\right) \longmapsto \left(\frac{x_i y_j}{x_s y_t}\right) = \left(\frac{u_{ij}}{u_{st}}\right)$$

If we set $i=s$, we recover $\frac{y_j}{y_t} = \frac{u_{sj}}{u_{st}}$.

If we set $j=t$, we recover $\frac{x_i}{x_s} = \frac{u_{it}}{u_{st}}$

$\therefore$ We get a left section $\mathbb{A}^{mn+m+n} \xrightarrow{\psi} \mathbb{A}^m \times \mathbb{A}^n$

from the two morphisms

$$\mathbb{A}^{mn+m+n} \longrightarrow \mathbb{A}^m \qquad \frac{u_{ij}}{u_{st}} \mapsto \begin{cases} \frac{x_i}{x_s} & \text{if } j=t \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{A}^{mn+m+n} \longrightarrow \mathbb{A}^{m} \qquad \frac{u_{ij}}{u_{st}} \mapsto \begin{cases} \frac{y_j}{y_t} & \text{if } i=s \\ 0 & \text{otherwise} \end{cases}$$

ψ is a projection $\Rightarrow$ σ is an embedding ✓

The image equals $Z(I)$:

Let $v \in Z(I)$ and suppose wlog that $v \in D_+(u_{00})$

so that $v = (1 : v_{01} : \cdots : v_{mn}) \in Z(I)$

(Changing order of rows / colums of M does not change I)

Let

$$m = \begin{pmatrix} 1 & * & * & * \\ * & - & - & - \\ * & - & - & - \end{pmatrix}$$

denote the matrix M with $u_{ij} \doteq v_{ij}$ substituted for the entries (chosen so that $v_{00}=1$)

$$\rightsquigarrow m = \begin{pmatrix} 1 & v_{01} & v_{02} & \cdots & v_{0n} \\ v_{10} & v_{01}v_{10} & \cdots & & \vdots \\ v_{20} & v_{01}v_{20} & \cdots & & \\ \vdots & & & & \\ v_{m0} & v_{01}v_{m0} & \cdots & & v_{m0}v_{0n} \end{pmatrix}$$

(since the rank $= 1$)

Define $x_i = v_{i0}$

$y_j = v_{0j}$ $\rightsquigarrow$ $x_i y_j = v_{i0} v_{0j}$

$\rightsquigarrow$ v is in the image of σ.

In fact, the minors generate all the relations, so

$$I(\mathbb{P}^m \times \mathbb{P}^n) = I.$$

Cor The product of two projective varieties is projective.

$X \subset \mathbb{P}^m$ $\rightsquigarrow$ $X \times Y \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n$ is a closed subset

$Y \subset \mathbb{P}^n$

$\rightsquigarrow$ $X \times Y \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n$ is a closed subvariety.

$\rightsquigarrow$ $X \times Y \hookrightarrow \mathbb{P}^m \times \mathbb{P}^n \hookrightarrow \mathbb{P}^{mn+m+n}$

is a closed embedding, so $X \times Y$ is projective.

A Nullstellensatz for $\mathbb{P}^m \times \mathbb{P}^n$

$\mathbb{Z}^2$-grading

Let $R = k[x_0 \dots x_m; y_0 \dots y_n]$.

R has a bigrading given by $\deg x_i = (1,0)$
$\deg y_j = (0,1)$

We say $F \in R$ is bihomogeneous, if $F \in R_{a,b}$ for some a,b.
This is $\iff$ $F(sx, ty) = s^a t^b F(x,y)$.

If $\mathfrak{a}$ is a bihomogeneous ideal, we get an algebraic set

$$Z_+(\mathfrak{a}) \subseteq \mathbb{P}^m \times \mathbb{P}^n$$

Conversely, for any subset $X \subset \mathbb{P}^m \times \mathbb{P}^n$ we get an ideal in R

$$I(X) = \left\langle f \in R \;\middle|\; \begin{array}{l} f \text{ bihomogeneous} \\ f(x) = 0 \quad \forall x \in X \end{array} \right\rangle$$

Note: The ideals

$$\mathfrak{m}_+ = (x_0, \dots, x_m)$$
$$\mathfrak{n}_+ = (y_0, \dots, y_n)$$

both define the empty set.
We call them the irrelevant ideals of R.

