MAT4210 – towards final exam.

Ten exam topics. (numbering according to the version in Syllabus):

1- Thm 1.12 Hilbert Nullstellensatz (with sketch of proof, from weak to strong version)
2- Thm 3.45: X prevariety, Y affine: Hom(X, Y) = Hom(A(Y), O_X (X)) (with sketch of proof)
3- Zariski topology on an affine and on a projective variety (2.1+4.2 with main properties)
4- Thm 4.43: O_X (X)=k, when X is a projective variety over k. (with sketch of proof)
5- Blowup of A^2 in the origin and the strict transform of the cuspidal cubic.(example 8.25)
6- Projective tangent space and Prop 8.15. (Jacobian criterion) for a projective variety
7- Prop 7.54 and 7.58: Dimension of intersection of affine and of projective varieties(with sketch of proof)
8- Thm 9.12:Extension of rational maps from nonsingular curves into projective varieties.(with sketch of proof)
9- Thm 10.2: Generic structure theorem for dominant morphisms (with sketch of proof)
10- Thm 11.5 (with n=2) Bezouts theorem for plane curves (explain ingredients in the statement and main ideas in the proof)

A few words about preparing for the exam:

In the first part (<15 min) of the exam you present a topic drawn from the list of ten above. Show up 1 hour before your exam time, to draw your topic. For this part of the exam you may have one page of handwritten notes.

In the second part you will answer questions on definitions, key results and examples. These are taken from the list below, which is an edited version of the exam preparation sheet two years ago.

The points marked with * are especially important.

The main examples:

Affine space, projective space, conics, plane curves, hypersurfaces, cuspidal curve, nodal cubic, twisted cubic, quadratic cone $Z(xz-y^2)$, quadric surface $Z(xw-yz)$, rational normal curves, Veronese varieties, the Veronese surface, Segre embedding, $A^n - 0$, blow-ups.

Chapter 1: Algebraic sets

Key definitions

A^n
Z(S)
I(X) and A(X)

Key results

* Hilbert's Nullstellensatz; the relationships between Z(I) and I(X).

* Bijection between closed subsets and radical ideals

Chapter 2: Zariski topology

Key definitions

* The Zariski topology

* D(f)
Irreducible spaces

Noetherian spaces

* Dimension of a topological space

* Polynomial maps

Key results

D(f) form a basis
X irreducible iff I(X) is prime

maximal ideals ↔ points
prime ideals ↔ irreducible spaces

radical ideals ↔ closed subsets

Decomposition into irreducible subsets

* dimX = dimA(X)

*.$Hom_{AS}(X, Y) = Hom_{alg}(A(Y), A(X))$

Chapter 3: Varieties

Key definitions

* $k(X)$, rational functions

The maximal open set of definition for a rational function

* $O_{X,p}$

* Ringed spaces; sheaf of k-valued functions

* Pullback ϕ^*

Morphisms of ringed spaces

* The structure sheaf of a variety O_X

Affine variety

Prevariety

Open subprevarieties; closed subprevarieties

Hausdorff axiom,

* Varieties

Products of varieties; universal property. Especially for affine varieties.

Hausdorff axiom vs. the diagonal Δ

Key results

*O_X is a sheaf
$O_X(D(f))$ and $O_{X,p} = A(X)_{m_p}$
$D(f)$ is an affine variety
Morphisms from prevarieties into affine space

* X prevariety, Y affine: $Hom(X, Y) = Hom(A(Y), O_X(X))$

Consequences of this formula
Affine varieties satisfy the Hausdorff axiom

Products of varieties exist

Chapter 4: Projective varieties

Key definitions

* P^n as a quotient space

* P^n as a (pre)variety

Homogeneous coordinates
Distinguished open sets; coordinates

Regular functions on projective varieties

* O_X for a projective variety

S(X)
How to define morphisms of (quasi)projective varieties

Key results

* Projective Nullstellensatz

Subvarieties of P^n vs cones vs homogeneous ideals

* Distinguished open sets are homomorphic/isomorphic to affine spaces

Projective varieties are varieties (affine cover; Hausdorff axiom)

* Global regular functions are constant: $O_X(X) = k$

Chapter 5: Segre and Veronese varieties

Key definitions

Closed embedding
Rational normal curves; affine and projective

Segre embeddings
Veronese embeddings

Key results

Criteria for being a closed embedding (local on target; left section which is a section)

There is a unique RNC through any n + 3 in linearly general position
The ideals of RNC, Segre and Veronese are generated by quadrics (minors of certain matrices). (Only superficial knowledge of the proof needed).

Nullstellensatz for $P^n \times P^m$; bihomogeneous ideals

* Veronese variety vs. space of hypersurfaces (e.g., conics)

Chapter 6: Rational maps

Key definitions

Rational maps
Birational maps
Maximal domain of definition

Blow-up at a point
Blow-up at a regular sequence

Key results

* Main theorem of rational maps: dominant rational maps $\phi : X \dashrightarrow Y$ = k-algebra homomorphisms $k(Y) \to k(X)$

Birational automorphisms of P^1 are automorphisms

Quadrics are rational

Chapter 7: More on dimension

Key definitions

Dimension
Dominant rational map

Finite maps
Systems of parameters

Key results

Lying Over-theorem for finite morphisms
Going Up-theorem for finite and dominating morphisms

Relate dimX and dimY for a morphism $\phi : X \to Y$ which is i) finite; or ii) dominating

* Noether normalization; geometric interpretation dimX vs $\mathrm{tr.deg}_k k(X)$.

Krull's PIT
*The dimension of the fibers of a morphism

* Dimensions of intersections (affine, and projective case)

Chapter 8: Non-singular varieties

Key definitions

Tangent space
* Tangent space and m/m^2

Regular local rings

* singular points

Normal varieties

Key Results

* Jacobian criterion (affine and projective case)

Non-singular points are dense
Normalization

Example of desingularizations via blow-ups

Chapter 9: Curves

Key results

Local rings of smooth curves are DVRs
* Extensions of rational maps of (non-singular) curves always extend to morphisms

Normalizations of curves exist
* Fields of tr.deg 1 over k = non-singular projective curves over k

Elliptic curves are irrational

Chapter 10: The structure of maps

Key result

Semicontinuity of fiber dimension.

Chevalley's theorem: The image of a morphism between varieties is constructible.

Chapter 11: Bezout's theorem

Key definitions

Local multiplicity

Hilbert functions

Degree
Regular sequences

Key results

* Bezout's theorem

Superficial knowledge of the proof of Bezout

Hilbert–Serre

Degree of hypersurface and of twisted cubic

Chapter 12: Applications of Bezout's theorem

Key results

Automorphisms of projective space

The theorems of Pappus and Pascal

Bound for number of singular points