MAT4210 - an overview

A few words about preparing for the exam: The exam will be oral, so you should focus on concepts, definitions, examples and counterexamples in your preparation. The oral exam will not be anything like the mandatory assignment: you may be asked for some basic computations, but nothing complicated. Think about how the topics fit together, and why terms are defined the way they are.
Example questions:

- "What is a sheaf ? Why do we introduce sheaves? Can you give an example? A non-example?"
- "What is the Hausdorff axiom? Why do we include it?"

Some of the main examples: Affine space, projective space, conics, plane curves, hypersurfaces, cuspidal curve, nodal cubic, twisted cubic, quadratic cone $Z(xz-y^2)$, quadric surface $Z(xw-yz)$, rational normal curves, Veronese varieties, the Veronese surface, Segre embedding, $\mathbb{A}^n - 0$, blow-ups.

Here is a checklist of the topics covered through the course. The points marked with * are especially important.

Chapter 1: Algebraic sets

Key definitions

$\mathbb{A}^n$
$Z(S)$
$I(X)$ and $A(X)$

Key results

* Hilbert's Nullstellensatz; the relationships between $Z(I)$ and $I(X)$.
* Bijection between closed subsets and radical ideals

Key examples

Hypersurfaces
Conics
Affine twisted cubic

Chapter 2: Zariski topology

Key definitions

* The Zariski topology
* $D(f)$
Irreducible spaces
Noetherian spaces
* Dimension of a topological space
* Polynomial maps

Key results

$D(f)$ form a basis
X irreducible iff $I(X)$ is prime
maximal ideals $\leftrightarrow$ points
prime ideals $\leftrightarrow$ irreducible spaces
radical ideals $\leftrightarrow$ closed subsets
Decomposition into irreducible subsets
* $\dim X = \dim A(X)$
*.$\mathrm{Hom}_{AS}(X,Y) = Hom_{alg}(A(Y), A(X))$

Key examples

Hyperelliptic curve $y^2 - P(x) = 0$
$(x,y) \mapsto (x,xy)$
Quadrics (and diagonalization)

Chapter 3: Varieties

Key definitions

* $k(X)$, rational functions
The maximal open set of definition for a rational function
* $\mathcal{O}_{X,p}$
* Ringed spaces; sheaf of k-valued functions
* Pullback ϕ^*
Morphisms of ringed spaces
* The structure sheaf of a variety $\mathcal{O}_X$
Affine variety
Prevariety
Open subprevarieties; closed subprevarieties
* Hausdorff axiom,
* Varieties
Products of varieties; universal property. Especially for affine varieties.
Hausdorff axiom vs. the diagonal Δ

Key results

* $\mathcal{O}_X$ is a sheaf
$\mathcal{O}_X(D(f))$ and $\mathcal{O}_{X,p} = A(X)_{m_p}$
$D(f)$ is an affine variety
Morphisms from prevarieties into affine space
* X prevariety, Y affine:

$$\mathrm{Hom}(X,Y) = \mathrm{Hom}(A(Y), \mathcal{O}_X(X))$$

Consequences of this formula
Affine varieties satisfy the Hausdorff axiom
Products of varieties exist

Key examples

$Z(xy - zw)$
Sheaf of continuous, differentiable, constant, holomorphic, regular functions.
$\mathbb{A}^n - 0$ not affine
The affine line with two origins
Nodal cubic
cuspidal cubic

Chapter 4: Projective varieties

Key definitions

* $\mathbb{P}^n$ as a quotient space
* $\mathbb{P}^n$ as a (pre)variety
Homogeneous coordinates
Distinguished open sets; coordinates
Regular functions on projective varieties
* $\mathcal{O}_X$ for a projective variety
$S(X)$
How to define morphisms of (quasi)projective varieties

Key results

* Projective Nullstellensatz
Subvarieties of $\mathbb{P}^n$ vs cones vs homogeneous ideals
* Distinguished open sets are homomorphic/isomorphic to affine spaces
Projective varieties are varieties (affine cover; Hausdorff axiom)
* Global regular functions are constant: $\mathcal{O}_X(X) = k$

Key examples

$\mathbb{P}^n$
Linear projection
Projective twisted cubic

Chapter 5: Segre and Veronese varieties

Key definitions

Closed embedding
Rational normal curves; affine and projective
Segre embeddings
Veronese embeddings

Key results

Criteria for being a closed embedding (local on target; left section which is a section)
There is a unique RNC through any $n+3$ in linearly general position
The ideals of RNC, Segre and Veronese are generated by quadrics (minors of certain matrices). (Only superficial knowledge of the proof needed).
Nullstellensatz for $\mathbb{P}^n \times \mathbb{P}^m$; bihomogeneous ideals
* Veronese variety vs. space of hypersurfaces (e.g., conics)

Key examples

The blow-up of $\mathbb{P}^2$ as a subvariety of $\mathbb{P}^1 \times \mathbb{P}^2$.
Spaces of polynomials

Chapter 6: Rational maps

Key definitions

Rational maps
Birational maps
Maximal domain of definition
Blow-up at a point
Blow-up at a regular sequence

Key results

* Main theorem of rational maps: dominant rational maps $\phi : X \dashrightarrow Y$ = k-algebra homomorphisms $k(Y) \to k(X)$
Birational automorphisms of P^1 are automorphisms

Key examples

Quadrics are rational
Linear projections

Chapter 7: More on dimension

Key definitions

Dimension
Dominant rational map
Finite maps
Systems of parameters

Key results

* Lying Over-theorem for finite morphisms
* Going Up-theorem for finite and dominanting morphisms
Relate $\dim X$ and $\dim Y$ for a morphism $\phi : X \to Y$ which is i) finite; or ii) dominanting
* Noether normalization; geometric interpretation
$\dim X$ vs $tr.deg_k k(X)$.
* Krull's PIT
The dimension of the fibers of a morphism
* Dimensions of intersections (affine, and projective case)

Key examples

blow-up
a parabola mapping to A^1.

Chapter 8: Non-singular varieties

Key definitions

Tangent space
* Tangent space and m/m^2
Regular local rings
* singular points
Normal varieties

Key results

* Jacobian criterion (affine and projective case)
Non-singular points are dense
Normalization

Key examples

nodal cubic; cuspidal cubic; quadric cone.
Desingularizations via blow-ups

Chapter 9: Curves

Key results

Local rings of curves are DVRs
* Extensions of rational maps of (non-singular) curves always extend to morphisms
Normalizations of curves exist
* Fields of tr.deg 1 over k = non-singular projective curves over k
Elliptic curves are irrational

Chapter 11: Bezout's theorem

Key definitions

Local multiplicity
Hilbert functions
Degree
Regular sequences

Key results

* Bezout's theorem
Superficial knowledge of the proof of Bezout
Hilbert–Serre

Key examples

* Computations of local multiplicities for plane curves and surfaces
Degree of hypersurfaces; twisted cubic

Chapter 12: Applications of Bezout's theorem

Key results

Automorphisms of projective space
The theorems of Pappus and Pascal
Bound for number of singular points