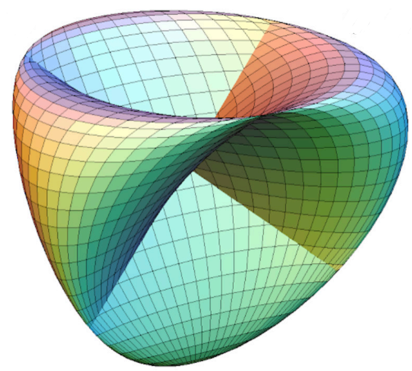
**PROBLEM 7.4** Compute the singular points of the *Steiner surface*

$$Z(x^2y^2 + y^2z^2 + z^2x^2 - xyz) \subset \mathbb{A}^3$$



$$F = x^2y^2 + y^2z^2 + z^2x^2 - xyz$$

$$\frac{\partial F}{\partial x} = 2xy^2 + 2xz^2 - yz$$

$$\frac{\partial F}{\partial y} = 2yx^2 + 2zy^2 - xz$$

$$\frac{\partial F}{\partial z} = 2y^2z + 2x^2z - xy$$

Note: 
$$\begin{aligned} xyz = & -4(x^2y^2 + y^2z^2 + z^2x^2 - xyz) \\ & + x(2xy^2 + 2xz^2 - yz) \\ & + y(2yx^2 + 2zy^2 - xz) \\ & + z(2y^2z + 2x^2z - xy) \end{aligned}$$

$\rightsquigarrow$ $\mathrm{sing}(X) \subseteq Z(xyz)$.

By symmetry, we consider $x=0$:

$$\left(F, \frac{\partial F}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial F}{\partial z}, x\right) = (y^2z^2, yz, 2zy^2, 2y^2z, x)$$
$$= (x, yz) = (x,y) \cap (x,z)$$

$\rightsquigarrow \quad \mathrm{sing}(X) = Z(x,y) \cup Z(x,z) \cup Z(y,z)$

**Problem 7.5** For $X = Z(f_1, \dots, f_r) \subset \mathbb{A}^n$ we define the *tangent bundle of* $X$ has the set

$$T(X) = \{(x,v) \in X \times \mathbb{A}^n | \sum_{i=1}^{n} \frac{\partial f_j}{\partial x_i}(x) \cdot v_i = 0 \text{ for all } j\}$$

Show that $T(X)$ is an affine variety, and describe the morphism $p : T(X) \to X$ given by the first projection. ★

$x_1 \dots x_n, v_1 \dots v_n$ coordinates on $\mathbb{A}^{2n}$:

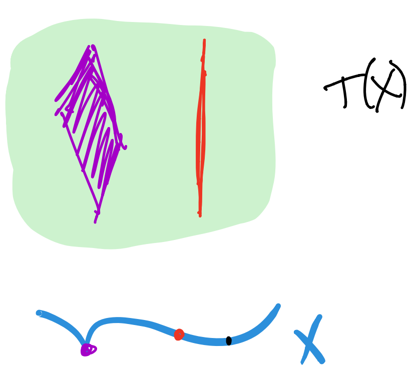
$$T(X) = Z(f_1, \dots, f_r, g_1, \dots, g_n) \subset \mathbb{A}^{2n}$$

$$g_k = \sum_{i=1}^{n} \frac{\partial f_k}{\partial x_i}(x) \cdot v_i$$

$\rightsquigarrow$ $T(X)$ is an affine variety.

For $a = (a_1, \dots, a_n) \in X$, the fiber of $p: T(X) \longrightarrow X$ is given by

$$p^{-1}(a) = Z\left(x_1 - a_1, \dots, x_n - a_n, \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(a) \cdot v_i, \dots, \sum_{i=1}^{n} \frac{\partial f_1}{\partial x_i}(a) \cdot v_i\right)$$




$= \bigcup \{\text{tangent lines passing through } a\}$

**8.3** Let $\phi \colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map that sends $(x_0 : x_1 : x_2)$ to $(x_2^2 : x_0x_1 : x_0x_2)$. Determine largest set of definition. Show that $\phi$ is birational , and determine what curves are collapsed.

