Problem 3.10 The x-axis minus the origin is a closed and irreducible subset Z of $\mathbb{A}^2 \backslash \{(0,0)\}$. Exhibit regular functions on Z that are not restrictions of regular functions on $\mathbb{A}^2 \backslash \{(0,0)\}$. This illustrates why the definition of $\mathcal{O}_Z$ is delicate; functions can locally be extended to ambient space without being globally extendable. Hint: Regular functions on then punctured affine plane $\mathbb{A}^2 \backslash \{(0,0)\}$ are polynomials (see Example 3.6 on page 46). ★

$$Z = \mathbb{A}^1 - 0 \hookrightarrow \mathbb{A}^2 - 0 =: U$$

Note that $Z = Z(y) \subset U$ is closed + irreducible.

The function $\frac{1}{x}$ is regular on $\mathbb{A}^1 - 0 = D(x) \subset \mathbb{A}^1$.

On the other hand, we saw that the inclusion

$$i : U \hookrightarrow \mathbb{A}^2$$

induces an isomorphism $i^* : \underset{\substack{\| \\ k[x,y]}}{\mathcal{O}_{\mathbb{A}^2}(\mathbb{A}^2)} \xrightarrow{\sim} \mathcal{O}_U(U)$.

So the polynomials in x and y are the only restrictions from $\mathbb{A}^2 - 0$.

Problem 3.11 Let X be a variety and let $Y \subseteq X$ be a closed irreducible subset. For any open $U \subseteq X$, let $\mathcal{I}_Y(U)$ be the subset of regular functions on U that vanish on $Y \cap U$. Show that $\mathcal{I}_Y(U)$ is an *ideal* in $\mathcal{O}_X(U)$. Show that if $V \subseteq U$ are two open sets, then ρ_{UV} takes $\mathcal{I}_Y(U)$ into $\mathcal{I}_Y(V)$. Show that $\mathcal{I}_Y$ is a sheaf (of abelian groups, in fact of rings without unit element). ★

$$I_Y(U) = \{ f \in \mathcal{O}_X(U) \mid f(x) = 0 \quad \forall x \in Y \cap U \}.$$

This is an ideal: $f, g \in I(U) \Rightarrow f+g \in I(U)$

$f \cdot g \in I(U)$

$a \cdot f \in I(U) \quad \forall a \in \mathcal{O}(U)$. ✓

$U \supseteq V$: If $f \in I(U)$

then $\rho_{UV}(f) = f|_V$ is also zero on $V \cap Y \subseteq U \cap Y$.

$\rightsquigarrow \rho_{UV}(f) \in I(V)$

I is a sheaf:

$I \hookrightarrow \mathcal{O}_X$ presheaf of a sheaf $\Rightarrow$ locality holds automatically

Gluing: Given U + covering U_i + $f_i \in I(U_i)$

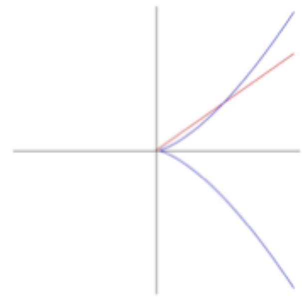
$\rightsquigarrow$ can glue f_i to a section $f \in \mathcal{O}_X(U)$ (since $\mathcal{O}_X$ is a sheaf)

This has the property that $f|_{U_i} = f_i$ is zero on $U_i \cap Y$.

$\rightsquigarrow f$ is zero on Y (since $Y \cap U_i$ is a covering of Y).

3.23 *(Rational cusp.)* Consider the curve C in $\mathbb{A}^2$ whose equation is $y^2 - x^3$. Show that C can be parametrized by the map $\phi \colon \mathbb{A}^1 \to \mathbb{A}^2$ defined as $\phi(t) = (t^2, t^3)$. Describe the map $\phi^* \colon A(C) \to A(\mathbb{A}^1)$. Show that ϕ is bijective but not an isomorphism. Show that the function field of C equals $k(t)$.

$$\begin{array}{ccc} \mathbb{A}^1 & \xrightarrow{\phi} & \mathbb{A}^2 \\ t & \longmapsto & (t^2, t^3) \end{array}$$



The rational cusp $y^2 = x^3$.

