GAGA Talk (SArS Fall 2004)

Outline:

I. Definitions / Theorems

II. Proof in the projective case

III. Proof in the proper case

IV. Applications

Refs: Serre's original paper "Géométrie Algébrique et Géométrie Analytique"
SGA 1 exp. $\overline{\text{XII}}$

I. Definitions / Theorems

Defs: 1) An alg. $\mathbb{C}$-scheme X is a scheme locally of fin. type $/\mathbb{C}$ (think of an alg. variety if this frightens you)

2) A $\mathbb{C}$-analytic space $(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ is a space locally ringed by $\mathbb{C}$-algebras ($\mathbb{C}$-LRS) that is locally isomorphic to a $\mathbb{C}$-LRS of the form $(C, i^*(\mathcal{O}_U/\mathcal{J}))$, where $C \subseteq U$ is closed, $U \subseteq \mathbb{C}^n$ is open, and $\mathcal{J} \subseteq \mathcal{O}_U$ is coherent. Maps of analytic spaces are just maps of $\mathbb{C}$-LRSs. (This is just a generalitation of $\mathbb{C}$-manifold which allows singularities, nilpotents, non-Hausdorffness, etc...)

Given an alg. $\mathbb{C}$-scheme, define a functor Φ_X: (an spaces) $\to$ (sets) by $\Phi_X(\mathfrak{X}) := \mathrm{Hom}_{\mathbb{C}\text{-LRS}}(\mathfrak{X}, X)$.

Thm 1: Φ_X is representable, i.e. $\exists$ an. space X^{an} & morphism $\phi: X^{an} \to X$ of $\mathbb{C}$-LRSs satisfying the universal property: $\forall$ an. space $\mathcal{Y}$ & morphism of $\mathbb{C}$-LRSs $\psi: \mathcal{Y} \to X$, $\exists!$ morphism $\chi: \mathcal{Y} \to X$ making
$$\begin{tikzcd} X^{an} \arrow[r, "\phi"] & X \\ \mathcal{Y} \arrow[u, dashed, "\chi"] \arrow[ur, "\psi"'] & \end{tikzcd}$$
commute.

Moreover, ϕ induces a bijection between X^{an} and $X(\mathbb{C})$ and for any $x \in X^{an}$, the map ϕ_x $\mathcal{O}_{X,\phi(x)} \to \mathcal{O}_{X^{an},x}$ of local rings induces an isomorphism on completions $\hat{\phi}_x: \hat{\mathcal{O}}_{X,\phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{X^{an},x}$. Thus ϕ_x is faithfully flat.

Rmk: The term "flat" was invented by Serre to describe this map.

This gives a functor $\Phi : (\text{alg. } \mathbb{C}\text{-sch.}) \to (\text{an.sp.}) : X \mapsto X^{an}$

This functor commutes with fiber products, i.e. $(X \times_Y Z)^{an} = X^{an} \times_{Y^{an}} Z^{an}$.
[This follows from the fact that the scheme $X \times_Z Y$ is also the fiber product in the category of LRSs.]

Now take $\phi : X^{an} \to X$. Then for any $\mathcal{O}_X$-module $\mathcal{F}$, we can consider the $\mathcal{O}_{X^{an}}$-module $\phi^* \mathcal{F} =: \mathcal{F}^{an}$. A thm. of Oka states that $\mathcal{O}_{X^{an}}$ is coherent, so that $\mathcal{F}$ coh. $\Rightarrow \mathcal{F}^{an}$ coh.

Thm 2: The functor $\mathcal{F} \mapsto \mathcal{F}^{an}$ is exact & faithful.

Thm 3: X has one of the following properties $\Leftrightarrow$ X^{an} does

(i) non-empty
(ii) discrete
(iii) Cohen-Macaulay
(iv) regular
(v) normal
(vi) reduced
(vii) n-dimensional
(viii) connected
(ix) irreducible.

Thm 4: $f : X \to Y$ (f.type) is one of the following $\Leftrightarrow$ $f^{an} : X^{an} \to Y^{an}$ is

(i) flat
(ii) unramified
(iii) étale (i.e. locally biholom.)
(iv) smooth (i.e. submersive)
(v) normal
(vi) reduced
(vii) injective
(viii) separated
(x) an isomorphism
(x) a monomorphism
(xi) an open immersion
(xii) surjective

(xiii) dominant
(xiv) a closed immersion
(xv) an immersion
(xvi) proper
(xvii) finite

We can now state Serre's famous GAGA thms. which compare the (coherent) sheaves on X to the sheaves on X^{an} when $X/\mathbb{C}$ is proper.