$$\mathfrak{a}_\phi = \sqrt{(x_2^2, x_0x_1, x_0x_2)} = (x_2, x_0x_1) = (x_0, x_2) \cap (x_1, x_2)$$

$$\rightsquigarrow \quad U_\phi = \mathbb{P}^2 - \{(0:1:0), (1:0:0)\}$$

$\phi$ is birational: On the open set $D_+(u_0 u_2)$:

$$\begin{aligned} u_0 &= x_2^2 \\ u_1 &= x_0x_1 \\ u_2 &= x_0x_2 \end{aligned} \qquad \begin{aligned} \frac{x_1}{x_2} &= \frac{u_1}{u_2} \\ \frac{x_0}{x_2} &= \frac{u_2}{u_0} \end{aligned}$$

$\rightsquigarrow$ reconstruct $(x_0 : x_1 : x_2)$ as $\left(\frac{u_2}{u_0} : \frac{u_1}{u_2} : 1\right)$

on $u_0 = 0$: $Z(x_2)$ gets contracted to $(0:1:0)$

$u_2 = 0$: $Z(x_0)$ gets contracted to $(1:0:0)$

8.4 Let $\phi\colon \mathbb{P}^2 \dashrightarrow \mathbb{P}^2$ be the rational map that sends $(x_0 : x_1 : x_2)$ to $(x_2^2 - x_0x_1 : x_1^2 : x_1x_2)$. Determine the set largest set where $\phi$ is defined. Show that $\phi$ is birational, and determine what curves are collapsed.

$$\sqrt{(x_2^2 - x_0x_1,\ x_1^2,\ x_1x_2)} = (x_2, x_1)$$

$$\rightsquigarrow U = \mathbb{P}^2 - \{(1:0:0)\}.$$

$\phi$ birational:

over $u_1 \neq 0$: $\quad \dfrac{x_2}{x_1} = \dfrac{u_2}{u_1}$

$$\frac{u_0}{u_1} = \frac{x_2^2 - x_0x_1}{x_1^2} = \frac{u_2^2}{u_1^2} - \frac{x_0}{x_1} \rightsquigarrow \frac{x_0}{x_1} = \frac{u_2^2 - u_1u_0}{u_1^2}$$

birational.

on $u_1 = 0$: $\quad x_1 = 0$

$Z(x_1)$ gets contracted to $(1:0:0)$ ✓

8.5 Let $\phi\colon \mathbb{A}^2 \to \mathbb{A}^4$ be the map defined by

$$\phi(x,y) = (x, xy, y(y-1), y^2(y-1)).$$

$$\quad u_1 \; u_2 \quad u_3 \qquad u_4$$

a) Show that $\phi(0,0) = \phi(0,1) = (0,0,0,0)$, and that $\phi$ is injective on $U = \mathbb{A}^2 \backslash \{(0,0),(0,1)\}$.

b) Show that $\phi|_U$ is an isomorphism between $U$ and its image. HINT: $\phi|_U$ takes values in $V = \mathbb{A}^4 \backslash Z(x)$ and the map $V \to \mathbb{A}^2$ sending $(u,v,w,t)$ to $(u,v)$ is a left section for $\phi|_U$.

$(u, vu^{-1})$

c) Show that the image of $\phi$ is given by the polynomials $ut - vw$, $w^3 - t(t-w)$ and $u^2w - v(v-u)$.

a) $\phi(0,0) = (0,0,0,0)$ ✓

$\phi(0,1) = (0,0,0,0)$ ✓

$\phi$ injective on $\mathbb{A}^2 - \{(0,0),(0,1)\}$:

If $x \neq 0$ then we can recover $x = u_1$
$y = u_2/u_1$

If $x = 0$, the morphism is

$(0, 0, y(y-1), y^2(y-1)) \rightsquigarrow y = \dfrac{u_3}{u_2}$ ✓

b)

$$\begin{tikzcd} \phi|_U : U \arrow[r] \arrow[dr] & V = \mathbb{A}^4 - Z(x) \arrow[d] \\ & (u, vu^{-1}) \end{tikzcd}$$

is a left section

$\Rightarrow \phi|_U$ is an embedding

c) $f_1 = ut - vw$ $\quad X = \phi(\mathbb{A}^2)$ $\quad I = (f_1, f_2, f_3)$

$f_2 = w^3 - t(t-w)$ $\quad \rightsquigarrow$ codim $I = 2$ since $X$ has dim 2.

$f_3 = u^2 w - v(v-u)$ $\quad \mathbb{A}^4$

<u>Claim:</u> $Z(I) =: W$ is irreducible.

In $D_+(u)$ set $u=1$ $\quad \mathbb{A}^3$

$$I|_{u=1} = (t - vw,\ v^2 - v - w,\ w^3 + wt - t^2)$$

$t = vw$

eliminate $t$: $\quad (v^2 - v - w)$ $\quad \mathbb{A}^2$ $\quad$ irreducible ✓

There could be an irred component in $u=0$.

