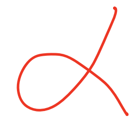
Chapter 9 - Curves



Defn A curve is a variety of dimension 1.

If $p \in X$ is a point, then $A = \mathcal{O}_{X,p}$ is a local ring of dimension 1.

Lemma Let (A, m) be a local ring of dim 1. TFAE:

(1) The maximal ideal m is principal

(2) A is a PID and all ideals are powers of m

(3) A is integrally closed (in $K = K(A)$).

(1) $\Rightarrow$ (2): Let $m = (x)$ and let $a \subseteq A$ be an ideal.

$\rightsquigarrow$ $a \subseteq m$ since A is local.

Knull's intersection theorem $\Rightarrow$ $\bigcap_{i \geq 0} m^i = 0$

$\rightsquigarrow$ pick n such that $a \subseteq m^n \wedge a \nsubseteq m^{n+1}$

$\rightsquigarrow$ Pick $c \cdot x^n \in a$ such that $x^n \in m^n - m^{n+1}$

$\rightsquigarrow$ $c \notin m$ $\Rightarrow$ c is a unit (A is local)

$\rightsquigarrow \quad (x^n) \subseteq a \quad \longrightarrow \quad a = (x^n).$

$(2) \Rightarrow (3)$: PID $\Rightarrow$ UFD $\Rightarrow$ integrally closed

$(3) \Rightarrow (1)$: Suppose A is integrally closed in $K = K(A)$.

Pick any element $x \in m$.

A noetherian + dim 1 $\Rightarrow$ $\exists\, y \in A - (x)$

such that $(x : y) = m$

$\Longrightarrow \quad yx^{-1} m \subseteq A$

<u>Claim</u> $yx^{-1} m = A$.

If not, $yx^{-1} m = m$.

m f.g. + faithful A-module $\Rightarrow$ yx^{-1} integral over A

$\Rightarrow \quad yx^{-1} \in A$ (A integrally closed)

$\Rightarrow \quad y \in (x)$

$\Rightarrow$ contradicting the assumption on y.

$\Rightarrow$ claim ok $\square$

Discrete valuation rings

A ring satisfying (1) – (3) is also a discrete valuation ring.

Defn For a local ring (A, m) as above, we call $t \in A$ a uniformizing parameter if $m = (t)$.

Any element $a \in A$ can be written uniquely as

$$a = c \cdot t^n$$

where $c \in A - m = A^\times$ is a unit and $n \in \mathbb{Z}$.

$\rightsquigarrow$ the same is true for elements in $K = K(A)$

$\rightsquigarrow$ we get a function

$$v : K^\times \longrightarrow \mathbb{Z}$$
$$a \longmapsto n$$

"order of vanishing along $t=0$"

Note that $v(a) \geq 0$ if and only if $a \in A$.

Defn A function $v : A - 0 \longrightarrow \mathbb{Z}$ is called a
discrete valuation if

(1) $v(fg) = v(f) + v(g)$

(2) $v(f+g) \geq \min\big(v(f), v(g)\big)$
with $=$ if $v(f) \neq v(g)$.

We sometimes define $v(0) = \infty$.

A ring A admitting a discrete valuation is called a discrete valuation ring (DVR).

ex $A = k[t]_{(t)}$

$\rightsquigarrow$ t is the uniformizing parameter

$v(f) =$ unique n s.t $f = c(t) \cdot t^n$.
$c(0) \neq 0$

$f = c \cdot t^n$
$g = d \cdot t^m$
$m \geqslant n$

$v(f \cdot g) = v(cd\,t^{m+n}) = m+n$

$v(f+g) = v(ct^n + dt^m) \geqslant n$

order of vanishing at $t=0$.

ex

Similarly, each $A = k[t]_{(t-a)}$ is a DVR

via $v(f) = n$ s.t $f = c(t) \cdot t^n$

ex $\mathbb{Z}_{(p)} = \left\{ \frac{a}{b} \mid p \nmid b \right\}$

$\rightsquigarrow$ $v : \mathbb{Z}_p - 0 \longrightarrow \mathbb{Z}$

$\frac{a}{b} \longmapsto$ largest power k s.t $p^k \mid a$,

ex X a curve

$p \in X$ a non-singular point

$\rightsquigarrow$ $A = \mathcal{O}_{X,p}$ is a regular local ring

$\rightsquigarrow$ A is a DVR.