<u>Thm 5</u>: (Serre-Grothendieck) Let $X/\mathbb{C}$ be proper. Then:

(1) For every coherent $\mathcal{O}_X$-module $\mathcal{F}$, there is a functorial isomorphism $\varepsilon: H^q(X,\mathcal{F}) \to H^q(X^{an}, \mathcal{F}^{an})$ commuting with the long exact sequence.

(2) The functor (coh. $\mathcal{O}_X$-mods) $\longrightarrow$ (coh. $\mathcal{O}_{X^{an}}$-mods): $\mathcal{F} \mapsto \mathcal{F}^{an}$ is an equivalence of (abelian) categories.

<u>Rmk</u>: More generally, if $f: X \to Y$ is a proper morphism of alg. $\mathbb{C}$-sch., then there is an isom. $\theta_q: (R^q f_* \mathcal{F})^{an} \to R^q f_*^{an}(\mathcal{F}^{an})$ (same proof).

<u>Cor</u>: (Chow's thm.) Every closed analytic subspace of $\mathbb{P}^n_{\mathbb{C}}$ is algebraic.

<u>Pf</u>: Note that $\mathbb{C}$-projective space actually is $(\mathbb{P}^n_{\mathbb{C}})^{an}$ (they represent the same functors ...). For clarity, write $X = \mathbb{P}^n_{\mathbb{C}}$ (alg. side).
Suppose that $Z^{an} \subseteq X^{an}$ is a closed subspace. Then it corresponds to a coherent ideal sheaf $\mathcal{J}_{Z^{an}} \subseteq \mathcal{O}_{X^{an}}$. By (2) above, we get $\mathcal{J}_Z \subseteq \mathcal{O}_X$ coh. & a corresp. $Z \subseteq X$ closed subvar. whose analytification is easily seen to be Z^{an}. □.

Time permitting, we'll also prove:

<u>Thm. 6</u>: Let X be an alg. $\mathbb{C}$-scheme, $\phi: X^{an} \to X$ the canonical map. Then for $x \in X^{an}$, there is a canonical isom. between $\pi_1^{ét}(X, \phi(x))$ and the profinite completion of $\pi_1(X^{an}, x)$.

<u>Cor</u>: The functor (proper $\mathbb{C}$-sch.) $\to$ ($\mathbb{C}$-an. sp.) is fully faithful, i.e. $\mathrm{Hom}_{\mathbb{C}}(X,Y) = \mathrm{Hom}(X^{an}, Y^{an})$.

II. Proof in the projective case.

We first need to prove the existence of X^{an}, i.e., that $\Phi_X := \mathrm{Hom}_{LRS}(-,X)$ is representable. We also want that $\phi_x: \hat{\mathcal{O}}_{X,\phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{X^{an},x}$.

Pf: (of Thm. 1) Let X = alg. $\mathbb{C}$-sch.

First we show that the problem is local on X, i.e., X^{an} exists $\Leftrightarrow$ for any open cover $\{U_i\}$, each U_i^{an} exists.

"$\Rightarrow$": Say we have $\phi: X^{an} \to X$. We claim $U_i^{an} = \phi^{-1}(U_i)$. In fact, a map from an an. sp. $\mathfrak{X} \to U_i$ gives a comm. diagram

$$\begin{tikzcd} \mathfrak{X} \arrow[r] \arrow[drr] & X^{an} \arrow[r, "\phi"] & X \\ & & U_i \arrow[u] \end{tikzcd}$$

Thus set-theoretically $\mathfrak{X} \to X^{an}$ factors through $\phi^{-1}(U_i)$, thus it does so as a map of analytic spaces.

"$\Leftarrow$": By above, solutions on the U_i restrict to sol'ns on the overlaps, so the universal property allows us to glue uniquely to get X^{an}.

Now we directly construct $(\mathbb{A}^n_{\mathbb{C}})^{an}$. No surprise, it's $\mathbb{C}^n$:
Giving a map of LRSs $\mathfrak{X} \to \mathbb{A}^n_{\mathbb{C}}$ is the same as choosing n global sections in $\Gamma(\mathfrak{X}, \mathcal{O}_{\mathfrak{X}})$ (cf. [H] Ch. II Ex.), i.e. n holomorphic fcns. on $\mathfrak{X}$. This is the same as a map of an. spaces $\mathfrak{X} \to \mathbb{C}^n$. $\therefore \mathbb{C}^n \cong (\mathbb{A}^n_{\mathbb{C}})^{an}$.

