Mandatory assignment MAT4210 – Spring 2022

The assignment must be submitted via Canvas by **14:30, Thursday February 24th**.

You need to solve at least 3.5 problems to pass. If you have any questions or comments about the problems, feel free to email me at `johnco@math.uio.no`.

All varieties are over the field $k = \mathbb{C}$.

Problem 1. Show that a quasiaffine variety is quasiprojective. Is the converse true?

By definition, a quasiaffine variety V is an open set in some affine variety $X \subset \mathbb{A}^n$. Let $X - V = Z(I)$,where $I \subset A(X)$ is an ideal. Embed $\mathbb{A}^n$ in $\mathbb{P}^n$ as the distinguished open $D_+(x_0)$. Denoting I^h by the homogenization of I with respect to x_0 (a homogeneous ideal in $x_0, \dots, x_n$), we have $V = \overline{X} - Z(x_0) \cup Z(I^h)$, so V is quasiprojective.

The converse is not true: $\mathbb{P}^1$ or $\mathbb{P}^2 - (0:0:1)$ are counterexamples (open sets in affine varieties can not contain $\mathbb{P}^1$'s).

Problem 2. a) Which of the following varieties are isomorphic?

(1) $\mathbb{A}^2$.
(2) $\mathbb{P}^1 \times \mathbb{P}^1$.
(3) $Z(f) \subset \mathbb{A}^3$, where $f = x + y + z + 1$
(4) $Z(f) \subset \mathbb{A}^3$, where $f = x^2 + y^2 + z^2 + 1$.
(5) $Z_+(F) \subset \mathbb{P}^3$, where $F = x_0^2 + x_1^2 + x_2^2 + x_3^2$.
(6) $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ defined by $x_0y_0 - x_1y_1 = 0$.

b) For each variety X in problem a), compute $\mathcal{O}_X(X)$.

Label the varieties as $X_1, \dots, X_6$.
X_1, X_3, X_4 are affine varieties, with

- $A(X_1) = k[x, y]$
- $A(X_3) = k[x, y, z]/(x + y + z + 1) \simeq k[x, y]$
- $A(X_4) = k[x, y, z]/(x^2 + y^2 + z^2 + 1)$

So $X_1 \simeq X_3$, but $X_1 \not\simeq X_4$.

The varieties X_2 and X_5 are projective, and in fact isomorphic, since $\mathbb{P}^1 \times \mathbb{P}^1$ embeds as a non-singular quadric in $\mathbb{P}^3$ and all non-singular quadrics are isomorphic (via a linear coordinate change).

X_6 is neither affine nor projective. Not affine: X_6 contains the $\mathbb{P}^1$ given by $(0, 0) \times \mathbb{P}^1$. Not projective: X_6 admits a non-constant global regular function: $X_6 \to \mathbb{A}^2 \to \mathbb{A}^1$.

b) X_1, X_3, X_4 are affine, so $\mathscr{O}_X(X)$ is given by $A(X_1), A(X_2), A(X_4)$ above.
X_2, X_5 are projective, so they have $\mathscr{O}_X(X) = k$.
For $X = X_6$, note that the first projection induces a map

$$k[x_0, x_1] \to \mathscr{O}_X(X)$$

We claim that this is in fact an isomorphism.
To prove this, note that $\mathbb{A}^2 \times \mathbb{P}^1$ has a covering by two open subsets

$$D_+(y_0) \simeq \mathbb{A}^3, \quad \text{and} \quad D_+(y_1) \simeq \mathbb{A}^3,$$

On $D_+(y_0)$, we have coordinates x_0, x_1 and $u = \frac{y_1}{y_0}$, while on $D_+(y_1)$ we have coordinates x_0, x_1 and $v = \frac{y_0}{y_1}$. In U, X is defined by the equation $x_0 - x_1 u = 0$, so X is isomorphic to $\mathbb{A}^2$ with coordinates x_1, u. Similarly, $V \simeq \mathbb{A}^2$ with coordinates x_0, v.

