Mandatory assignment MAT4210 – Spring 2022

The assignment must be submitted via Canvas by 14:30, Thursday February 24th. You need to solve at least 3.5 problems to pass. If you have any questions or comments about the problems, feel free to email me at johnco@math.uio.no.

All varieties are over the field $k = \mathbb{C}$.

Problem 1. Show that a quasiaffine variety is quasiprojective. Is the converse true?

By definition, a quasiaffine variety V is an open set in some affine variety $X \subset \mathbb{A}^n$. Let X - V = Z(I), where $I \subset A(X)$ is an ideal. Embed \mathbb{A}^n in \mathbb{P}^n as the distinguished open $D_+(x_0)$. Denoting I^h by the homogenization of I with respect to x_0 (a homogeneous ideal in x_0, \ldots, x_n), we have $V = \overline{X} - Z(x_0) \cup Z(I^h)$, so V is quasiprojective.

The converse is not true: \mathbb{P}^1 or $\mathbb{P}^2 - (0:0:1)$ are counterexamples (open sets in affine varieties can not contain \mathbb{P}^1 's).

Problem 2. a) Which of the following varieties are isomorphic?

(1) \mathbb{A}^2 . (2) $\mathbb{P}^1 \times \mathbb{P}^1$. (3) $Z(f) \subset \mathbb{A}^3$, where f = x + y + z + 1(4) $Z(f) \subset \mathbb{A}^3$, where $f = x^2 + y^2 + z^2 + 1$. (5) $Z_+(F) \subset \mathbb{P}^3$, where $F = x_0^2 + x_1^2 + x_2^2 + x_3^2$. (6) $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ defined by $x_0y_0 - x_1y_1 = 0$.

b) For each variety X in problem a), compute $\mathcal{O}_X(X)$.

Label the varieties as X_1, \ldots, X_6 . X_1, X_3, X_4 are affine varieties, with

- $A(X_1) = k[x, y]$
- $A(X_3) = k[x, y, z]/(x + y + z + 1) \simeq k[x, y]$ $A(X_4) = k[x, y, z]/(x^2 + y^2 + z^2 + 1)$

So $X_1 \simeq X_3$, but $X_1 \not\simeq X_4$.

The varieties X_2 and X_5 are projective, and in fact isomorphic, since $\mathbb{P}^1 \times \mathbb{P}^1$ embeds as a non-singular quadric in \mathbb{P}^3 and all non-singular quadrics are isomorphic (via a linear coordinate change).

 X_6 is neither affine nor projective. Not affine: X_6 contains the \mathbb{P}^1 given by $(0,0) \times \mathbb{P}^1$. Not projective: X_6 admits a non-constant global regular function: $X_6 \to \mathbb{A}^2 \to \mathbb{A}^1$.

b) X_1, X_3, X_4 are affine, so $\mathscr{O}_X(X)$ is given by $A(X_1), A(X_2), A(X_4)$ above. X_2, X_5 are projective, so they have $\mathscr{O}_X(X) = k$. For $X = X_6$, note that the first projection induces a map

 $k[x_0, x_1] \to \mathscr{O}_X(X)$

We claim that this is in fact an isomorphism.

To prove this, note that $\mathbb{A}^2 \times \mathbb{P}^1$ has a covering by two open subsets

$$D_+(y_0) \simeq \mathbb{A}^3$$
, and $D_+(y_1) \simeq \mathbb{A}^3$,

On $D_+(y_0)$, we have coordinates x_0, x_1 and $u = \frac{y_1}{y_0}$, while on $D_+(y_1)$ we have coordinates x_0, x_1 and $v = \frac{y_0}{y_1}$. In U, X is defined by the equation $x_0 - x_1 u = 0$, so X is isomorphic to \mathbb{A}^2 with coordinates x_1, u . Similarly, $V \simeq \mathbb{A}^2$ with coordinates x_0, v .