Finally, we just need to show that if X^{an} exists, then so does Z^{an} for any closed subscheme $Z \subseteq X$. (Since every alg. $\mathbb{C}$-sch. has a covering by closed subschemes of $\mathbb{A}^n_{\mathbb{C}}$). This corresp. to a coh. ideal $\mathcal{J}_Z \subseteq \mathcal{O}_X$. The ideal $\phi^{-1}\mathcal{J}_Z \cdot \mathcal{O}_{X^{an}}$ is also coherent, so defines $Z' \subseteq X^{an}$ closed subspace. Easy to check $Z' = Z^{an}$.

The question of $\hat{\phi}_x: \hat{\mathcal{O}}_{X,\phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{X^{an},x}$ is clearly local, so we may assume that $X \subseteq \mathbb{A}^n_{\mathbb{C}}$ (closed subvar.). Clearly $\hat{\phi}_x: \hat{\mathcal{O}}_{\mathbb{A}^n,\phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{\mathbb{C}^n,x}$ (this is just saying that $\mathbb{C}[x_1,\dots,x_n]^\wedge \cong \mathbb{C}\{x_1,\dots,x_n\}^\wedge$). The general case follows from this, since for any ideal $I \subseteq \mathcal{O}_{\mathbb{A}^n,\phi(x)}$, we get $\hat{\mathcal{O}}_{\mathbb{A}^n,\phi(x)} / I\hat{\mathcal{O}}_{\mathbb{A}^n,\phi(x)} \xrightarrow{\cong} \hat{\mathcal{O}}_{\mathbb{C}^n,x} / I \cdot \hat{\mathcal{O}}_{\mathbb{C}^n,x}$.

(The fact that $X^{an} \leftrightarrow X(\mathbb{C})$ is clear from the univ. property). $\square$

Pf: (of Thm. 2) Exactness of $\mathcal{F} \mapsto \mathcal{F}^{an}$ follows from the flatness of $\phi: X^{an} \to X$.
To check faithfulness, $\therefore$ only need check $\mathcal{F}^{an} = 0 \Rightarrow \mathcal{F} = 0$. But to check that $\mathcal{F} = 0$, it suffices to check on stalks at closed points (X is Jacobson), so this follows from the fact that $\phi_x : \mathcal{O}_{X,\phi(x)} \to \mathcal{O}_{X^{an},x}$ is faithfully flat. $\square$.

Now we give a complete sketch of Serre's thm. (Thm. 5) in the case that $X/\mathbb{C}$ is projective. We start with (1).
Firstly, we need to describe $\varepsilon : H^q(X, \mathcal{F}) \to H^q(X^{an}, \mathcal{F}^{an})$. These maps just come from the fact that $\mathcal{F} \mapsto \mathcal{F}^{an}$ is exact & that $H^q(X, -)$ is a universal δ-functor.
Note that $(\mathbb{P}^n_{\mathbb{C}})^{an} = \mathbb{P}^n$ and that $\mathcal{O}(1)^{an} = \mathcal{O}(1)$ ($\mathcal{O}(1)$ is the univ. inv. sheaf on $\mathbb{P}^n$).
We claim it suffices to prove (1) in the case that $X = \mathbb{P}^n$:
In fact, if we have $i : X \hookrightarrow \mathbb{P}^n$, then $i^{an} : X^{an} \to \mathbb{P}^n_{an}$ is also a closed immersion (we showed this in our existence proof). Thus there are canonical ~~functorial~~ isomorphisms $H^q(X, \mathcal{F}) \cong H^q(\mathbb{P}^n_{\mathbb{C}}, i_* \mathcal{F})$ (note that $i_* \mathcal{F}$ is again coherent $\because$ i proper) and $H^q(X^{an}, \mathcal{F}^{an}) \cong H^q(\mathbb{P}^n_{an}, i^{an}_*(\mathcal{F}^{an}))$
Thus just need $i^{an}_*(\mathcal{F}^{an}) = (i_* \mathcal{F})^{an}$ (canonically). (This is easy if you think about it right). Claim is proved by uniqueness of ε.
$\therefore$ WLOG, assume $X = \mathbb{P}^n$.
First note that $\varepsilon : H^q(X, \mathcal{F}) \to H^q(X^{an}, \mathcal{F}^{an})$ gives an isomorphism when $\mathcal{F} = \mathcal{O}_X$:
indeed $\varepsilon : \underset{\mathbb{C}}{\underset{\cong}{H^0(X, \mathcal{O}_X)}} \xrightarrow{\sim} \underset{\mathbb{C}}{\underset{\cong}{H^0(X^{an}, \mathcal{O}_{X^{an}})}}$ and $H^q(X, \mathcal{O}_X) = 0 = H^q(X^{an}, \mathcal{O}_{X^{an}})$ $q \geq 1$
(thm of Dolbeault $\Rightarrow H^q(X^{an}, \mathcal{O}_{X^{an}}) \cong H^{0,q}(X)$)