Check $D_+(v)$ $\quad v=1$

$$I|_{v=1} = (w - ut,\ w^3 + wt - t^2,\ u^2 w + u - 1)$$

eliminate $w$: $\quad (u^3 t + u - 1)$ $\quad$ irreducible ✓

If $Z(I)$ is not irreducible, there would be a component in $u = v = 0$.

In $D_+(w)$: $\quad w = 1$

$$(ut - v,\ -t^2 + t + 1,\ u^2 + uv - v^2)$$

eliminate $v$: $\quad (t^2 - t - 1)$ $\quad$ irreducible.

We end at the point $u = v = w = 0$ — but this is clearly in the closure of $Z(I|_{u=1})$ ✓

**9.1** Assume that $\nu$ is a discrete valuation on a field $K$. Show that the set $A = \{ x \in K \mid \nu(x) \geqslant 0 \}$ is discrete valuation ring by showing that $\{ x \in K \mid \nu(x) > 0 \}$ is a maximal ideal generated by one element.

$A$ is a subring of $K$ ✓

$v: K \longrightarrow \mathbb{Z}$

Let $m = \{ x \in K \mid v(x) > 0 \}$.

$v(xy) = v(x) + v(y)$

$m$ is an ideal ✓

$v(x+y) = \min(v(x), v(y))$

$A - m = \{ x \in K \mid v(x) = 0 \}$

$v(0) = \infty$

if $y \in A - m$, then also $y^{-1} \in A - m \Rightarrow A - m$ consists of units

$\Rightarrow A$ is a local ring with maximal ideal $m$.

$m$ is principal:

wlog $v: K \to \mathbb{Z}$ is surjective (otherwise let $v' = \frac{v}{N}$, where $\langle v(K) \rangle = (N) \subseteq \mathbb{Z}$)

Let $t \in K$ be an element with $v(t) = 1$

$\longrightarrow t \in m \subset A$

Claim $m = (t)$.

If $a \in m \Rightarrow v(a) \geqslant 1 \Rightarrow v(at^{-1}) = v(a) - v(t^{-1}) \geqslant 0$

$\Rightarrow at^{-1} \in A \Rightarrow a = (at^{-1}) \cdot t \in (t) \Rightarrow$ ok.

10.4 Let $\phi\colon X \to Y$ be a morphism of varieties and $r \in \mathbb{N}_0$ a non-negative integer. Show that the set $\{ y \in Y \mid \dim \phi^{-1}(y) = r \}$ is locally closed.

open $\cap$ closed

We showed that the sets

$$W_r = \{ y \in Y \mid \dim \phi^{-1}(y) \geq r \}$$

are closed.

We are interested in $W_r - W_{r+1}$; this is open in $W_r$

induced top.

$\Rightarrow$ $W_r - W_{r+1} = U \cap W_r$ is locally closed

**Problem 10.9** Asume that $p$ and $q$ are two relatively prime numbers. Let $C \subseteq \mathbb{A}^2$ be the image of the map $\phi\colon \mathbb{A}^1 \to \mathbb{A}^2$ given as $t \mapsto (t^p, t^q)$. Show that $C = Z(x^q - y^p)$. Prove that $\phi$ is a finite map and determine all fibres of $\phi$. ★

The map is induced by

$$k[x,y] \xrightarrow{\theta} k[t]$$
$$x \longrightarrow t^p$$
$$y \longrightarrow t^q$$

$(p,q)=1 \rightsquigarrow \ker\theta = (y^p - x^q)$ is prime

$$\therefore \frac{k[x,y]}{y^p - x^q} \simeq k[t^p, t^q] \subseteq k[t]$$

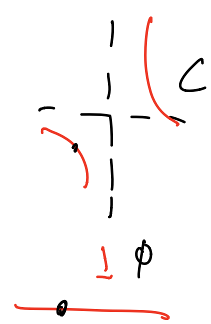
and $k[t]$ is finite as a module over $k[t^p, t^q]$ ✓

$\phi$ quasifinite if $\phi^{-1}(q)$ finite $\forall q \in Y$.

$\not\Downarrow$

finite

$xy = 1$




$\phi$ quasi finite ✓

$\phi$ not finite: $A(C) = \dfrac{k[x,y]}{xy-1} \simeq k[x, x^{-1}]$

not finite as a $k[x]$-module