Now, take an element $f \in \mathscr{O}_X(X)$. $f|_U$ is an element of $\mathscr{O}_X(U) = k[x_1, u]$, thus a polynomial $P(x_1, u)$, whereas $f|_V = Q(x_0, v)$ is an element of $k[x_0, v]$. These polynomials have to agree on the intersection $U \cap V$; and here we are allowed to write $v = u^{-1}$ and $x_0 = x_1 u$. Thus

$$P(x_1, u) = Q(x_1 u, u^{-1})$$

If we write $Q(z, w) = \sum a_{ij} z^i w^j$, the only way that the right hand side is a polynomial in x_1 and u, is that each monomial $z^i w^j$ satisfies $i \geq j$. Thus Q is a polynomial in $z (= x_1 u = x_0)$ and $zw \ (= x_1 u \cdot u^{-1} = x_1)$. This implies that f is in fact given by a polynomial in x_0, x_1 .

Problem 3. Consider the closed algebraic set $Z_+(I) \subset \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$x_1y_0 - x_0y_1 = x_2y_0 - y_2x_0 = 0$$

Let W = Z(I) where $I = (x_1y_0 - x_0y_1, x_2y_0 - y_2x_0)$ in $R = k[x_0, x_1, x_2, y_0, y_1, y_2]$. Let $p = (x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2) \in W$. If $x_0 \neq 0$, then wlog $x_0 = 1$, so $x_1y_0 - y_1 = x_2y_0 - y_2 = 0$, thus

$$p = (1:x_1:x_2) \times (y_0:x_1y_0:x_2y_0) = (1:x_1:x_2) \times (1:x_1:x_2)$$

(the equations imply that $y_0 \neq 0$ also). Thus $p \in \Delta$, where $\Delta \subset \mathbb{P}^2 \times \mathbb{P}^2$ is the diagonal, defined by the (prime) ideal

$$I(\Delta) = (x_1y_0 - x_0y_1, x_2y_0 - y_2x_0, x_1y_2 - x_2y_1)$$

If $x_0 = 0$, then $x_1y_0 = x_2y_0 = 0$. The only possibility here is that $y_0 = 0$, so $p \in Z(x_0, y_0)$. We have shown that $W \subset \Delta \cup Z(x_0, y_0)$. Conversely, we clearly have $\Delta \cup Z(x_0, y_0) \subseteq W$. These components are clearly irreducible, so

$$W = \Delta \cup Z(x_0, y_0)$$

is the decomposition into irreducibles.

 $\dim \Delta = \dim \mathbb{P}^2 = 2$ and $\dim Z(x_0, y_0) = \dim \mathbb{P}^1 \times \mathbb{P}^1 = 2$, so $\dim W = 2$.

Problem 4. Consider the cubic surface

$$X = Z_{+}(x_{0}x_{1}^{2} - x_{2}x_{3}^{2}) \subset \mathbb{P}^{3}$$

- i) Compute all singular points of X;
- ii) Show that X is rational.

We note that X is irreducible. The jacobian is given by

$$J = \begin{pmatrix} x_1^2 & 2x_0x_1 & -x_3^2 & -2x_2x_3 \end{pmatrix}$$

 $J = (x_1 \quad 2x_0x_1 \quad -x_3 \quad -2x_2x_3)$ Thus $p \in X$ is a singular point $\Leftrightarrow rk J = 0 \Leftrightarrow p \in Z(x_1^2, 2x_0x_1, -x_3^2, -2x_2x_3) = Z(x_0, x_3).$ Thus

$$\operatorname{sing}(X) = X(x_0, x_3)$$

Consider $D_+(x_3) \simeq \mathbb{A}^3$ and the corresponding open set $U = Z(xy^2 - z) \subset X$. Hence a birational parameterization is given by

 $\begin{array}{cccc} \mathbb{A}^2 & \dashrightarrow & U \\ (u,v) & \mapsto & (u,v,uv^2) \end{array}$

The inverse is given by the projection $U \to \mathbb{A}^2$ onto the two first factors.