Now, take an element $f \in \mathscr{O}_X(X)$. $f|_U$ is an element of $\mathscr{O}_X(U) = k[x_1, u]$, thus a polynomial $P(x_1, u)$, whereas $f|_V = Q(x_0, v)$ is an element of $k[x_0, v]$. These polynomials have to agree on the intersection $U \cap V$; and here we are allowed to write $v = u^{-1}$ and $x_0 = x_1 u$. Thus

$$P(x_1, u) = Q(x_1 u, u^{-1})$$

If we write $Q(z, w) = \sum a_{ij} z^i w^j$, the only way that the right hand side is a polynomial in x_1 and u, is that each monomial $z^i w^j$ satisfies $i \geq j$. Thus Q is a polynomial in z $(= x_1 u = x_0)$ and zw $(= x_1 u \cdot u^{-1} = x_1)$. This implies that f is in fact given by a polynomial in x_0, x_1.

Problem 3. Consider the closed algebraic set $Z_+(I) \subset \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$x_1 y_0 - x_0 y_1 = x_2 y_0 - y_2 x_0 = 0$$

Compute its dimension and describe its irreducible components.

Let $W = Z(I)$ where $I = (x_1y_0 - x_0y_1, x_2y_0 - y_2x_0)$ in $R = k[x_0, x_1, x_2, y_0, y_1, y_2]$.
Let $p = (x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2) \in W$. If $x_0 \neq 0$, then wlog $x_0 = 1$, so $x_1y_0 - y_1 = x_2y_0 - y_2 = 0$, thus

$$p = (1 : x_1 : x_2) \times (y_0 : x_1y_0 : x_2y_0) = (1 : x_1 : x_2) \times (1 : x_1 : x_2)$$

(the equations imply that $y_0 \neq 0$ also). Thus $p \in \Delta$, where $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the diagonal, defined by the (prime) ideal

$$I(\Delta) = (x_1y_0 - x_0y_1, x_2y_0 - y_2x_0, x_1y_2 - x_2y_1)$$

If $x_0 = 0$, then $x_1y_0 = x_2y_0 = 0$. The only possibility here is that $y_0 = 0$, so $p \in Z(x_0, y_0)$. We have shown that $W \subset \Delta \cup Z(x_0, y_0)$. Conversely, we clearly have $\Delta \cup Z(x_0, y_0) \subseteq W$. These components are clearly irreducible, so

$$W = \Delta \cup Z(x_0, y_0)$$

is the decomposition into irreducibles.

$\dim \Delta = \dim \mathbb{P}^2 = 2$ and $\dim Z(x_0, y_0) = \dim \mathbb{P}^1 \times \mathbb{P}^1 = 2$, so $\dim W = 2$.

Problem 4. Consider the cubic surface

$$X = Z_+(x_0x_1^2 - x_2x_3^2) \subset \mathbb{P}^3$$

i) Compute all singular points of X;
ii) Show that X is rational.

We note that X is irreducible. The jacobian is given by

$$J = \begin{pmatrix} x_1^2 & 2x_0x_1 & -x_3^2 & -2x_2x_3 \end{pmatrix}$$

Thus $p \in X$ is a singular point $\Leftrightarrow rk\, J = 0 \Leftrightarrow p \in Z(x_1^2, 2x_0x_1, -x_3^2, -2x_2x_3) = Z(x_0, x_3)$. Thus

$$\text{sing}(X) = X(x_0, x_3)$$

Consider $D_+(x_3) \simeq \mathbb{A}^3$ and the corresponding open set $U = Z(xy^2 - z) \subset X$. Hence a birational parameterization is given by

$$\begin{array}{ccc} \mathbb{A}^2 & \dashrightarrow & U \\ (u, v) & \mapsto & (u, v, uv^2) \end{array}$$

The inverse is given by the projection $U \to \mathbb{A}^2$ onto the two first factors.

Problem 5. Consider the quotient space $X = \mathbb{A}^3 - 0/\sim$ where the equivalence relation is defined by

$$(x, y, z) \sim (tx, ty, t^2z).$$

a)* Show that X has the structure of a variety.
b) Compute $\mathscr{O}_X(X)$.
c) Show that X admits an embedding $X \hookrightarrow \mathbb{P}^3$ as a quadric surface. Deduce that X is projective.
d) Find all singular points of X.

(Possible hints: Work on the "distinguished open sets" $D_+(x), D_+(y), D_+(z)$. In one of the charts, you will see the quadric cone $v^2 = uw$. There is also a convenient map $\mathbb{A}^3 - 0 \to \mathbb{P}^3$. You may solve these problems in any order, if that makes it easier.)