system of parameters $\rightsquigarrow$ $\exists$ uniformizer $t \in m$

$$v : \mathcal{O}_{X,p} - 0 \longrightarrow \mathbb{Z}$$

$$f = \alpha \cdot t^n \longmapsto n$$

$$\alpha \in \mathcal{O}_{X,p}^{\times}$$

The Extension Lemma

"Rational maps from non-singular curves extend to morphisms"

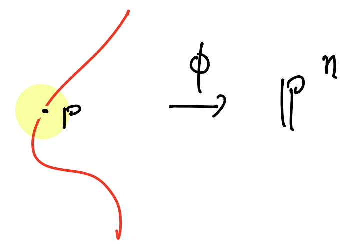
Lemma X a curve

$p \in X$ a non-singular point

$\phi : X - p \longrightarrow \mathbb{P}^n$ a morphism

$\rightsquigarrow$ $\exists$ unique morphism

$\bar{\phi} : X \longrightarrow \mathbb{P}^n$



extending ϕ.

First reduce to the case when X is affine :

If $U \subseteq X$ contains p, and $\phi|_{U-p}$ extends to $\psi : U \to \mathbb{P}^n$

then ψ and ϕ agree on $U - p$ $\Rightarrow$ glue to a

morphism $\bar{\phi} : X \to \mathbb{P}^n$.

$x_0, \dots, x_n$ coordinates on $\mathbb{P}^n$

$\rightsquigarrow$ may assume that $\phi(U - p)$ meets $D = D_+(x_0)$

$\rightsquigarrow$ $V = \phi^{-1}(D)$ non-empty open in X

and $\phi : V \longrightarrow D \simeq \mathbb{A}^n$

is given by n rational functions $f_i = \dfrac{g_i}{g_0}$ which are regular on V.

Choose a common g_0 for each i.

Consider $$\begin{aligned} X &\xrightarrow{\ \Phi\ } \mathbb{A}^{n+1} \\ x &\longmapsto (g_0, g_1, \dots, g_n) \end{aligned}$$

Want to define $\overline{\phi}$ using Φ.

$\rightsquigarrow$ want to show that $\Phi(p) \neq (0, \dots, 0)$.

If so, we get an extension of ϕ by

$$\begin{tikzcd} X \arrow[r, "\Phi"] \arrow[dr, bend right, "\overline{\phi}"'] & \mathbb{A}^{n+1} - 0 \arrow[d] \\ & \mathbb{P}^n \end{tikzcd}$$

Let t be the uniformizer of $\mathcal{O}_{X,p}$ (viewed as a rational function).

$g_i = \alpha_i(t)\, t^{v_i}$ where $\alpha_i(t) \in \mathcal{O}_{X,p}$ does not vanish at p.

$v := \min(v_0, \dots, v_n)$

$\rightsquigarrow$ $\mu_i = v_i - v \geq 0$ and at least one $\mu_i = 0$.

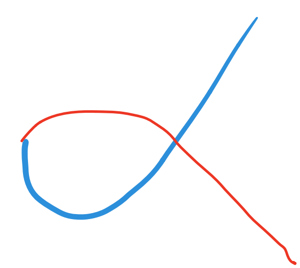
$\rightsquigarrow$ replace each g_i by $g_i t^{-v} = \alpha(t) t^{v_i - v}$

$\rightsquigarrow$ the g_i do not all vanish at p

$\rightsquigarrow$ get a morphism $\overline{\phi} = \pi \circ \Phi$ as above $\square$

ex $X = Z(y^2 - x^3 - x^2)$

$p = (0,0)$ singular!



$\rightsquigarrow$ $X \dashrightarrow \mathbb{A}^1$

$(x,y) \longmapsto \dfrac{y}{x}$

is a rational map with no extension $X \longrightarrow \mathbb{A}^1$.

Over $\mathbb{C}$: two branches at $p = (0,0)$

$y = x\sqrt{x+1}$ $\qquad \lim_{x \to 0} \dfrac{y}{x} = \lim_{x \to 0} \sqrt{x+1} = 1$

$y = -x\sqrt{x+1}$ $\qquad \lim_{x \to 0} \dfrac{y}{x} = \lim -\sqrt{x+1} = -1$ $\qquad \rightsquigarrow$ not continuous!

The Extension Theorems

Thm X a curve

$p \in X$ non-singular point

Y a projective variety

$\rightsquigarrow$ any rational map $\phi : X \dashrightarrow Y$

extends to a morphism near p.

$Y \subset \mathbb{P}^m \rightsquigarrow \phi : X \dashrightarrow Y \longrightarrow \mathbb{P}^m$

extends to $\overline{\phi} : X \longrightarrow \mathbb{P}^m$ at p

and $\overline{\phi}(p) \in Y$ since Y is closed.

