PROBLEM 11.1 Along the lines above, prove that a conic intersects a curve of degree n in $2n$ points multiplicities taken into account unless the conic is a component of the curve. HINT: Parameterize the conic as $(u^2, uv : v^2)$. ★

Since all conics Q in $\mathbb{P}^2$ are projectively equivalent, we may assume Q is given by $Q = Z_+(xz - y^2)$ that is, the image of

$$\begin{aligned} \mathbb{P}^1 &\xrightarrow{\phi} \mathbb{P}^2 \\ (u:v) &\longmapsto (u^2 : uv : v^2) \end{aligned}$$

← Veronese embedding of $\mathbb{P}^1 \hookrightarrow \mathbb{P}^2$

Now let $C = Z_+(F)$ be a curve of degree n.
Then $C \cap Q = Z_+(xz - y^2, F)$ is given by the solutions to the equation

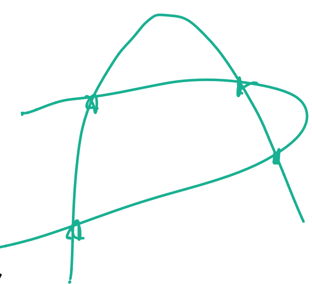
$$\phi^* F = F(u^2, uv, v^2) = 0 \quad = \prod_{i=1}^{2n} (a_i u + b_i v)$$

This is a binary form of degree $2n$ in u, v
$\rightsquigarrow$ exactly $2n$ solutions (up to scaling).

PROBLEM 11.3 Find the intersection and the local multiplicities of the three surfaces in $\mathbb{P}^3$ given by $xy - zw$, $xz - yw$ and $xw - yz$. ★

By Bezout, we expect 8 intersection pts (with multiplicities).
Note that the ideal I generated by the quadrics is symmetric w.r.t. permuting the variables.

We find some points by inspection:



$(1:0:0:0)$ ⇝ gives 4 points

$(1:1:1:1)$ ⇝ gives 1 point

$(-1:-1:1:1)$ ⇝ gives 3 points

⇝ 8 intersection pts and all have $\mu_p = 1$.

$D_+(x)$ $\quad x=1$

$$\mathcal{O}_p = \left(\frac{k[y,z,w]}{(y - zw,\ z - yw,\ w - yz)} \right)_{(y,z,w)} \cong k$$

$$z = yw = y^2 z$$
$$w = yz = y^2 w$$

$$w(\underbrace{1-y^2}_{\text{invertible}}) = 0 \implies w = 0$$

3 lines + one conic

PROBLEM 11.4 Prove that $xy - zw$ and $x^2y - z^2x$ intersect along ~~five lines~~. Find the intersection of $y - x$, $xy - zw$ and $x^2y - z^2x$. ★

$$xy = zw$$
$$x^2 y = z^2 x$$

If $x = 0$, then $x = zw = 0$ describe the two lines

$$\underset{C_1}{Z(x,w)} \cup \underset{C_2}{Z(x,z)}$$

If $x \neq 0$, then $y = x^{-1} zw$

$$\rightsquigarrow x^2(x^{-1}zw) = z^2 x$$

$$\longrightarrow x zw = z^2 x \longrightarrow zw - z^2 = 0$$
$$zw - z^2 = z(w - z)$$

$\rightsquigarrow$ two components

$$\underset{C_3}{Z(z, y)} \cup \underset{C_4}{Z(w - z, xy - z^2)}$$

Intersecting with $y - z = 0$:

$$C_1 \cap Z(y - z) = Z(x, y - z, w) \qquad (0:1:1:0)$$

$$C_2 \cap Z(y - z) = Z(x, z, y - z) \qquad (0:0:0:1)$$

$$C_3 \cap Z(y - z) = Z(z, y, y - z) \qquad \text{line } Z(y, z)$$

$$C_4 \cap Z(y - z) = Z(w - z, xy - z^2, y - z)$$
$$= \text{two pts: } (1:1:1:-1) \quad (1:0:0:0)$$

PROBLEM 11.5 Let $F = (x_0^2 + x_1^2)x_2 + x_0^3 + x_1^3$ and $G = x_0^3 + x_1^3 - 2x_0x_1x_2$. Find all intersection points of $C = Z_+(F)$ and $D = Z_+(G)$ in $\mathbb{P}^2$ and compute their multiplicities. ★

$$F = (x^2+y^2)z + x^3 + y^3$$
$$G = x^3 + y^3 - 2xyz$$

Consider $I = (F,G)$ and $X = Z_+(I) \subset \mathbb{P}^2$.

