Problem 4.12 Given an open $U \subseteq X$ and a continuous function $f\colon U \to \mathbb{A}^1$. Let $C_0(U)$ be punctured cone over U and denote by $\pi_U\colon C_0(U) \to U$ the (restriction of the) projection. Show that f is regular if and only if the composition $f \circ \pi$ is regular on $C_0(U)$. ★

$$\begin{tikzcd} C_0(U) \arrow[d, "\pi_U"'] & \\ U \arrow[r, "f"] & \mathbb{A}^1 \end{tikzcd}$$

f regular $\iff$ $\forall\, p \in U$ $\exists V$ nbh s.t.

$f|_V = g/h$ $\quad g, h \in \mathcal{O}_X(V)$
$h(x) = 0$
$\forall\, x \in V$

Consider $W = \pi^{-1} V$

$$\rightsquigarrow \quad f \circ \pi \,\Big|_W = \frac{g \circ \pi}{h \circ \pi} = \frac{\pi^* g}{\pi^* h}$$

$\pi^* g,\ \pi^* h \in \mathcal{O}_{C_0}(\pi^{-1} V)$ are regular

and $(\pi^* h)(x) = h(\pi(x)) \neq 0$ since $x \in \pi^{-1} V$.

$\Rightarrow$ $f \circ \pi$ is regular on $\pi^{-1} V$

$\Rightarrow$ $f \circ \pi$ is regular on $C_0(U)$ since $\pi^{-1} V$ form a cone of $\pi^{-1} U$.

Conversely, if $f \circ \pi$ is regular: pick $p \in U$.

$$\rightsquigarrow \quad f \circ \pi \,\Big|_{\pi^{-1} V} = \frac{g}{h} \quad \text{for some } V \ni p.$$

$g, h \in \mathcal{O}_{C_0}(\pi^{-1} V)$ $\quad h(x) \neq 0$ $\forall x \in \pi^{-1} V$

$\frac{g}{h}$ must be constant on lines though the origin $\to$ g, h homogeneous

$\Rightarrow$ $f|_V = \dfrac{g}{h}$ on V $\quad g, h$ homogeneous, and $h \neq 0$ on V

$\Rightarrow$ f is regular.

Problem 4.13 Let S be a graded ring. Show that the set T consisting of the homogeneous elements in S is a multiplicative system and that the localization S_T is a graded ring. Show S_T is an integral domain when S is, and in that case the homogeneous piece of degree zero $(S_T)_0$ is a field. ★

$T = \{ s \in S \mid s \text{ homogeneous} \}$

$1 \in T$

$s, t \in T$ homogeneous $\Rightarrow$ $s \cdot t$ homogenous $\Rightarrow s \cdot t \in T$

S integral domain: $S_T = T^{-1}S$

If $\frac{a}{t} \in T^{-1}S$ is a zerodivisor $\Rightarrow \exists \frac{b}{t'}$ s.t. $\frac{a}{t} \cdot \frac{b}{t'} = 0$

$\frac{a \cdot b}{t\, t'} = 0/1 \Leftrightarrow u \cdot (ab \cdot 1 - 0 \cdot tt') = u \cdot ab = 0$

$\Rightarrow ab = 0$ since S is an integral domain

$\Rightarrow a = 0$ or $b = 0$

$(T^{-1}S)_0 = \left\{ \frac{a}{t} \;\middle|\; \begin{array}{l} a \in S \\ t \in T \end{array} \quad \deg a = \deg t \right\}$

$\rightsquigarrow$ a is also homogeneous

is a field:

$\frac{a}{t} \in (T^{-1}S)_0 \Rightarrow$ inverse is given by $\frac{t}{a} \in (T^{-1}S)_0$

Problem 4.14 Let $S = k[x_0, x_1]$ and let T be the multiplicative system $T = \{ x_1^i \mid i \in \mathbb{N} \}$. Show that the homogeneous piece $(S_T)_0$ of degree zero of S_T equals $k[x_0 x_1^{-1}]$. Show furthermore that the decomposition of S into homogeneous pieces is given as

$$S = \bigoplus_{i \in \mathbb{Z}} k[x_0 x_1^{-1}] \cdot x_1^i.$$

$T = \{1, x_1, x_1^2, \ldots\}$

$(T^{-1}S)_0 = \left\{ \frac{a}{x_1^m} \mid \deg a = m \right\}$

write $a = a_m x_0^m + a_{m-1} x_0^{m-1} x_1 + \ldots + a_0 x_1^m$, then

$$\frac{a}{x_1^m} = a_m \left(\frac{x_0}{x_1}\right)^m + \ldots + a_0$$

$\rightsquigarrow$ define $(T^{-1}S)_0 \longrightarrow k[x_0 x_1^{-1}]$ $\rightsquigarrow$ isomorphism ✓