Nullstellensatz for $\mathbb{P}^m \times \mathbb{P}^n$

The assignments

$$\mathfrak{a} \longmapsto Z_+(\mathfrak{a})$$
$$Z \longmapsto I(Z)$$

give a 1-1 correspondence

$$\left\{ \begin{array}{l} \text{closed algebraic} \\ \text{subsets of } \mathbb{P}^m \times \mathbb{P}^n \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{l} \text{radical bihomogeneous} \\ \text{ideals } \mathfrak{a} \subseteq R \\ \text{not containing } \mathfrak{m}_+ \cap \mathfrak{n}_+ \end{array} \right\}$$

ex The polynomial $F = u_0 X_1 - u_1 X_0$ defines a closed algebraic set $Z(F) \subset \mathbb{P}^1 \times \mathbb{P}^2$.

$(u_0 X_1 - u_1 X_0)$ prime $\Rightarrow$ $Z(F)$ is irreducible $\quad$ ($u_0\ u_1 \quad x_0\ x_1\ x_2$) $\quad \leftarrow$ "the blow-up of $\mathbb{P}^2$ at $(0:0:1)$"

Can also verify this locally: In the $u_0 = 1$ chart, we have $F \cap D_+(x_0)$ isomorphic to the affine hypersurface

$$Z(X_1 - u X_0) \subset \mathbb{A}^3.$$

$X_1 - u X_0$ is irreducible $\Longrightarrow$ ok

Same holds by symmetry in the $u_1 = 1$ chart. $\Rightarrow Z(F)$ irreducible.

Moreover, $Z(F)$ has dimension 2, as $Z(X_1 - u X_0) \subseteq \mathbb{A}^2$ has dimension 2.

The Veronese embedding

$d > 0$

$$\phi : \mathbb{P}^n \longrightarrow \mathbb{P}^N \qquad N = \binom{n+d}{d} - 1$$

$$(x_0 : \cdots : x_n) \longmapsto \left(x_0^{i_0} \cdots x_n^{i_n} \right)_{i_0 + \cdots + i_n = d}$$

all monomials of degree d in lexicographic order.

ex $n=1$:

$$\mathbb{P}^1 \longrightarrow \mathbb{P}^{\binom{d+1}{1}-1}$$

$$(x_0 : x_1) \longmapsto (x_0^d : x_0^{d-1} x_1 : \cdots : x_1^d)$$

This is exactly the rational normal curve (which we know is an embedding)

ex $n = d = 2$

$$\mathbb{P}^2 \xrightarrow{\phi} \mathbb{P}^5$$

$$(x_0 : x_1 : x_2) \longmapsto (x_0^2 : x_0 x_1 : x_0 x_2 : x_1^2 : x_1 x_2 : x_2^2)$$

The image is the Veronese surface.

Notation

$R = k[x_0, \dots, x_n]$ with the usual grading

$\mathcal{I}$ = the set of multiindexes $I = (a_0, \dots, a_n)$ s.t $\sum a_i = d$ and $a_i \geq 0$. ← $N+1$ of these

$I \in \mathcal{I} \rightsquigarrow$ monomial $M_I = x_0^{a_0} \cdots x_n^{a_n}$ of degree d.
and a homogeneous coordinate u_I on $\mathbb{P}^N$.

$\mathbb{A}^{N+1}$ = affine space with underlying k-vector space R_d

For $F \in R_d \rightsquigarrow$ can express $F = \sum a_I \cdot M_I$.

$$\begin{aligned} \phi : \mathbb{P}^n &\longrightarrow \mathbb{P}^N \\ [x] &\longmapsto (m_I(x)) \end{aligned}$$

ϕ is well-defined : all m_I have the same degree

ϕ is a morphism : the m_I do not simultaneously vanish anywhere.

Prop The Veronese morphism $\phi : \mathbb{P}^n \longrightarrow \mathbb{P}^N$ is a closed embedding.

Consider $U = D_+(x_0) \subset \mathbb{P}^n$

Here the morphism is

$$\begin{array}{ccc} \mathbb{A}^n & \longrightarrow & D_+(m_{(d,0\dots0)}) \simeq \mathbb{A}^N \\ (z_1 .. z_n) & \longmapsto & (z_1, z_2, \dots, z_n, z_1^2, z_1 z_2, \dots, z_n^d) \end{array}$$

all monomials of degree $\leq d$

This is an embedding, because projection onto the first n coordinates gives a left section.

By symmetry, this holds also for the subsets $D_+(m_{(0,.0,d,0,..0)})$

These form an open cover of $\phi(\mathbb{A}^n)$ $\rightsquigarrow$ DONE $\square$

Prop The homogeneous ideal of the image X of the Veronese embedding is generated by the quadrics

$$m_I m_J - m_K m_L$$

for all multiindices I, J, K, L s.t $I + J = K + L$.

For the proof, see the Notes.