Thm X, Y projective, non-singular curves

X and Y are birational $\Longleftrightarrow X \simeq Y$.

Given
$$\begin{tikzcd} X \arrow[r, dashed] & Y \\ U \arrow[u, phantom, "\cup" description] \arrow[r, "\phi"] & V \arrow[u, phantom, "\cup" description] \end{tikzcd}$$

ϕ isomorphism with inverse ψ

X, Y non-singular + projective implies that

ϕ and ψ extend to morphisms on

X and Y respectively.

we have $\phi \circ \psi = id$

$\psi \circ \phi = id$

on an open set $\Rightarrow$ holds everywhere by Hausdorff axiom

$\Rightarrow \phi$ isomorphism

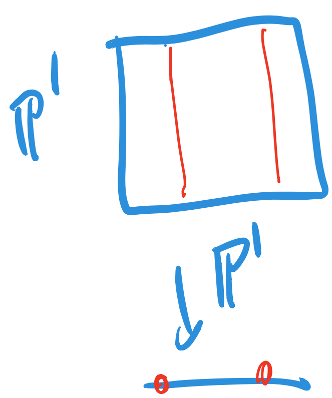
$\square$

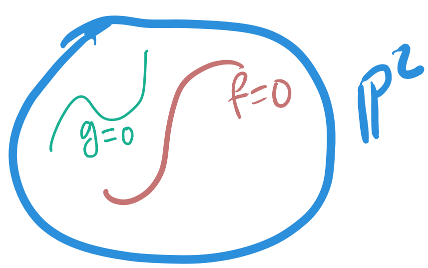
ex This fails for singular curves:
$C = Z_+(x_0^3 - x_1 x_2^2)$ is birational to $\mathbb{P}^1$
but not isomorphic. need non-singular

ex $\mathbb{A}^1$ and $\mathbb{P}^1$ are birational, but
not isomorphic $\rightsquigarrow$ need projective also

ex $\mathbb{P}^1 \times \mathbb{P}^1$ and $\mathbb{P}^2$ are non-singular + projective
They are birational, but not isomorphic

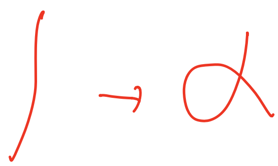
$\rightsquigarrow$ only works in dim 1.





Desingularizations of curves $A|^{N}$

Given a curve $X \rightsquigarrow$ want to construct $\tilde{X}$,
a non-singular curve,
+ birational morphism
$\pi : \tilde{X} \to X$



Lemma A f.g k-algebra which is an integral domain. $K = K(A)$.
For any finite field extension L/K
B = integral closure of A in L
is a <u>finite</u> A-module.

$$\begin{tikzcd} A \arrow[r, hook] \arrow[d] & B \arrow[d] \\ K \arrow[r, hook] & L \end{tikzcd}$$

In particular, $\bar{A}$ = integral closure of A in K
is finite over A.

ex $A = k[x,y]/(x^3 - y^2) \simeq k[t^2, t^3]$ has $\bar{A} = k[t]$

$$\bar{A} = A \oplus At.$$

X an affine variety

$\rightsquigarrow$ $B =$ integral closure of $A(X)$ in $k(X)$.

is a f.g k-algebra.

$\rightsquigarrow$ $\exists$ affine variety $\widetilde{X}$ s.t $A(\widetilde{X}) = B$.

$A(X) \subseteq B$ induces $\widetilde{X} \xrightarrow{\pi} X$

which is finite, since B is finite $/A(X)$.

These have the same function field

$\implies$ π is also birational.

Prop Any affine variety X has a normalization

This means: $\widetilde{X}$ is normal $\Leftrightarrow$ $\mathcal{O}_{\widetilde{X},p}$ normal $\forall p \in \widetilde{X}$

$\pi : \widetilde{X} \to X$ is finite + birational

Universal property of normalization: Any morphism $\sigma : Y \longrightarrow X$ from a normal variety factors as

$$\begin{tikzcd} Y \arrow[r, red] \arrow[dr, "\sigma"'] & \widetilde{X} \arrow[d, "\pi"] \\ & X \end{tikzcd}$$

For curves, we have more generally:

Thm Let X be a quasi-projective curve
$\rightsquigarrow$ X has a normalization.