We check the affine charts:

$U_0 = D_+(x) \simeq \mathbb{A}^2$ $\qquad X \cap U_0 \subset \mathbb{A}^2$ is defined by

$$f = (1+u^2)v + 1 + u^3 \qquad u = \frac{x_1}{x_0}, \quad v = \frac{x_2}{x_0}$$
$$g = 1 + u^3 - 2uv$$

$J = (f,g)$ has "primary decomposition"

$$J = \underbrace{(3u+3v+3,\ v^3)}_{q_1} \cap \underbrace{(v,\ u^2-u+1)}_{q_2}$$

If $p \in Z(q_1)$ then

$$\mathcal{O}_{X,p} \simeq \left(\frac{k[u,v]}{(3u+3v+3,\ v^3)}\right)_{m_p} \overset{\text{eliminate } u}{\simeq} \frac{k[v]}{v^3} \qquad \text{one point.} \rightsquigarrow \mu_p = \underline{3}$$

If $p \in Z(q_2)$ then

$$\mathcal{O}_{X,p} \simeq \left(\frac{k[u,v]}{(v,\ u^2-u+1)}\right)_{m_p} \overset{\text{elimate } v}{\simeq} \left(\frac{k[u]}{(u-\alpha)(u-\beta)}\right)_{m_p} \qquad \text{two points:} \rightsquigarrow \mu_p = 1$$

$\rightsquigarrow$ Checking the remaining points where $x=0$:

$\left(u^2+v^2+u^3+v^3,\ u^3+v^3-2uv\right)$

In $D_+(x_2)$, X is given by

$Z\left(u^2+2uv+v^2,\ u^3+v^3-2uv\right) = Z(u,v)$

$u = x_0/x_2$

$v = x_1/x_2$

$$\left(\frac{k[u,v]}{\left((u+v)^2,\ u^3+v^3-2uv\right)}\right)_m = \left(\frac{k[u,v]}{\left((u+v)^2,\ uv\right)}\right)_m = \left(\frac{k[u,v]}{(u^2+v^2,\ uv)}\right)_m \cong \left(\frac{k[u,v]}{(uv,\ u^2+v^2,\ v^3,\ u^3)}\right)_m$$

$v^3 = v(u^2+v^2) - u(uv)$

$\left((u+v)^2,\ u^3+v^3-2uv\right)$

$= \left((u+v)^2,\ uv(3u+3v+2)\right)$

invertible in

$= k \oplus ku \oplus kv \oplus ku^2$

$\rightsquigarrow \mu_p = 4$

$\therefore$ one point of multiplicity 4

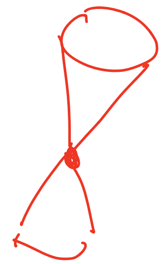
one point of multiplicity 3

two points of multiplicity 1

$\left(\text{so} \quad \sum \mu_p = 9 = 3\cdot 3\right)$

Problem 11.6 With the set up og Examle 11.5, show that any line through the origin that is not contained in $Z(xy - z^2)$, meets $Z(xy - z^2)$ with multiplicy two there. ★

(u^2, uv, v^2)



$$L = (ax + \underline{by} + cz,\ ex + \underline{fy} + gz)$$

$$ax + by + cz = 0$$
$$ex + fy + gz = 0$$

wlog $af - be \neq 0$

$\rightsquigarrow$ may assume $L = (x - cz,\ y - gz)$

$$\frac{k[x,y,z]}{(x - cz,\ y - gz,\ xy - z^2)} \cong \frac{k[z]}{(cz)(gz) - z^2} = \frac{k[z]}{((cg-1)z^2)}$$