Now we prove (1) when $\mathcal{F} \cong \mathcal{O}_X(k)$
Proceed by induction on n ($X = \mathbb{P}^n$). The case $n = 0$ is clear.
We have an exact sequence $0 \to \mathcal{O}_X(-1) \to \mathcal{O}_X \to \mathcal{O}_E \to 0$ ($\mathcal{O}_E \subseteq \mathcal{O}_X$ a hyperplane)
Get $\forall k \in \mathbb{Z}$ an e.s. $0 \to \mathcal{O}_X(k-1) \to \mathcal{O}_X(k) \to \mathcal{O}_E(k) \to 0$
$\therefore$ have comm. diagram

$$\begin{array}{ccccccccc} & & & & \swarrow \text{(inductive hypoth.)} & & & & \\ \cdots \to & H^q(X, \mathcal{O}(k-1)) & \to & H^q(X,\mathcal{O}(k)) & \to H^q(E, \mathcal{O}_E(k)) & \to & H^{q+1}(X, \mathcal{O}(k-1)) & \to \cdots \\ & \downarrow & & \downarrow & \downarrow \cong & & \downarrow & \\ \cdots \to & H^q(X^{an}, \mathcal{O}(k-1)^{an}) & \to & H^q(X^{an}, \mathcal{O}(k)^{an}) & \to H^q(E^{an}, \mathcal{O}_E(k)^{an}) & \to & H^{q+1}(X^{an}, \mathcal{O}(k-1)^{an}) & \to \cdots \end{array}$$

$\therefore$ by the 5-lemma (1) is true for $\mathcal{O}(k) \Leftrightarrow$ it is true for $\mathcal{O}(k-1)$

$\therefore$ It's true $\forall k$ since it's true for $k=0$.

Finally, any coh. sheaf $\mathcal{F}$ on X is the quotient of a sheaf $\mathcal{L} \cong \bigoplus_i \mathcal{O}(n_i)$

$0 \to \mathcal{H} \to \mathcal{L} \to \mathcal{F} \to 0$ ($\mathcal{H}$ coherent)

Get a comm. diagram

$$\begin{array}{ccccccccccc} \cdots \to & H^q(X,\mathcal{H}) & \to & H^q(X,\mathcal{L}) & \to & H^q(X,\mathcal{F}) & \to & H^{q+1}(X,\mathcal{H}) & \to & H^{q+1}(X,\mathcal{L}) & \to \cdots \\ & \varepsilon_1 \downarrow & & \varepsilon_2 \downarrow & & \varepsilon_3 \downarrow & & \varepsilon_4 \downarrow & & \varepsilon_5 \downarrow & \\ \cdots \to & H^q(X^{an},\mathcal{H}^{an}) & \to & H^q(X^{an},\mathcal{L}^{an}) & \to & H^q(X^{an},\mathcal{F}^{an}) & \to & H^{q+1}(X^{an},\mathcal{H}^{an}) & \to & H^{q+1}(X^{an},\mathcal{L}^{an}) & \to \cdots \end{array}$$

We know that ε_2 & ε_5 are isomorphisms.

Argue by descending induction on q: if $q \gg 0$, ε_4 is an isomorphism since
$H^{q+1}(X,\mathcal{H}) = 0 = H^{q+1}(X^{an},\mathcal{H}^{an})$.

If $\varepsilon_2, \varepsilon_4, \varepsilon_5$ are isomorphisms, then the 5-lemma $\Rightarrow$ ε_3 is surjective
But then we've proved this for all coh. sheaves, so that ε_1 is also surjective
Another application of the 5-lemma $\Rightarrow$ ε_3 is injective $\therefore$ an isom.
$\therefore$ (1) is true for all coherent $\mathcal{F}$.

