COROLLARY 5.16 *The product of two projective varieties is projective.*

PROOF: Let the two projective varieties be X and Y with $X \subset \mathbb{P}^n$ and $Y \subset \mathbb{P}^m$. The Zariski topology on a product is stronger than the product topology, and hence the product $X \times Y$ is closed in $\mathbb{P}^n \times \mathbb{P}^m$. The topological space $X \times Y$ carries both the sheaf of regular functions considered a product variety and the sheaf of regular functions considered a closed subvariety of $\mathbb{P}^n \times \mathbb{P}^m$, but of course, the two coincide. By the proposition $\mathbb{P}^n \times \mathbb{P}^m$ is isomorphic to a closed subvariety of $\mathbb{P}^{mn+n+m}$ via the Segre embedding, and consequently the product $X \times Y$ is as well isomorphic to a closed subvariety. ❑

PROBLEM 5.6 Verify that the two sheaves alluded to in the proof are equal. HINT: This is a local question. ★

$X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m$ closed ✓

$\rightsquigarrow$ two sheaves : $\mathcal{F} = \mathcal{O}_{X \times Y}$ (product variety)

$\mathcal{G} = \mathcal{O}_{X \times Y \subset \mathbb{P}^n \times \mathbb{P}^m}$ (subvariety)

The sheaf $\mathcal{F}$ was constructed as follows :

Take affine $U \subseteq X$ and $V \subseteq Y$.

If $U \subseteq \mathbb{A}^n$ and $V \subseteq \mathbb{A}^m$ have affine coordinate rings

$$A(U) = k[x_1 \dots x_n]/(f_1 \dots f_s) \qquad A(V) = k[y_1 \dots y_m]/(g_1 \dots g_t)$$

then
$$\mathcal{F}(U \times V) = \frac{k[x_1 \dots x_n, y_1 \dots y_m]}{(f_1 \dots f_s, g_1 \dots g_t)}.$$

Now consider

$$D(x_i) \simeq \mathbb{A}^n$$
$$D(y_i) \simeq \mathbb{A}^m$$

$\rightsquigarrow D(x_i) \times D(y_j) \simeq \mathbb{A}^{m+n}$ is an affine open in $\mathbb{P}^n \times \mathbb{P}^m$

and $U = X \cap D(x_i)$, $V = Y \cap D(y_j)$ are affine subsets

By construction, $U \times V \subseteq \mathbb{A}^n \times \mathbb{A}^m$ is a subvariety

$$\rightsquigarrow \mathcal{G}(U \times V) = \{ f : U \times V \to k \text{ regular} \}$$

Gitt $\varphi_i : \mathcal{F}|_{U_i} \to \mathcal{G}|_{U_i}$
U_i overdekning av X
s.a. $\varphi_i = \varphi_j$ på $U_i \cap U_j$
$\Rightarrow \exists\ \varphi : \mathcal{F} \to$

$$= \frac{k[x_1 \ldots x_n, y_1 \ldots y_m]}{(f_1 \ldots f_s\ g_1 \ldots g_t)}$$

$$= \mathcal{F}(U \times V)$$

$\therefore$ There are natural isomorphisms $\mathcal{G}(U \times V) \xrightarrow{\varphi_{ij}} \mathcal{F}(U \times V)$

for $U = D_+(x_i)$ $i = 0, \ldots n$, $V = D_+(y_j)$ $j = 0, \ldots m$ $\rightsquigarrow U \times V$ form an open cover of $X \times Y$

The φ_{ij} agree on the overlaps $(U_i \times V_j) \cap (U_k \times V_\ell)$

$\longrightarrow$ glue to an isomorphism $\mathcal{G} \xrightarrow{\sim} \mathcal{F}$.

5.8 Show that under the Segre map the fibres of the two projections from $\mathbb{P}^1 \times \mathbb{P}^1$ onto $\mathbb{P}^1$ embed as lines in $\mathbb{P}^3$. ~~Show that if Z is an effective divisor in $\mathbb{P}^1 \times \mathbb{P}^1$ of bidegree $(0, n)$ or $(n, 0)$, then Z is a union of lines.~~

1) Any subvariety $F \subset \mathbb{P}^1 \times \mathbb{P}^1$ is given by $F = Z(f)$ for some bihomogeneous $f \in k[x_0, x_1, y_0, y_1]$.