Note that X is a topological space via the quotient topology. We let $\pi : \mathbb{A}^3 - 0 \to X$ for the quotient map.

We will equip X with a sheaf to make it a ringed space, and subsequently show that X is a prevariety and finally a variety.

Let $R = k[x, y, z]$ with the grading $\deg x = \deg y = 1$ and $\deg z = 2$. We will use the notation $D(x)$, $D(y)$, $D(z)$ for the distinguished open sets.

Define $\mathscr{O}_X$ by letting $\mathscr{O}_X(U)$ consist of the continuous functions $f : U \to k$ which are locally of the form g/h where $g, h \in R$ are homogeneous elements of the same degree. Thus $\frac{x^2}{z} \in \mathscr{O}_X(D(z))$, but $\frac{x}{z} \notin \mathscr{O}_X(D(z))$. It is clear that this satisfies the Gluing axiom (same proof as for $\mathbb{P}^2$), so we have a ringed space $(X, \mathscr{O}_X)$.

Consider the map

$$\begin{array}{rcl} \Phi : \mathbb{A}^3 - 0 & \to & \mathbb{P}^3 \\ (x, y, z) & \mapsto & (x^2 : xy : y^2 : z) \end{array}$$

Note that $\Phi(tx, ty, t^2z) = \Phi(x, y, z)$, so we get an induced continuous map

$$\phi : X \to \mathbb{P}^3.$$

Write u_0, u_1, u_2, u_3 for the homogeneous coordinates on $\mathbb{P}^3$.

Claim: ϕ is a homeomorphism onto the quadric $Q = Z(u_1^2 - u_0u_2)$.
We check this on the charts $D(x)$, $D(y)$, $D(z)$.
$D(x)$:
Let us restrict Φ to the subvariety $A_1 = Z(x - 1) \subset \mathbb{A}^3 - 0$ which is isomorphic to $\mathbb{A}^2$. We note that $\Phi(A_1) \subset D_+(u_0) \simeq \mathbb{A}^3$.
Write $\pi_1 = \pi|_{A_1}$ and $\Phi_1 = \Phi|_{A_1}$ so that we have a diagram

$$\begin{array}{ccc} A_1 & \xrightarrow{\Phi_1} & Q \cap D_+(u_0) \\ & {\scriptstyle \pi_1} \searrow \quad \nearrow {\scriptstyle \phi_1} & \\ & A_1/\sim & \end{array}$$

and $Q \cap D_+(u_0) = Z(u_1^2 - u_2) \subset \mathbb{A}^3$.
Thus the restriction can be identified with the map

$$\begin{array}{rcl} \Phi_1 : \mathbb{A}^2 & \to & Z(u_1^2 - u_2) \\ (y, z) & \mapsto & (y, y^2, z) \end{array}$$

This is clearly an isomorphism, with inverse $(u_1, u_2, u_3) \mapsto (u_1, u_3)$. Note that π_1 is surjective; this implies that ϕ_1 is also a homeomorphism.
$D(y)$: The same argument as applies.
$D(z)$: Here the situation is a little different: Defining $A_3 = Z(z - 1) \subset \mathbb{A}^3 - 0$ as above, we have a diagram

$$\begin{array}{ccc} A_3 & \xrightarrow{\Phi_3} & Q \cap D_+(u_3) \\ & {\scriptstyle \pi_3} \searrow \quad \nearrow {\scriptstyle \phi_3} & \\ & A_3/\sim & \end{array}$$

The restriction π is no longer a homeomorphism: It identifies $(x, y, 1)$ and $(-x, -y, 1)$.
Here Φ restricted to $Z(z - 1)$ can be identified with the map

$$\begin{array}{rcl} \Phi : \mathbb{A}^2 & \to & Z(u_1^2 - u_2u_0) \subset \mathbb{A}^3 \\ (x, y) & \mapsto & (x^2, xy, y^2) \end{array}$$