Cor Let $f(x_0, \dots, x_n)$ be a non-zero homogeneous polynomial. Then

!

$$D_+(f) = \mathbb{P}^n - Z_+(f)$$ ← know this is ok when degree $f = 1$.

is an affine variety.

Let $d = \deg f$ and consider the Veronese embedding

$$\mathbb{P}^n \xrightarrow{\phi} \mathbb{P}^{\binom{n+d}{d}-1}$$

If $f = \sum a_I m_I \in k[x_0, \dots, x_n]$, then

$$Z_+(f) = \phi^{-1}(Z(L))$$ where $L = \sum a_I u_I$ is a linear polynomial in u_I

Note that $\mathbb{P}^N - Z(L) \simeq D_+(L) \cong \mathbb{A}^N$ is affine.

We have

$$D_+(f) = \mathbb{P}^n - Z(f) = \phi^{-1}(\mathbb{P}^N - Z(L))$$

so $D_+(f)$ is also affine. $\square$

Cor $X \subseteq \mathbb{P}^n$ a subvariety which is not a point

↑ or more generally algebraic set..

If $f \underset{\neq 0}{\in} k[x_0, \dots, x_n]$ is homogeneous

then $Z_+(f) \cap X \neq \emptyset$

"projective varieties of positive dimension intersect any hypersurface"

If $Z_+(f) \cap X = \emptyset$, then
$X \subseteq D_+(f)$.
Hence X is a closed subvariety of an affine variety $\Rightarrow$ X is itself affine.
However, if X is both affine and projective, then it is a point $\square$

Note that this is not true for $\mathbb{A}^n$

$\rightsquigarrow$ projective varieties are better!

Conics and the Veronese surface

Recall that a conic is a curve $C \subset \mathbb{P}^2$ defined by a quadratic form in x_0, x_1, x_2

$$q = a_{00}x_0^2 + a_{01}x_0x_1 + a_{02}x_0x_2 + a_{11}x_1^2 + a_{12}x_1x_2 + a_{22}x_2^2 \qquad (*)$$

Let $\mathbb{P}^5$ denote the projective space with homogeneous coordinates

$$u_{00}, u_{01}, u_{02}, u_{11}, u_{12}, u_{22}.$$

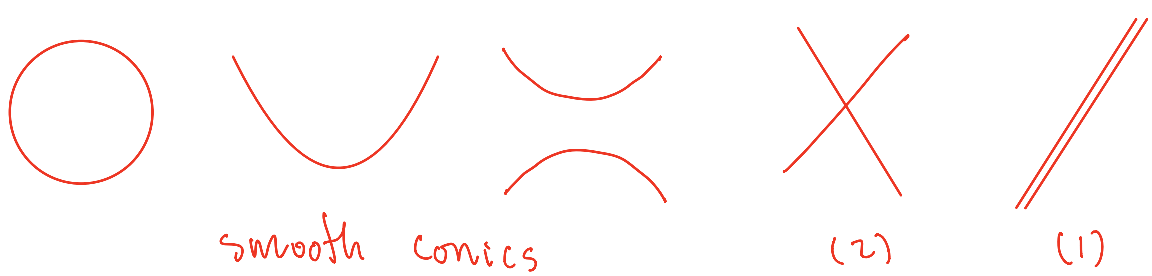
We may then view $\mathbb{P}^5$ as the variety parameterizing conics

That is, a point $(a_{00} : a_{01} : a_{02} : a_{11} : a_{12} : a_{22}) \in \mathbb{P}^5$ corresponds to the conic $(*)$ (and the correspondence is clearly 1-1 if we note that scaling $(*)$ does not change C).

Note that some q's above give degenerate conics, i.e

(1) A double line $(a_0x_0 + a_1x_1 + a_2x_2)^2$

(2) A union of two lines $(a_0x_0 + a_1x_1 + a_2x_2)(b_0x_0 + b_1x_1 + b_2x_2)$



These two give rise to subvarieties of the $\mathbb{P}^5$:

(1) Note that

$$(a_0x_0+a_1x_1+a_2x_2)^2 = a_0^2\,x_0^2 + 2a_0a_1\,x_0x_1 + 2a_0a_2\,x_0x_2 + a_1^2\,x_1^2 + 2a_1a_2\,x_1x_2 + a_2^2\,x_2^2.$$

Hence the subvariety of $\mathbb{P}^5$ corresponding to conics (1) is parameterized by

$$\begin{aligned} \mathbb{P}^2 &\xrightarrow{\ \phi\ } \mathbb{P}^5 \\ (a_0 : a_1 : a_2) &\longmapsto (a_0^2 : 2a_0a_1 : 2a_0a_2 : a_1^2 : 2a_1a_2 : a_2^2) \end{aligned}$$

Up to a linear change of variables, this is exactly the Veronese surface! this requires char $k \neq 2$

$\therefore$ The Veronese surface is the algebraic variety which parameterizes double lines in $\mathbb{P}^2$.