$$\left(\frac{x_0}{x_1}\right)^m \longmapsto \left(\frac{x_0}{x_1}\right)^m$$

$$k[x_0 x_1^{-1}] x_1^i \xrightarrow{\sim} (T^{-1}S)_i$$

$$\bigoplus k[x_0 x_1^{-1}] x_i^i \xrightarrow{\sim} T^{-1}S$$

4.16 Let the projection $\mathbb{P}^3$ to $\mathbb{P}^2$ be given as $(x : y : z : w) \mapsto (x : x+z : w+y)$. Determine the centre and describe the projection of the twisted cubic parametrized as $(u : v) \mapsto (u^3 : u^2v : uv^2 : v^3)$. Hint: The key words is "rational node" (see Problem 3.23 on page 60).

Centre $\Leftrightarrow$ map is not defined $\Leftrightarrow$ $\begin{matrix} x=0 \\ x+z=0 \\ w+y=0 \end{matrix}$ $\Leftrightarrow$ $(0:1:0:-1)$

projection: $(u:v) \to (u^3 : u^2v : uv^2 : v^3)$

$\rightsquigarrow (u^3 : u^3 + uv^2 : v^3 + u^2v)$

The image: $\begin{aligned} x_0 &= u^3 \\ x_1 &= u^3 + uv^2 \\ x_2 &= v^3 + u^2v \end{aligned}$

Follow the hint: $x_2^2 x_0 = x_1^3 + c x_0 x_1^2$ for some c?

Yes, pick $c = -1$:

$$(v^3+u^2v)^2 u^3 = (u^3+uv^2)^3 + c \cdot u^3 (u^3+uv^2)^2$$

$$\parallel$$

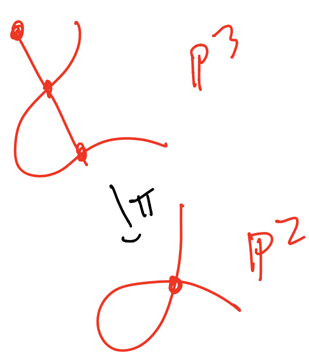
$$u^3v^6 + 2u^5v^4 + u^7v^2 = u^9 + 3u^7v^2 + 3u^5v^4 + u^3v^6$$

$$- u^3(u^6 + 2u^4v^2 + u^2v^4)$$

$$= \cancel{u^9} + \cancel{3}u^7v^2 + \overset{2}{\cancel{3}}u^5v^4 + u^3v^6$$

$$- (\cancel{u^9} + 2\cancel{u^7v^2} + \cancel{u^5v^4})$$

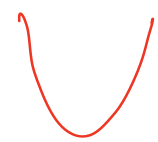
$$= u^7v^2 + 2u^5v^4 + u^3v^6 \checkmark$$



4.18 Describe (by giving an equation) the image of the rational normal quartic under the projection $\mathbb{P}^4 \dashrightarrow \mathbb{P}^2$ that forgets the third and the forth coordinate. Accomplish the same task but with the projection that forgets the second and the forth coordinate.

(u^4, u^3v, v^4) $\quad$ $\underline{X_1^4 = X_2 \cdot X_0^3}$

$$(u:v) \longrightarrow (u^4 : u^3v : \underline{u^2v^2} : \underline{uv^3} : \underline{v^4})$$
$$\downarrow$$
$$(u^4 : u^3v : u^2v^2)$$

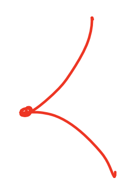


$(u^4 : \underline{u^2v^2} : v^4)$ $\quad$ $y^2 = xz$

extends as $(u:v) \to (u^2 : uv : v^2)$ = conic

$\underline{y^2 = xz}$

$$(u:v) \longrightarrow (u^4 : u^3v : \cancel{u^2v^2} : uv^3 : \cancel{v^4})$$
$$\downarrow$$
$$(u^4 : u^3v : uv^3)$$



extends as $(u^3 : u^2v : v^3)$

image is $\quad y^3 = x^2 z \quad$ ✓

$$\begin{array}{ccc} C & \longrightarrow & \mathbb{P}^2 \\ \cup & \nearrow & \\ U & & \end{array}$$

$\underline{(u^2 : uv : v^2)}$ $\quad u \neq 0$

4.19 Let C_d be the rational normal curve in $\mathbb{P}^d$ whose parametrization is

$$\phi_d(u:v) = (u^d : u^{d-1}v : \ldots : uv^{d-1} : v^d).$$

Let $\pi\colon \mathbb{P}^d \backslash \{q\} \to \mathbb{P}^{d-1}$ be the projection with centre $q = (0:0:\ldots:0:1)$. Prove that $q \in C_d$ and that the closure in $\mathbb{P}^{d-1}$ of $\pi(C_d \backslash \{q\})$ is equal to C_{d-1}.