This also identifies (x, y) with $(-x, -y)$. This means that the induced ϕ is bijective.
ϕ is also a closed map: Let $V \subset A_3/\sim$ be a closed set. By definition this means that $\pi^{-1}(V)$ is closed in $\mathbb{A}^3 - 0$, and hence $W = \pi^{-1}(V) \cap A_3$ is closed in $A_3 = \mathbb{A}^2$. We write this as $W = Z(I)$ where $I \subset k[x, y]$ is some ideal. Since W is invariant under $(x, y) \mapsto (-x, -y)$, the polynomials

that define I must be polynomials in x^2, xy, y^2. Thus $W = \Phi^{-1}(Z)$ for some closed subset $Z \subset Z(u_1^2 - u_0 u_2)$, and we have $\phi(V) = Z$ by the diagram.
(Alternatively, $k[x^2, xy, y^2] \subset k[x, y]$ is an integral extension, so by Going-Up $\operatorname{Spec} k[x, y] \to \operatorname{Spec} k[x^2, xy, y^2]$ is closed. The map Φ is simply the restriction of this map of Spec's restricted to the maximal ideals, so it is again closed. Thus ϕ is also closed.)
This means that ϕ is a homeomorphism.

Next we prove that X is a prevariety, by showing that it is locally isomorphic to an affine variety. Of course we will use the open sets $D(x)$, $D(y)$ and $D(z)$ as the affine cover, and use the homeomorphisms above. To proceed, we need to show that the homeomorphisms ϕ_i $i = 1, 2, 3$ induce isomorphisms between the structure sheaves $\mathscr{O}_X|_{D_+(x)}$ and $\mathscr{O}_{A_1}$ etc.
The proof of this is almost exactly as in the case for $\mathbb{P}^2$. For instance, for $D_+(x)$, let $U \subset D_+(x)$ be an open subset and $f \in \mathscr{O}_X(U)$. By definition, f can locally (in some $V \subset U$) be written as $f = g/h$, where $g, h \in R$ are homogeneous of the same degree. Then the homeomorphism $\pi_1 : \mathbb{A}^2 \to D_(x)$ has the property that $\pi_1^*(g(x, y, z)/(h(x, y, z)) = g(1, y, z)/h(1, y, z)$, which is regular, since $h(1, y, z) \neq 0$ in $\pi_1^{-1}(V)$. The inverse $\alpha : D(x) \to \mathbb{A}^2$ is given by $(x : y : z) \mapsto (y/x, z/x^2)$. If $f \in \mathscr{O}_{\mathbb{A}^2}(W)$ is some regular function of the form $g(y, z)/h(y, z)$ then $\alpha^* f = G(x, y, z)/H(x, y, z)$ where G, H are the homogenizations of g, h with respect to x. A similar argument works for $D(y)$ and $D(z)$.

What we have shown is that X can be covered by 3 open sets $D(x)$, $D(y)$, $D(z)$, and these are naturally isomorphic to affine varieties. We also have diagrams

$$\begin{array}{ccc} A_i & \xrightarrow{\quad\Phi_i\quad} & Q \cap D_+(u_i) \\ & {\scriptstyle\pi_i}\searrow \quad \nearrow{\scriptstyle\phi_i} & \\ & A_i/\sim & \end{array}$$

so that ϕ_i is an isomorphism for each i. But then $X \simeq Q$ as prevarieties. Consequently, X satisfies the Hausdorff axiom, since Q does (being a projective variety).

This completes the proof for a). Along the way, we have also shown c).

b) $\mathscr{O}_X(X) = k$ follows since X is isomorphic to a projective variety. But here is a more direct argument to see this. If $f : X \to k$ is a global regular function, then $\pi^* f$ is a global regular function on $\mathbb{A}^3 - 0$, hence a polynomial in $k[x, y, z]$ (for $U = \mathbb{A}^n - 0$, we have $\mathscr{O}_{\mathbb{A}^n}(U) = \mathscr{O}_{\mathbb{A}^n}(\mathbb{A}^n)$ for all $n \geq 2$). But $\pi^* f$ is also invariant under the scaling $x \mapsto tx$ $y \mapsto ty$ and $z \mapsto t^2 z$ for $r \in k^\times$. This can only happen if $\phi^* f$ is in fact constant. But then f is also constant.

d) By a) X is the union of $D(x) = \mathbb{A}^2$ $D(y) = \mathbb{A}^2$ and $D(z) = Z(uw - v^2)$. Thus the only singular points can lie in $D(z)$, and there is only one, namely the point $p = (0, 0, 1)$.

Alternatively, one can use the projective embedding $X \subset \mathbb{P}^3$, plus the jacobian criterion to get the same conclusion.