Problem 5. Consider the quotient space $X = \mathbb{A}^3 - 0 / \sim$ where the equivalence relation is defined by

$$(x, y, z) \sim (tx, ty, t^2 z).$$

a)* Show that X has the structure of a variety.

b) Compute $\mathscr{O}_X(X)$.

c) Show that X admits an embedding $X \hookrightarrow \mathbb{P}^3$ as a quadric surface. Deduce that X is projective. d) Find all singular points of X.

(Possible hints: Work on the "distinguished open sets" $D_+(x), D_+(y), D_+(z)$. In one of the charts, you will see the quadric cone $v^2 = uw$. There is also a convenient map $\mathbb{A}^3 - 0 \to \mathbb{P}^3$. You may solve these problems in any order, if that makes it easier.)

Note that X is a topological space via the quotient topology. We let $\pi : \mathbb{A}^3 - 0 \to X$ for the quotient map.

We will equip X with a sheaf to make it a ringed space, and subsequently show that X is a prevariety and finally a variety.

Let R = k[x, y, z] with the grading deg x = deg y = 1 and deg z = 2. We will use the notation D(x), D(y), D(z) for the distinguished open sets.

Define \mathscr{O}_X by letting $\mathscr{O}_X(U)$ consist of the continuous functions $f: U \to k$ which are locally of the form g/h where $g, h \in \mathbb{R}$ are homogeneous elements of the same degree. Thus $\frac{x^2}{z} \in \mathscr{O}_X(D(z))$, but $\frac{x}{z} \notin \mathscr{O}_X(D(z))$. It is clear that this satisfies the Gluing axiom (same proof as for \mathbb{P}^2), so we have a ringed space (X, \mathscr{O}_X) .

Consider the map

$$\begin{array}{rcl} \Phi: \mathbb{A}^3 - 0 & \rightarrow & \mathbb{P}^3 \\ (x, y, z) & \mapsto & (x^2 : xy : y^2 : z) \end{array}$$

Note that $\Phi(tx, ty, t^2z) = \Phi(x, y, z)$, so we get an induced continuous map

 $\phi: X \to \mathbb{P}^3.$

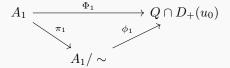
Write u_0, u_1, u_2, u_3 for the homogeneous coordinates on \mathbb{P}^3 .

Claim: ϕ is a homeomorphism onto the quadric $Q = Z(u_1^2 - u_0 u_2)$. We check this on the charts D(x), D(y), D(z).

D(x):

Let us restrict Φ to the subvariety $A_1 = Z(x-1) \subset \mathbb{A}^3 - 0$ which is isomorphic to \mathbb{A}^2 . We note that $\Phi(A_1) \subset D_+(u_0) \simeq \mathbb{A}^3$.

Write $\pi_1 = \pi|_{A_1}$ and $\Phi_1 = \Phi|_{A_1}$ so that we have a diagram



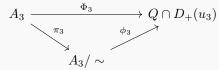
and $Q \cap D_+(u_0) = Z(u_1^2 - u_2) \subset \mathbb{A}^3$. Thus the restriction can be identified with the map

$$\begin{array}{rcl} \Phi_1: \mathbb{A}^2 & \to & Z(u_1^2 - u_2) \\ (y, z) & \mapsto & (y, y^2, z) \end{array}$$

This is clearly an isomorphism, with inverse $(u_1, u_2, u_3) \mapsto (u_1, u_3)$. Note that π_1 is surjective; this implies that ϕ_1 is also a homeomorphism.

D(y): The same argument as applies.