(2) The conics which are unions of lines are determined by

$$(a_0x_0 + a_1x_1 + a_2x_2)(b_0x_0 + b_1x_1 + b_2x_2) = (a_0 b_0)\, x_0^2 + (a_0b_1 + a_1b_0)\, x_0x_1 + (a_0b_2 + a_2b_0)\, x_0x_2 + (a_1 b_1)\, x_1^2 + (a_1b_2 + a_2b_1)\, x_1x_2 + (a_2b_2)\, x_2^2$$

We thus consider the map

$$\begin{aligned} \mathbb{P}^2 \times \mathbb{P}^2 &\longrightarrow \mathbb{P}^5 \\ [a] \times [b] &\longmapsto [a_0b_0 : a_0b_1 + a_1b_0 : \cdots : a_2b_2] \end{aligned}$$

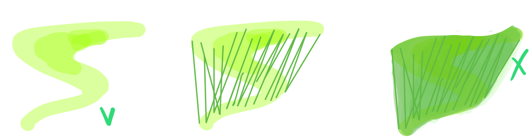
The image is a cubic hypersurface $X = Z_+(F) \subset \mathbb{P}^5$.

Here
$$F = \det \begin{pmatrix} u_{00} & u_{01} & u_{02} \\ u_{10} & u_{11} & u_{12} \\ u_{20} & u_{21} & u_{22} \end{pmatrix}$$

Note that X contains the Veronese surface $V \subset \mathbb{P}^5$.
In fact the two are very closely related:

- X is singular, and $V = \mathrm{Sing}(X)$ (more on this later)
- V is given by the 2×2 minors of the 3×3 matrix defining X.
- X is the secant variety of V.

  In other words,
$$X = \overline{\bigcup_{p \neq q \in V} L_{pq}}$$
  L_{pq} = line through p and q



The geometry of spaces of polynomials

Let $\mathbb{P}^d$ = projective space on the vector space V with basis $\{1, x, \dots, x^d\}$

$\therefore$ $(a_0 : \dots : a_d) \in \mathbb{P}^d \longleftrightarrow a_0 + a_1 x + \dots + a_d x^d$ modulo scalars

λ a partition of d (i.e. $\lambda = (\lambda_1, \dots, \lambda_r)$ $\sum \lambda_i = d$)

$\rightsquigarrow V_\lambda = \{ p(x) \mid p$ has r roots $x_1 \dots x_r$ with multiplicities $\lambda_1 \dots \lambda_r \}$

This is the image of the "multiplication map"

$$\mathbb{P}^1 \times \dots \times \mathbb{P}^1 \longrightarrow \mathbb{P}^d$$
$$(u_1 : v_1) \times \dots \times (u_r : v_r) \longmapsto (u_1 x + v_1)^{\lambda_1} \cdots (u_r x + v_r)^{\lambda_r}$$

Hence V_λ is irreducible $\Rightarrow$ V_λ is a projective variety.

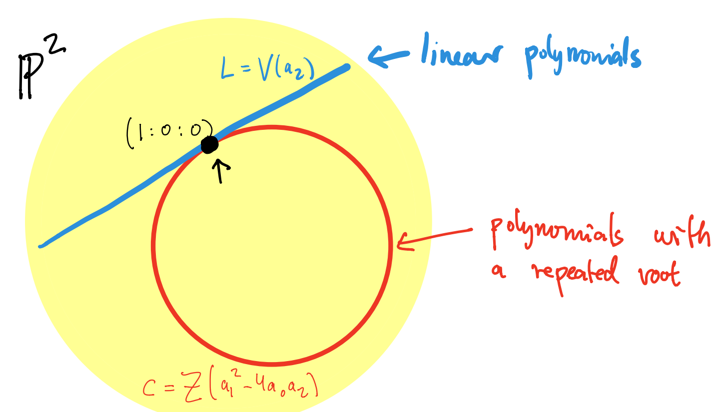
Degree 2

$a_0 + a_1 x + a_2 x^2 \longleftrightarrow a = (a_0 : a_1 : a_2) \in \mathbb{P}^2$

$a_0 + a_1 x \longleftrightarrow (a_0 : a_1 : 0) \rightsquigarrow$ a $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