If F is a fiber of $\mathbb{P}^1 \times \mathbb{P}^1 \xrightarrow{pr_1} \mathbb{P}^1$, then

$F = Z_f(\ell)$ where $\ell = a_0 x_0 + a_1 x_1$ [i.e. $f = pr_1^* \ell$]

This embeds as a line in $\mathbb{P}^3$: the ideal of the image is given by the kernel of

$$k[u_0, u_1, u_2, u_3] \xrightarrow{\theta} k[x_0, x_1, y_0, y_1]/\ell$$

$$\begin{aligned} u_0 &\to x_0 y_0 \\ u_1 &\to x_1 y_0 \\ u_2 &\to x_0 y_1 \\ u_3 &\to x_1 y_1 \end{aligned}$$

$\mathbb{P}^1 \times \mathbb{P}^1 \to \mathbb{P}^3$

$x_i \; y_j \mapsto (x_i x_j)$

$$r(x_0 y_0, x_1 y_0, x_0 y_1, x_1 y_1) = 0 \mod a x_0 + b x_1$$

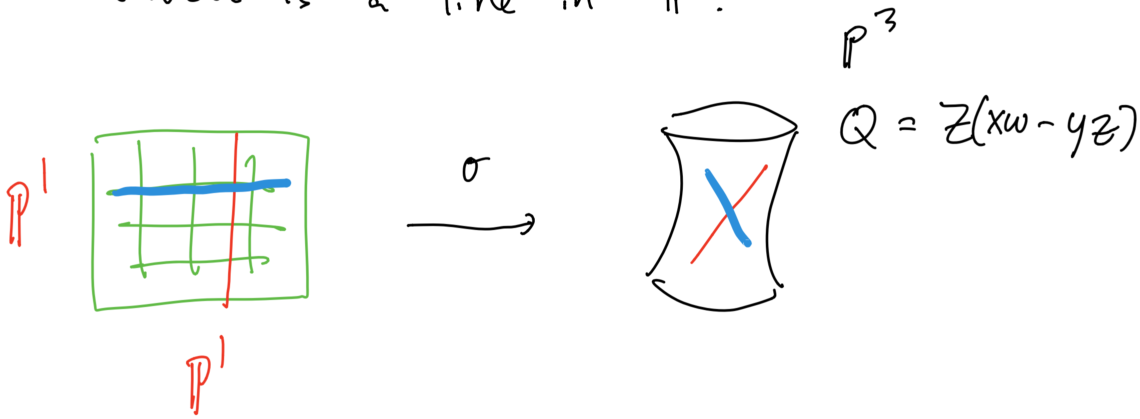
$$a_0 x_0 y_0 + a_1 x_1 y_0 = a_0 u_0 + a_1 u_1 = 0$$

$$a_0 x_0 y_1 + a_1 x_1 y_1 = a_0 u_2 + a_1 u_3 = 0$$

$$(a_0 u_0 + a_1 u_1, \; a_0 u_2 + a_1 u_3) \subseteq \ker \theta$$

This is an equality, since both have ht $2 = \operatorname{codim} \ell$.

$\Rightarrow$ F embeds as $Z_+(a_0 u_0 + a_1 u_1, a_0 u_2 + a_1 u_3)$
which is a line in $\mathbb{P}^3$.



6.2 Let $\psi\colon \mathbb{A}^2 \to \mathbb{A}^2$ be the map $\psi(x,y) = (x, xy)$. Determine the ideals $\psi^*\mathfrak{m}_{(a,b)}$ and the fibres $\psi^{-1}(a,b)$ for all points $(a,b) \in \mathbb{A}^2$.

$$\mathfrak{m}_{ab} = (u-a,\ v-b)$$

$$\psi^*\mathfrak{m}_{ab} = (x-a,\ xy-b) = (x-a,\ ay-b)$$

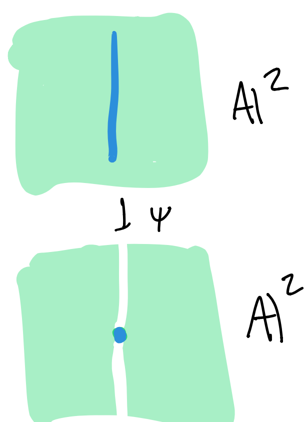
$$\psi^{-1}(a,b) = Z(x-a,\ ay-b)$$

$$a=0,\ b\neq 0 \Rightarrow \psi^{-1}(a,b) = Z(x,\ b) = \emptyset$$

$$a=0,\ b=0 \Rightarrow \psi^{-1}(a,b) = Z(x) = y\text{-axis}$$

$$a\neq 0 \Rightarrow \psi^{-1}(a,b) = Z(x-a,\ y-\tfrac{b}{a}) = \{(a, \tfrac{b}{a})\}.$$