Step 1 X has an affine cover with two affines

$X \subseteq \mathbb{P}^n \longrightarrow$ choose a hyperplane h_1 not containg X

$\rightsquigarrow X \cap h_1$ is finite

$\rightsquigarrow$ choose h_2 avoiding this finite set

$\rightsquigarrow$ X is covered by $U_1 = D_+(h_1) \cap X$ and $U_2 = D_+(h_2) \cap X$

Step 2 Pick an affine open $U \subseteq U_1 \cap U_2$
s.t. every $p \in U$ is non-singular.

Step 3 Construct normalizations

$$\begin{tikzcd} V_1 \arrow[dr, "\pi_1"'] & U \arrow[l, hook'] \arrow[r, hook] \arrow[d, "\simeq"] & V_2 \arrow[dl, "\pi_2"] \\ & U_1 \supseteq U \subseteq U_2 & \end{tikzcd}$$

Step 4 "Glue V_1 and V_2 along U"

$V_1 \hookrightarrow \mathbb{P}^N$
$V_2 \hookrightarrow \mathbb{P}^N$ some projective embedding

$\rightsquigarrow$ $W_1 := \overline{V_1}$ $\quad W_2 := \overline{V_2}$ projective curves

We have

$$\begin{tikzcd} & V_1 \arrow[r] \arrow[d, dashed, "\text{rational map}"] & W_1 \\ U \arrow[ur, hook] \arrow[r, hook] & V_2 \arrow[r] & W_2 \end{tikzcd}$$

$\rightsquigarrow$ get morphisms $\phi_1 : V_1 \longrightarrow W_2$
$\phi_2 : V_2 \longrightarrow W_1$ (by the extension theorem)

$\rightsquigarrow$ morphisms $V_1 \hookrightarrow W_1 \times W_2$ $(x, \phi_1(x))$ (graphs of ϕ_1 and ϕ_2)

$V_2 \hookrightarrow W_1 \times W_2$

these agree on $U \subseteq V_1 \cap V_2$ and $\overline{U} = \overline{V_1} = \overline{V_2}$ in W.

Now define $\tilde{X} = V_1 \cup V_2 \subset W$

V_1, V_2 both normal ✓

V_1, V_2 cover $\tilde{X}$ ✓

$\tilde{X}$ birational to X ✓

π_1, π_2 agree on $U \longrightarrow$ glue to a morphism $\tilde{X} \to X$.

$\square$

Note: $\tilde{X} \subset W = W_1 \times W_2$ is quasi-projective since W is.

In fact, analysing the above proof shows that

Prop If X is projective, then so is $\tilde{X}$.

We have $\tilde{X} \subset \overline{V_1} = \overline{V_2}$

let $\pi_{V_1} : \Gamma \to \overline{V_1}$ be the normalization of $\overline{V_1}$

π_{V_1} induces a rational map $\Gamma \dashrightarrow X$

Γ non-singular, X projective $\rightsquigarrow$ this extends

to a morphism $\psi : \Gamma \longrightarrow X$

$\rightsquigarrow$ factors as

$$\begin{tikzcd}
\Gamma \arrow[r, "\phi"] \arrow[d, "\pi_{V_1}"'] \arrow[dr, "\psi"] & \widetilde{X} \arrow[d, "\pi_X"] \\
\overline{V}_1 & X \arrow[l, bend left]
\end{tikzcd}$$

by universal property

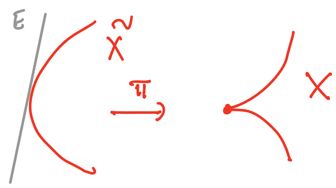
$\therefore\ X \to \overline{V}_1$ is surjective

$\Longrightarrow\ X \simeq \overline{V}_1$. $\square$

Rmk One can also find $\widetilde{X}$ via blow ups e.g. if $X \subset \mathbb{P}^2$ is a plane curve.

ex $X = Z(x^3 - y^2 z) \subset \mathbb{P}^2 \rightsquigarrow \widetilde{X} =$ strict transform of X under $\pi : Bl_p \mathbb{P}^2 \longrightarrow \mathbb{P}^2$

$\cong \mathbb{P}^1$.



The Fundamental theorem for curves

Thm Given a field K of $\mathrm{tr.deg}_k K = 1$

$\rightsquigarrow$ $\exists$ non-singular projective curve X (unique up to unique isomorphism)

s.t $\qquad k(X) = K$

Field theory: Can find an $x \in K$ s.t K is finite, separable over $k(x)$.

