Mandatory assignment MAT4210 – Spring 2022

The assignment must be submitted via Canvas by **14:30, Thursday February 24th**.

You need to solve at least 3.5 problems to pass. If you have any questions or comments about the problems, feel free to email me at `johnco@math.uio.no`.

All varieties are over the field $k = \mathbb{C}$.

Problem 1. Show that a quasiaffine variety is quasiprojective. Is the converse true?

Problem 2. a) Which of the following varieties are isomorphic?

(1) $\mathbb{A}^2$.
(2) $\mathbb{P}^1 \times \mathbb{P}^1$.
(3) $Z(f) \subset \mathbb{A}^3$, where $f = x + y + z + 1$
(4) $Z(f) \subset \mathbb{A}^3$, where $f = x^2 + y^2 + z^2 + 1$.
(5) $Z_+(F) \subset \mathbb{P}^3$, where $F = x_0^2 + x_1^2 + x_2^2 + x_3^2$.
(6) $X \subset \mathbb{A}^2 \times \mathbb{P}^1$ defined by $x_0y_0 - x_1y_1 = 0$.

b) For each variety X in problem a), compute $\mathcal{O}_X(X)$.

Problem 3. Consider the closed algebraic set $Z_+(I) \subset \mathbb{P}^2 \times \mathbb{P}^2$ defined by

$$x_1y_0 - x_0y_1 = x_2y_0 - y_2x_0 = 0$$

Compute its dimension and describe its irreducible components.

Problem 4. Consider the cubic surface

$$X = Z_+(x_0x_1^2 - x_2x_3^2) \subset \mathbb{P}^3$$

i) Compute all singular points of X;
ii) Show that X is rational.

Problem 5. Consider the quotient space $X = \mathbb{A}^3 - 0/\sim$ where the equivalence relation is defined by

$$(x, y, z) \sim (tx, ty, t^2z).$$

a)* Show that X has the structure of a variety.
b) Compute $\mathcal{O}_X(X)$.
c) Show that X admits an embedding $X \hookrightarrow \mathbb{P}^3$ as a quadric surface. Deduce that X is projective.
d) Find all singular points of X.

(Possible hints: Work on the "distinguished open sets" $D_+(x), D_+(y), D_+(z)$. In one of the charts, you will see the quadric cone $v^2 = uw$. There is also a convenient map $\mathbb{A}^3 - 0 \to \mathbb{P}^3$. You may solve these problems in any order, if that makes it easier.)