If $cg - 1 = 0$, then $L = (x - cz,\ y - c^{-1}z)$

lies on Q: $(cz)(c^{-1}z) - z^2 = 0$.

$$\therefore\ cg - 1 \neq 0 \implies \frac{k[z]}{((cg-1)z^2)} \cong \frac{k[z]}{z^2}$$

$$\therefore\ \mu_p = \dim_k \left(\frac{k[z]}{z^2} \right)_m = \underline{2}$$

PROBLEM 11.10 Let $n > m$ be two natural numbers and let $\alpha(x)$ and $\beta(x)$ be two polynomials which do not vanish at $x = 0$. Determine the local intersection multiplicity at the origin of the two curves defined respectively by $y - \alpha(x)x^n$ and $y - \beta(x)x^m$. If $m = n$, show by exhibiting an example that the local multiplicity can take any integral value larger than n. ★

$m \leq n$:

$$\left(\frac{k[x,y]}{(y - a(x)x^n,\ y - \beta(x)x^m)}\right)_{(x,y)} = \left(\frac{k[x]}{a(x)x^n - \beta(x)x^m}\right)_{(x)}$$

$a(0) \neq 0 \quad b(0) \neq 0 \implies a, b$ invertible in $k[x]_{(x)}$

$$= \frac{k[x]_{(x)}}{(x^m + ab^{-1}(x)\,x^n)}$$

$m < n$

$1, x, x^2, \dots, x^{m-1}$

$\rightsquigarrow$ multiplicity $= m$

$m = n$

Take a, b so that $a(x)x^n - b(x)x^n = x^{n+t}$

$$\| $$

$(a(x) - b(x))\,x^n$

$\rightsquigarrow$ $b(x) = 1$
$a(x) = 1 + x^t$ works.

Problem 11.11 Find all intersection points of the two cubic curves defined by the forms $zy^2 - x^3$ and $zy^2 + x^3$ (we assume the characteristic of the ground field to be different from two). Determine all the local intersection multiplicities of the two curves. ★

$$X = Z(F,G)$$

$$F = zy^2 - x^3$$
$$G = zy^2 + x^3$$

$\rightsquigarrow$ ≤ 9 intersection points with multiplicity.

If $z=0$, then $x=0$ $\rightsquigarrow$ $(0:1:0) \in X$

$\therefore$ Suffices to check $D_+(z)$ and $D_+(y)$

$D_+(z) \cong \mathbb{A}^2$

$$f = y^2 - x^3$$
$$g = y^2 + x^3$$

$\to$ intersect only at $0 = (0,0)$!

$$\frac{k[x,y]}{(y^2-x^3, y^2+x^3)} \underset{\text{char } k \neq 2}{\cong} \frac{k[x,y]}{(y^2+x^3, 2y^2)} \cong \frac{k[x,y]}{(x^3, y^2)}$$

$\therefore \mathcal{O}_{X,0}$ has length 6 : basis $1, x, y, x^2, xy, x^2y$

$\to$ mult 6 at $(0:0:1)$

$D_+(y)$:

$$f = z - x^3$$
$$g = z + x^3$$

$$\frac{k[x,z]}{(z-x^3, z+x^3)} \cong \frac{k[x]}{(2x^3)}$$
$$\shortparallel$$
$$\mathcal{O}_{X,0}$$

$\to$ multiplicity 3 at $(0:1:0)$

PROBLEM 11.12 Let X and Y be two curves in $\mathbb{P}^2$ being the zero loci of the polynomials $z^5y^2 - x^3(z^2 - x^2)(2z^2 - x^2)$ and $z^5y^2 + x^3(z^2 - x^2)(2z^2 - x^2)$. Determine all intersection points and the local multiplicities in all the intersection points of X and Y ★

$$F = z^5y^2 - x^3(z^2 - x^2)(2z^2 - x^2)$$
$$G = z^5y^2 + x^3(z^2 - x^2)(2z^2 - x^2)$$
$\rightarrow$ at most **49** points in the intersection

$$F - G = 2x^3(z^2 - x^2)(2z^2 - x^2)$$

$\rightarrow$ $x = 0$ or $x = z$ or $x = -z$ or $x = \sqrt{2}z$ or $x = -\sqrt{2}z$