This is a polynomial map, hence a morphism.

Given a point $(u,v) \in Z(y^2 - x^3)$ we have $u^3 = v^2$ $\leadsto$ we may find $a \in k$ so that $u = a^2$, $v = a^3$

$\leadsto$ $(u,v) = \phi(a)$. $\leadsto$ ϕ surjective

The map ϕ is also injective: $(s^2, s^3) = (t^2, t^3)$

$\Rightarrow$ $s^2 = t^2$ $(s = \pm t)$

$s^3 = t^3$ $\Rightarrow$ $s = t$.

The map $\phi^* : A(C) \longrightarrow A(\mathbb{A}^1)$ is given by

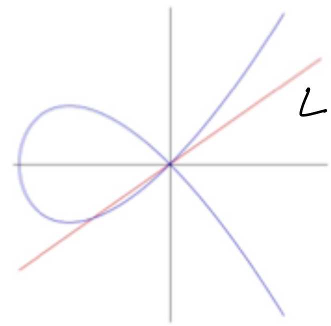
$$\frac{k[x,y]}{y^2 - x^3} \longrightarrow k[t]$$
$$x \longmapsto t^2$$
$$y \longmapsto t^3$$

ϕ is not an isomorphism: If it were, then ϕ would be an isomorphism, but ϕ is not surjective $(\operatorname{im} \phi = k[t^2, t^3])$

$A(C) \simeq k[t^2, t^3]$ which has $k[t]$ as a fraction field.

3.24 *(Rational node.)* In this exercise we let C be the curve in $\mathbb{A}^2$ whose equation is $y^2 - x^2(x+1)$. Define a map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ by $\phi(t) = (t^2-1, t(t^2-1))$. Show that $\phi(\mathbb{A}^1) = C$, and describe the map $\phi^*\colon A(C) \to A(\mathbb{A}^1)$. Show that ϕ is not an isomorphism, but induces an isomorphism $\mathbb{A}^1 \backslash \{\pm 1\} \to C \backslash \{0\}$. Show that the function field of C equals $k(t)$.

$$\phi : \mathbb{A}^1 \longrightarrow \mathbb{A}^2$$
$$t \longmapsto (t^2-1,\ t^3-t)$$



The rational node $y^2 = x^2(x+1)$.

$\phi(\mathbb{A}^1) = C$: $\subseteq$ is easy:

$$(t^3-t)^2 - (t^2-1)^2(t^2-1+1) = t^2(t^2-1)^2 - (t^2-1)^2 t^2$$
$$= 0$$

Let L be the line $Z(y-ax)$.

We have $L \cap C = Z(y-ax,\ y^2-x^3-x^2)$

$$y^2 - x^3 - x^2 = a^2x^2 - x^3 - x^2 = x^2(a^2 - x - 1)$$
$$\rightsquigarrow \quad x = a^2 - 1$$
$$y = a(a^2-1)$$

$\rightsquigarrow$ any point $(u,v) \in C$ is the image of $u^{-1}v \in \mathbb{A}^1$

(let L be the line $vx - uy = 0$)

The map ϕ is also injective:

$$(s^2-1,\ s^3-s) = (t^2-1,\ t^3-t)$$

$s^2-1 = t^2-1 \quad \rightsquigarrow \quad s = \pm t$

$s^3 - s = t^3 - t \quad \rightsquigarrow \pm(t^3-t) = t^3-t \quad \rightsquigarrow \quad s = t.$