$$\begin{tikzcd} X \arrow[r, "\varphi"] \arrow[dr, "\varphi^*f = f\circ\varphi"'] & Y \supset U \arrow[d, "f"] \\ & k \end{tikzcd}$$



$$(X, \mathcal{O}_X) \xrightarrow{\varphi} (Y, \mathcal{O}_Y)$$

$\varphi: X \to Y$ kontinuerlig avb.

$+$ for alle $U \subseteq Y \rightsquigarrow \phi^*: \mathcal{O}_Y(U) \to \mathcal{O}_X(\phi^{-1}U)$

$$f \longmapsto \phi^* f$$

$J = (x_0 - x_1, x_2, x_3)$

Problem 6.13 Show that $x_0 - x_1, x_2, x_3$ is a system of parameters at the origin for $Z(x_0x_1 - x_2x_3)$. A_1^4 ★

Enough to show $x_0, x_1 \in \sqrt{J}$

(then $\sqrt{J} = (x_0, x_1, x_2, x_3)$ is the maximal ideal)

$$(x_0 - x_1, x_2, x_3, x_0x_1 - x_2x_3)$$

$$= (x_0 - x_1, x_2, x_3, x_0x_1)$$

$$= (x_0 - x_1, x_2, x_3, x_0^2)$$

$\rightsquigarrow$ $x_0 \in \sqrt{J}$ and $x_1 \in \sqrt{J}$ also.

systems of parameters:

A dim n

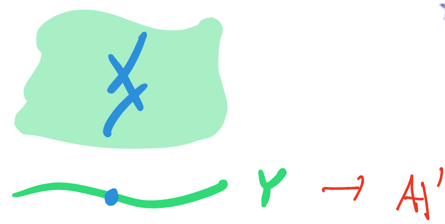
$\mathfrak{m} \subset A$ maximal ideal

$x_1 \dots x_n$ system of parameters if

$\underbrace{(x_1 \dots x_n)}$ is $\mathfrak{m}$-primary

$\sqrt{(x_1 \dots x_n)} = \mathfrak{m}$

Problem 6.15 Assume that X is a (irreducible) variety and that Y is a curve. Show that all components of all fibres of a dominant morphism $\phi\colon X \to Y$ are of codimension one in X. ★



Wlog Y is affine.

Pick a NNL $Y \xrightarrow{\pi} \mathbb{A}^1$ (finite)

$$\begin{tikzcd} X \arrow[d, "\phi"] \arrow[dr, bend left] & \\ Y \arrow[r, "\pi", "\text{finite}"'] & \mathbb{A}^1 \end{tikzcd}$$

$\rightsquigarrow$ wlog $Y = \mathbb{A}^1$.

If $\underset{a}{\overset{\|}{p}} \in \mathbb{A}^1$, then $\phi^{-1}(p) = Z(\phi^* m_p)$

$= Z(\phi^*(x-a))$

$\uparrow$

$\neq$ konst

ϕ^* injective.

This is defined by 1 equation

$\implies$ codim 1 by Krull.

PROBLEM 7.1 Show that for $X = Z(I) \subset \mathbb{A}^n$ and $p = (0,\ldots,0)$,

$$T_pX = Z\left(f^{(1)} \mid f \in I\right)$$

where $f^{(1)}$ denotes the degree 1 ~~linear part~~ of $f \in k[x_1,\ldots,x_n]$. ★

Take $f \in I$ and write $f = f^{(1)} + g$ where $g \in m^2$

$m = (x_1 \ldots x_n)$

$$v \in T_pX \iff \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(0) \cdot v_i = 0 \quad \forall f \in I$$

$f^{(1)} = a_1x_1 + \ldots + a_nx_n$

$$\iff \sum_{i=1}^{n} a_i v_1 = 0$$

$$\iff f^{(1)}(v_i) = 0$$

$$\iff v_i \in Z(f^{(1)})$$