D(z): Here the situation is a little different: Defining $A_3 = Z(z-1) \subset \mathbb{A}^3 - 0$ as above, we have a diagram



The restriction π is no longer a homeomorphism: It identifies (x, y, 1) and (-x, -y, 1). Here Φ restricted to Z(z-1) can be identified with the map

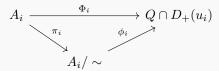
$$\Phi: \mathbb{A}^2 \to Z(u_1^2 - u_2 u_0) \subset \mathbb{A}^2$$
$$(x, y) \mapsto (x^2, xy, y^2)$$

This also identifies (x, y) with (-x, -y). This means that the induced ϕ is bijective. ϕ is also a closed map: Let $V \subset A_3/\sim$ be a closed set. By definition this means that $\pi^{-1}(V)$ is closed in $\mathbb{A}^3 - 0$, and hence $W = \pi^{-1}(V) \cap A_3$ is closed in $A_3 = \mathbb{A}^2$. We write this as W = Z(I) where $I \subset k[x, y]$ is some ideal. Since W is invariant under $(x, y) \mapsto (-x, -y)$, the polynomials (Alternatively, $k[x^2, xy, y^2] \subset k[x, y]$ is an integral extension, so by Going-Up Spec $k[x, y] \rightarrow$ Spec $k[x^2, xy, y^2]$ is closed. The map Φ is simply the restriction of this map of Spec's restricted to the maximal ideals, so it is again closed. Thus ϕ is also closed.) This means that ϕ is a homeomorphism.

Next we prove that X is a prevariety, by showing that it is locally isomorphic to an affine variety. Of course we will use the open sets D(x), D(y) and D(z) as the affine cover, and use the homeomorphisms above. To proceed, we need to show that the homeomorphisms ϕ_i

i = 1, 2, 3 induce isomorphisms between the structure sheaves $\mathcal{O}_X|_{D_+(x)}$ and \mathcal{O}_{A_1} etc. The proof of this is almost exactly as in the case for \mathbb{P}^2 . For instance, for $D_+(x)$, let $U \subset D_+(x)$ be an open subset and $f \in \mathcal{O}_X(U)$. By definition, f can locally (in some $V \subset U$) be written as f = g/h, where $g, h \in R$ are homogeneous of the same degree. Then the homeomorphism $\pi_1 : \mathbb{A}^2 \to D_(x)$ has the property that $\pi_1^*(g(x, y, z)/(h(x, y, z)) = g(1, y, z)/h(1, y, z)$, which is regular, since $h(1, y, z) \neq 0$ in $\pi_1^{-1}(V)$. The inverse $\alpha : D(x) \to \mathbb{A}^2$ is given by $(x : y : z) \mapsto (y/x, z/x^2)$. If $f \in \mathcal{O}_{\mathbb{A}^2}(W)$ is some regular function of the form g(y, z)/h(y, z) then $\alpha^* f = G(x, y, z)/H(x, y, z)$ where G, H are the homogenizations of g, h with respect to x. A similar argument works for D(y) and D(z).

What we have shown is that X can be covered by 3 open sets D(x), D(y), D(z), and these are naturally isomorphic to affine varieties. We also have diagrams



so that ϕ_i is an isomorphism for each *i*. But then $X \simeq Q$ as prevarieties. Consequently, X satisfies the Hausdorff axiom, since Q does (being a projective variety).

This completes the proof for a). Along the way, we have also shown c).

b) $\mathscr{O}_X(X) = k$ follows since X is isomorphic to a projective variety. But here is a more direct argument to see this. If $f: X \to k$ is a global regular function, then $\pi^* f$ is a global regular function on $\mathbb{A}^3 - 0$, hence a polynomial in k[x, y, z] (for $U = \mathbb{A}^n - 0$, we have $\mathscr{O}_{\mathbb{A}^n}(U) = \mathscr{O}_{\mathbb{A}^n}(\mathbb{A}^n)$ for all $n \geq 2$). But $\pi^* f$ is also invariant under the scaling $x \mapsto tx \ y \mapsto ty$ and $z \mapsto t^2 z$ for $r \in k^{\times}$. This can only happen if $\phi^* f$ is in fact constant. But then f is also constant.

d) By a) X is the union of $D(x) = \mathbb{A}^2 D(y) = \mathbb{A}^2$ and $D(z) = Z(uw - v^2)$. Thus the only singular points can lie in D(z), and there is only one, namely the point p = (0, 0, 1).

Alternatively, one can use the projective embedding $X \subset \mathbb{P}^3$, plus the jacobian criterion to get the same conclusion.