Now to prove (2): $\mathcal{F} \mapsto \mathcal{F}^{an}$ gives an equiv. of categories (coh. $\mathcal{O}_X$-mods.) $\cong$ (coh. $\mathcal{O}_{X^{an}}$-mods.)

$\mathcal{F} \mapsto \mathcal{F}^{an}$ is fully faithful: already know exact & faithful.

We can identify the map $\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F},\mathcal{G}) \to \mathrm{Hom}_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an},\mathcal{G}^{an})$

with $H^0(X, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})) \to H^0(X^{an}, \mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})^{an})$

The natural map $\mathcal{H}om_{\mathcal{O}_X}(\mathcal{F},\mathcal{G})^{an} \to \mathcal{H}om_{\mathcal{O}_{X^{an}}}(\mathcal{F}^{an},\mathcal{G}^{an})$ is an isomorphism, since the map $\phi: X^{an} \to X$ is flat. Thus (1) gives the result, since $\mathcal{H}om(\mathcal{F},\mathcal{G})$ is coh.

We now just need to prove that $\mathcal{F} \mapsto \mathcal{F}^{an}$ is essentially surjective (in the case X is projective).

By the same reasoning used in the pf. of (1), suff. to consider $X=\mathbb{P}^n$.
We proceed by induction on n. (The case $n=0$ is clear).

Claim 1: If $E \subseteq X$ is a hyperplane and $\mathcal{A}$ is a coh. sheaf on E^{an}, then for all $q>0$, and $k \gg 0$, $H^q(E^{an}, \mathcal{A}(k)) = 0$.

Pf: by ind. hyp. $\exists\ \mathcal{F}$ coh. on E s.t. $\mathcal{F}^{an} = \mathcal{A}$, so that $\mathcal{F}(k)^{an} = \mathcal{A}(k)$ & then pt. (1) gives the claim.

Claim 2: If $\mathcal{M}$ is coh. on X^{an}, then $\exists\ k_0 = k_0(\mathcal{M})$ s.t. $\forall k \geq k_0$ $\mathcal{M}(k)$ is globally generated.

Pf: In fact if $\mathcal{M}(k_0)$ is gl. gen., then $\mathcal{M}(k)$ is too $\forall k \geq k_0$, since $\forall i\ \exists$ map $\mathcal{M}(k_0) \to \mathcal{M}(k)$ which is an isom on U_i (mult. by t_i).
Also, by compactness of X^{an}, it suffices to prove that for any fixed $x \in X^{an}$, $\exists\ k_0$ s.t. $H^0(X^{an}, \mathcal{M}(k_0)) \twoheadrightarrow \mathcal{M}(k_0)_x$.
Choosing a hyperplane E through x gives an e.s.

$$0 \to \mathcal{O}(-1) \to \mathcal{O}_{X^{an}} \to \mathcal{O}_{E^{an}} \to 0$$

Tensor w/ $\mathcal{M}$:

$$0 \to \mathcal{K} \to \mathcal{M}(-1) \to \mathcal{M} \to i^*\mathcal{M} \to 0 \quad (i : E^{an} \hookrightarrow X^{an})$$

$\therefore \forall k$ get

$$0 \to \mathcal{K}(k) \to \mathcal{M}(k-1) \to \mathcal{M}(k) \to i^*\mathcal{M}(k) \to 0$$

Which breaks up as

$$0 \to \mathcal{K}(k) \to \mathcal{M}(k-1) \to \mathcal{F}_k \to 0, \quad 0 \to \mathcal{F}_k \to \mathcal{M}(k) \to i^*\mathcal{M}(k) \to 0.$$

So we get from the l.e.s.

$$H^1(X^{an}, \mathcal{M}(k-1)) \to H^1(X^{an}, \mathcal{F}_k) \to H^2(X^{an}, \mathcal{K}(k))$$
$$\& \quad H^1(X^{an}, \mathcal{F}_k) \to H^1(X^{an}, \mathcal{M}(k)) \to H^1(X^{an}, i^*\mathcal{M}(k))$$

Both $\mathcal{K}$ & $i^*\mathcal{M}$ are sheaves on E^{an}, so by claim (1), the higher cohom. of $\mathcal{K}(k)$ & $i^*\mathcal{M}(k)$ vanishes $\forall k \gg 0$.