The map $A(C) \xrightarrow{\phi^*} A(\mathbb{A}^1)$

$$k[x,y]/_{y^2-x^3-x^2} \longrightarrow k[t]$$

$$x \longmapsto t^2-1$$

$$y \longmapsto t^3-t$$

Not surjective: image equals $k[t^2-1, t^3-t] \subsetneq k[t]$

On $\mathbb{A}^1 - \{\pm 1\}$ we can define an inverse to ϕ:

$$C - 0 \xrightarrow{\psi} \mathbb{A}^1 - \{\pm 1\}$$
$$(x,y) \longmapsto y/x$$

$x = t^2-1$
$y = t(t^2-1) \leadsto t = y/x$

y/x is regular on $C-0$, and $\phi \circ \psi = id_{C-0}$:

$$(x,y) \to y/x \to \left(\left(\tfrac{y}{x}\right)^2 - 1, \left(\tfrac{y}{x}\right)^3 - \left(\tfrac{y}{x}\right)\right)$$

$$= \left(\frac{x^3+x^2}{x^2} - 1, \; \underbrace{\frac{y\,(x^3+x^2)}{x^3} - \frac{y}{x}}_{= y}\right)$$

Also $\psi \circ \phi$

$$t \to (t^2-1, t^3-t) \to \frac{t^3-t}{t^2-1} = t \qquad (t \neq \pm 1!!).$$

$\leadsto$ ϕ defines an isomorphism $C - 0 \simeq \mathbb{A}^1 - \{\pm 1\}$.

$\leadsto$ $C-0$ and $\mathbb{A}^1 - \{\pm 1\}$ have isomorphic function fields

$\leadsto$ $k(C) = k(C-0) = k(\mathbb{A}^1 - \{\pm 1\}) = k(\mathbb{A}^1) = k(t)$.

3.25 Let C be one of the curves from the two previous exercises. Show that, except for finitely many, every line through the origin intersects C in exactly one other point. What are the exceptional lines in the two cases? Use this to give a geometric interpretation of the parametrizations in the previous exercises.

For $\quad C = Z(y^2 - x^3) \quad L = Z(y - ax)$

$\rightsquigarrow \quad C \cap L = Z(y^2 - x^3, y - ax) = Z(x^2(a^2 - x), y - ax)$

2nd intersection point $= (a^2, a^3)$ for $a \neq 0$

exceptional lines: $y = 0$ and $x = 0$

parametrization from previous exercise = projection from $(0,0)$.

4.4 Show that two different lines in $\mathbb{P}^2$ meet in exactly one point.

4.5 Show that n hyperplanes in $\mathbb{P}^n$ always have a common point of intersection. Show that n general hyperplanes meet in exactly one point.

A linear hyperplane in $\mathbb{P}^n$ is a hypersurface $H = Z(\lambda)$, where
$$\lambda = a_0 x_0 + \dots + a_n x_n \qquad \text{where } a_0, \dots, a_n \in k$$

4.4 Given two $\overset{\text{distinct}}{\vee}$ lines
$$L_1 = Z(a_0 x_0 + a_1 x_1 + a_2 x_2)$$
$$L_2 = Z(b_0 x_0 + b_1 x_1 + b_2 x_2)$$

$\rightsquigarrow$ Perform a linear change of coordinates so that $L_2 = Z(x_2)$.

Then
$$L_1 \cap L_2 = Z(a_0 x_0 + a_1 x_1 + a_2 x_2, x_2)$$
$$= Z(a_0 x_0 + a_1 x_1, x_2)$$
$$= \{(-a_1 : a_0 : 0)\} \qquad (L_1 \neq L_2 \Leftrightarrow \text{one of } a_0, a_1 \text{ is non-zero})$$

4.5 Let $H_i = Z(\lambda_i)$ where λ_i is a linear form

Note that $\lambda_1, \dots, \lambda_n$ generate a proper prime ideal I in $k[x_0, \dots x_n]$. We also have codim $I \leq n$ by Krull's Hauptidealsatz, so $Z(I) \subseteq \mathbb{A}^{n+1}$ has dimension $\geq n$. Hence $Z_+(I) = \pi(Z(I) - 0)$ is non-empty.

If the $\lambda_1, \ldots, \lambda_n$ are general, the coordinate change

$$x_0' = x_0$$
$$x_i = \lambda_i \qquad i = 1 \ldots n$$

defines an isomorphism $k[x_0, \ldots x_n] \xrightarrow{\phi} k[x_0 \ldots x_n]$.

Hence $Z(\lambda_1 \ldots \lambda_n) = \phi(Z(x_1, \ldots x_n)) = \phi((1:0: \ldots :0))$.