$\rightsquigarrow$ $\exists f \in K$ s.t $\qquad K = k(x)[f]$

and f satisfies an irreducible polynomial

$$y^n + a(x)\, y^{n-1} + \dots + a_0(x) = 0$$

This is the equation of a plane curve Y (possibly singular) with $k(X) = K$.

Now take the projective closure $\overline{Y} \subseteq \mathbb{P}^2$

and let X = normalization of $\overline{Y}$.

Then X is non-singular, projective and $k(\tilde{X}) = k(Y) = K$.

This means that there is an equivalence of categories

$$\left\{ \begin{array}{c} \text{non-singular curves } X \\ \\ \text{dominant rational maps} \\ X \dashrightarrow Y \end{array} \right\} \longleftrightarrow \left\{ \begin{array}{c} \text{fields } K \text{ of tr.deg } 1 \\ \\ k\text{-alg homomorphisms} \\ \alpha : K \longrightarrow L \end{array} \right\}$$

Rational curves

A curve X is rational if it is birational to $\mathbb{P}^1$

$$\iff k(X) \simeq k(t)$$

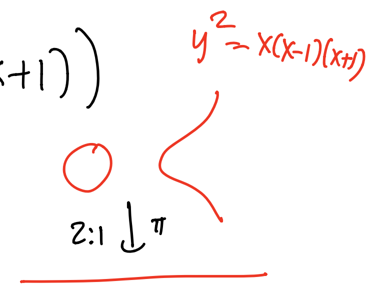
$\mathbb{P}^1$ is the only non-singular curve in its birational equivalence class.

Thm (Lüroth) If $L \subset k(t)$ is a subfield of tr.deg $=1$ then $\exists\, x \in L$ such that $L = k(x)$.

Geometric meaning: If $\mathbb{P}^1 \dashrightarrow C$ is a dominant rational map, then C is also rational.

Irrationality of some cubic curves

Thm The curve $C = Z\left(y^2 - x(x-1)(x+1)\right)$ is not rational.



Let $L = k(C) = k(x)(y)$ $\quad y^2 = x(x-1)(x+1)$

we need to show that $L \not\cong k(t)$.

Step A

Claim For a valuation $v : L \to \mathbb{Z}$

$\rightsquigarrow$ $v(x)$ is even.

$y^2 = x(x-1)(x+1)$

1) $v(x) = 0 \quad \Rightarrow \quad$ ok

2) $v(x) > 0 \quad \Rightarrow \quad v(x-1) = \min\left(v(x), v(-1)\right) = v(-1) = 0$

$\qquad v(x+1) = v(1) = 0$

$\Rightarrow \quad 2\,v(y) = v(x) \quad \rightsquigarrow \quad v(x)$ even

3) $v(x) < 0 \quad \Rightarrow \quad v(x-1) = \min\left(v(x), v(-1)\right) = v(x)$

$\qquad v(x+1) = v(x)$

$\Rightarrow \quad 2v(y) = 3\,v(x) \quad \Rightarrow \quad v(x)$ even

Step B : x is not a square in L

If $x = (a+by)^2$ in L 1, y basis for L as a k-vector space

$\longrightarrow$ $x = a^2 + 2ab\,y + b^2 y^2$

$= a^2 + 2ab\,y + b^2\, x(x-1)(x+1)$

Char $\neq 2$

$\rightsquigarrow$ either $a = 0$ $\Rightarrow$ $x = b^2\, x(x-1)(x+1)$ (I)

or $b = 0$ $\Rightarrow$ $x = a^2$ is a square in $k(x)$ (II)

(II): x is not a square in $\mathbb{C}(x)$ since $v(x) = 1$ where $v = \mathrm{ord}_0$

(I): $\Rightarrow$ $(x-1)(x+1) = x(x-1)(x+1)/x = b^2$ is a square

but $v(x^2-1) = 1$ where $v = \mathrm{ord}_1$.

$\therefore$ Claim OK.

Step C In $\mathbb{C}(t)$, any element f which has $v(f)$ even $\forall\, v : \mathbb{C}(t) \to \mathbb{Z}$ is a square.

$$f = \frac{\prod (t-a_i)^{n_i}}{\prod (t-b_i)^{m_i}}$$

Consider $\mathrm{ord}_{a_i}(f)$ and $\mathrm{ord}_{b_i}(f)$. If there are even, then n_i and m_i are all even

$\Longrightarrow$ f is a square.

This completes the proof. $\square$

In fact, the same argument works for any curve of the form

$$y^2 = p(x)$$

where $p(x) = (x-a)(x-b)(x-c)$ is a separable cubic.

$\therefore$ all elliptic curves are irrational.