$x = 0 \Rightarrow y = 0$ or $z = 0 \Rightarrow (0:1:0)$ ✓ 35; $(0:0:1)$ ✓ 6

$x = z \Rightarrow y = 0$ or $z = 0 \Rightarrow (1:0:1)$ ✓ 2; $(0:1:0)$

$x = -z \Rightarrow y = 0$ or $z = 0 \Rightarrow (1:0:-1)$ ✓ 2; $(0:1:0)$

$x = \pm\sqrt{2}z$ $\quad (\pm\sqrt{2} : 0 : 1)$ ✓ 2+2

$\Rightarrow$ 6 points in total $\qquad$ 49

$P = (0:1:0)$:

$$D(y):\quad (F, F-G)\big|_{y=1} = \big(z^5 - x^3(z^2-x^2)(2z^2-x^2),$$
$$2x^3(z^2-x^2)(2z^2-x^2)\big)$$
$$= \big(z^5,\ 2x^3(z^2-x^2)(2z^2-x^2)\big)$$

Additivity:

$$\mu_p = \mu_p(z^5, x^3) + \mu_p(z^5, z-x) + \mu_p(z^5, z+x)$$
$$+ \mu_p(z^5, \sqrt{2}z - x) + \mu_p(z^5, \sqrt{2}z + x)$$

$m=(x,z)$:

$\dfrac{k[x,z]_m}{(x^3, z^5)_m}$ has length $3 \cdot 5 = 15$

$\dfrac{k[x,z]}{(z^5, z-x)} = \dfrac{k[x]}{(x^5)}$ has length 5 etc

$$\rightsquigarrow \quad \mu_p = 15 + 5 + 5 + 5 + 5 = \underline{35}$$

$D(z)$:
$$\big(y^2 - x^3(1-x^2)(2-x^2),\ y^2\big)$$
$$= \big(y^2,\ x^3(1-x)(1+x)(\sqrt{2}-x)(\sqrt{2}+x)\big)$$

Additivity: $p=(0,0)$:

$$\mu_p = \mu_p(x^3, y^2) = 6$$

$p = (x, y) = (1, 0)$

$$\mu_p = \mu_p(y^2, 1-x) = 2$$

$p = (x, y) = (-1, 0)$

$$\mu_p = \mu_p(y^2, 1+x) = 2$$

PROBLEM 11.13 Let C be the curve given as $zy^2 - x(x-z)(x-2z)$. Determine the intersection points and the local multiplicities that X has withe the line $z = 0$. Same task, but with the line $x - z = 0$. ★

$z = 0 \Rightarrow x = 0 \Rightarrow p = (0:1:0)$ (this must have $\mu_p = 3$, by Bezout)

$D(y)$:

$$z - x(x-z)(x-2z)$$

$$\left(\frac{k[x,z]}{(z - x(x-z)(x-2z), z)}\right)_{(x,z)} \simeq \left(\frac{k[x]}{(x^3)}\right)_{(x)} \quad \text{has length } 3.$$

$x = z$

$\Rightarrow$ $y = 0$ $(1:0:1)$ or

$z = 0$ $(0:1:0)$

$D(z)$: $y^2 - x(x-1)(x-2) = 0$

$$\frac{k[x,y]}{(y^2 - x(x-1)(x-2), x-1)} \simeq \frac{k[y]}{y^2} \qquad \mu_{(1:0:1)} = 2$$

$D(y)$: $$\frac{k[x,z]}{(z - x(x-z)(x-2z), x-z)} \simeq \frac{k[x]}{x} \qquad \underline{\mu = 1}$$