$a_0 + a_1 x + a_2 x^2$ has a repeated root $\Leftrightarrow$ $a \in Z(a_1^2 - 4a_0 a_2)$

Constants

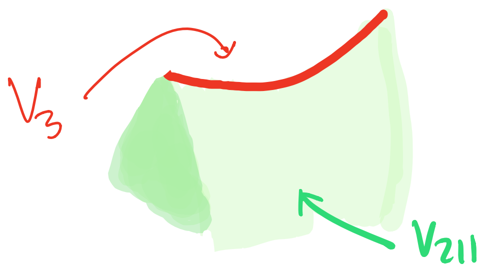


Degree 3

$p(x) = ax^3 + bx^2 + cx + d \quad \longleftrightarrow \quad p = (a:b:c:d) \in \mathbb{P}^3$

$p(x)$ has a repeated root $\iff$ the discriminant

$$a_2^2 a_1^2 - 4 a_3 a_1^3 - 4 a_2^3 a_0 - 27 a_3^2 a_0^2 + 18 a_0 a_1 a_2 a_3 = 0$$



$p(x)$ has a triple root $\iff$ $p(x) = (cx+d)^3$ for $c, d \in k$

$\iff a \in (c^3 : 3c^2 d : 3cd^2 : d^3)$

$\iff$ a lies on the twisted cubic C

$C = Z(I)$

I = 2x2 minors of $\begin{pmatrix} 3u_0 & u_1 & u_2 \\ u_1 & u_2 & 3u_3 \end{pmatrix}$

Degree 4

$$p(x) = ax^4 + bx^3 + cx^2 + dx + e \leftrightarrow (a:b:c:d:e) \in \mathbb{P}^4$$

We have 4 partitions of 4 : expected dimension :

1 1 1 1	(all roots distinct)	4
2 1	double root)	3
2 2	double roots)	2
3 1	triple root)	2
4	quadruple root)	1

V_{21}

p(x) has a double root $\Longleftrightarrow$ $p(x) = (ux+v)^2 (w_2 x^2 + w_1 x + w_0)$

$$= x^4 (u^2 w_2) + (u^2 w_1 + 2uvw_2) x^3 + (u^2 w_0 + 2uvw_1 + v^2 w_2) x^2 + (2uvw_0 + v^2 w_1) x + v^2 w_0$$

$\Longleftrightarrow$ p lies on the image of

$$\mathbb{P}^1 \times \mathbb{P}^2 \dashrightarrow \mathbb{P}^4$$

$$(u,v) \times (w_0 : w_1 : w_2) \mapsto (u^2 w_2 : u^2 w_1 + 2uvw_2 : \ldots)$$

$\Longleftrightarrow$ $p \in Z(\Delta)$ where

$$\Delta = 256 a^3 e^3 - 192 a^2 bde^2 + \ldots$$

(huge polynomial)

V_{31}

$p(x)$ has a triple root $\Leftrightarrow$ $p(x) = (ux+v)^3 (w_1 x + w_0)$

$\Leftrightarrow$ p lies on the image of
$\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^4$
$(u:v) \times (w_0 : w_1) \mapsto (u^3 w_1 : u^3 w_0 + 3u^2 w_1 : \ldots)$

$\Leftrightarrow$ $p \in Z(q, c)$
$p = 12ae - 3bd + c^2$
$c = 27ad^2 + 27b^2 e - 27bdc + 8c^3$

V_{22}

$p(x)$ has two repeated roots $\Leftrightarrow$ $p(x) = (u_1 x + u_0)^2 (v_1 x + v_0)^2$

$= x^4 (u_1^2 v_1^2) + x^3 (2u_1^2 v_0 v_1 + 2u_0 u_1 v_1^2)$

$+ x^2 (u_1^2 v_0^2 + 4u_0 u_1 v_0 v_1 + u_0^2 v_1^2)$

$+ x(2u_0 u_1 v_0^2 + 2u_0^2 v_0 v_1) + u_0^2 v_0^2$

$\Leftrightarrow$ p is in the image of
$\mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^4$
$(u_0 : u_1) \times (v_0 : v_1) \mapsto (u_1^2 v_1^2 : \ldots)$

$\Leftrightarrow$ $p \in Z(I)$
I = ideal of 5 cubics !

Rmk $Z(I)$ is the projected Veronese surface in $\mathbb{P}^4$

V_4

$p(x)$ has a quadruple root $\iff$ $p \in V_4 = C_4$ the rational normal curve of degree 4.

$$\mathbb{P}^4 \supset V_{2,1,1} \supset \begin{matrix} V_{2,2} \\ \\ V_{3,1} \end{matrix} \supset V_4$$