$q = \phi(0:1) \Rightarrow q \in C_d$. ✓

The rational map $\mathbb{P}^1 \to \mathbb{P}^{d-1}$ is given by

$\phi(u:v) = (u^d : u^{d-1}v : \ldots : uv^{d-1})$

if $u \neq 0$ $\quad \phi(u:v) = (u^{d-1} : u^{d-2}v : \ldots : v^{d-1}) \in C_{d-1}$

$\rightsquigarrow \pi(C_d - q) \subseteq C_{d-1}$

The closure of $\pi(C_d - q)$ is irreducible of dim 1 $\rightsquigarrow$ equal to C_{d-1}.

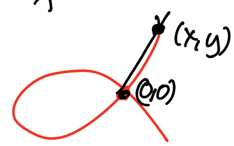
5.3 Being a closed embedding is a not a local property on the source. Consider the map $\phi: \mathbb{A}^1 \to \mathbb{A}^2$ given as $\phi(t) = \big(t^2 - 1, t(t^2 - 1)\big)$.

a) Show that ϕ is a closed map, but not a closed embedding.

b) Exhibit an open covering $\{U_i\}$ of $\mathbb{A}^1$ such each restriction $\phi|_{U_i}$ is a closed embedding into some open subset V_i of $\mathbb{A}^2$.

a) Note that the image of ϕ equals the curve C

$$y^2 = x^3 + x^2$$



(we showed this in a previous exercise)

ϕ is closed: closed subsets of $\mathbb{A}^1$: $\emptyset$ ✓, $\mathbb{A}^1$, $a \in k$

$\phi(\mathbb{A}^1) = C$ is closed

$\phi(a) =$ closed $\rightsquigarrow$ ϕ closed ✓

ϕ not an embedding since $\phi(1) = \phi(-1) = (0,0)$

$\rightsquigarrow$ ϕ is not even injective!

b) This does not appear to be correct as stated.

If there is a covering U_i of $\mathbb{A}^1$

then say $U_1 \xrightarrow{\phi_1} \mathbb{A}^2$

will be an embedding and $\phi_1(1) = (0,0)$.

However, then there should be an inverse of ϕ,
given by $\psi(x,y) = y/x$ defined in a
nbh V_1 of $(0,0)$ (since $\psi = y/x$ for $(x,y) \neq (0,0)$)
However, y/x is not regular in any nbh of $(0,0)$:
the local ring is given by

$$\mathcal{O}_{C,P} = \left(k[x,y] / y^2 - x^3 - x^2 \right)_{(x,y)}$$

and here x belongs to the maximal ideal $m = (x,y)$.

5.4 Assume that the ground field k is of positive characteristic 2. Show that the morphism $\mathbb{P}^1 \to \mathbb{P}^2$ that sends $(x_0 : x_1)$ to $(x_0^4 : x_0^2 x_1^2 : x_1^4)$ is a closed map which is a homeomorphism onto its image, but which is not a closed embedding.

This is a local question: restrict to $D_+(u_2) \subseteq \mathbb{P}^2$:

$\pi^{-1} D_+(u_2) = D_+(x_1^4) = D_+(x_1) \subseteq \mathbb{P}^1$

$\rightsquigarrow$ the map is given by $\mathbb{A}^1 \xrightarrow{\phi} \mathbb{A}^2$, $x \mapsto (x^4, x^2)$

this is a homeomorphism onto $C = Z(y - x^2) \subseteq \mathbb{A}^2$:

continuous ✓

bijective: given $p = (a, a^2) \in C \rightsquigarrow \sqrt{a} \in k$ unique square root ($\pm\sqrt{a} = \sqrt{a}$)

$\rightsquigarrow p = \phi(\sqrt{a})$ ✓

inverse is also continuous on $D_f(x) \longrightarrow$ homeomorphism.

But: ϕ closed embedding $\Leftrightarrow \phi^* : A(C) \to A(\mathbb{A}^1)$ surjective

But this map is

$$\frac{k[x,y]}{y - x^2} \longrightarrow k[t]$$

$$x \longrightarrow t^2$$

$$y \longrightarrow t^4$$

$\rightsquigarrow$ not surjective...