The dimension of these groups is <u>finite</u> by an analytic result of Serre-Cartan.
$\therefore$ we see that the dim. of $H^1(X^{an}, \mathcal{M}(k))$ is a decreasing function of k for $k \gg 0$, hence constant for $k \gg 0$, and we get that the map $H^1(X^{an}, \mathcal{F}_k) \to H^1(X^{an}, \mathcal{M}(k))$ is an isom. $\forall k \gg 0$, so the map $H^0(X^{an}, \mathcal{M}(k)) \twoheadrightarrow H^0(X^{an}, i^*\mathcal{M}(k))$ is surjective $\forall k \gg 0$.

We'll need to assume that we have proved the rmk. after Thm. 5 (i.e. that if $f: X \to Y$ is projective, then $(R^p f_* \mathcal{F})^{an} \cong R^p f_*^{an} \mathcal{F}^{an}$ functorially and that if Y is Spec $\mathbb{C}$, then this isom. is just ε of Thm. 5 (1)).

By this rmk., we have that $R^p g_*^{an} \mathcal{O}_{X'}(n)^{an} \cong (R^p g_* \mathcal{O}_{X'}(n))^{an} = 0$, so the same argument as above shows that $H^q(X, \mathcal{F}^{an}) \cong H^q(X'^{an}, \mathcal{O}_{X'}(n)^{an})$

(edge maps in the Leray ss. are functorial)

Thus we can just look at the commutative diagram

$$\begin{array}{ccc} H^q(X, \mathcal{F}) & \cong & H^q(X', \mathcal{O}_{X'}(n)) \\ \varepsilon \downarrow & & \downarrow \varepsilon \\ H^q(X^{an}, \mathcal{F}^{an}) & \cong & H^q(X'^{an}, \mathcal{O}_{X'}(n)^{an}) \end{array}$$

The right-hand map is an isom. by what we've already proved. Thus the left-hand map is as well.

Now we prove Thm. 5 (2) (that $\mathcal{F} \mapsto \mathcal{F}^{an}$ is an equivalence of categories (coh. $\mathcal{O}_X$-mods) $\to$ (coh. $\mathcal{O}_{X^{an}}$-mods.)).

We have shown unconditionally that $\mathcal{F} \mapsto \mathcal{F}^{an}$ is fully faithful.

Thus we need to deduce essential surjectivity from the projective case.

(this should really be located on the next page after "wlog")

Again use Chow's lemma to find $f: X' \to X$ projective, surjective, birational with $X/\mathbb{C}$ projective. Let $U \subseteq X$ be a dense open such that $f|_{f^{-1}(U)}$ is an isomorphism. We procede by Noeth. induction on X: for a closed set $Y \subseteq X$, let $P(Y)$ be the statement: if $\mathcal{G}$ is a coherent sheaf on X^{an} such that Supp $\mathcal{G} \subseteq Y^{an}$, then $\exists$ coh. sheaf $\mathcal{H}$ on X s.t. $\mathcal{H}^{an} \cong \mathcal{G}$.

We need to show that $P(Y')$ holds $\forall Y' \subsetneq Y \Rightarrow P(Y)$ holds.

But note that if $\mathcal{G}$ is any sheaf on X with Supp $\mathcal{G} \subseteq Y$, then by the analytic Nullstellensatz, there is a power of $\mathcal{J}_Y^{an}$ that kills $\mathcal{G}$, so that we may view $\mathcal{G}$ as a sheaf on $(Y, \mathcal{O}_X/\mathcal{J}_Y^n)^{an}$.

Thus we may assume WLOG that $X = Y$.

Let $\mathcal{F}^h$ be a coh. $\mathcal{O}_{X^{an}}$-mod. & consider the sequence (exact)

$$0 \to \mathcal{K}^h \to \mathcal{F}^h \to f_*^{an} f^{an*} \mathcal{F}^h \to \mathcal{L}^h \to 0$$

Since X' is projective, $f^{an*}\mathcal{F}^h \cong \mathcal{G}$ for some $\mathcal{G}$ coh. $\mathcal{O}_{X'}$-mod

By hypothesis, $\mathcal{K}^h \cong \mathcal{K}^{an}$ and $\mathcal{L}^h \cong \mathcal{L}^{an}$ for some $\mathcal{L}$ & $\mathcal{K}$ coh $\mathcal{O}_X$-mods, since $\mathcal{K}^h|_U = \mathcal{L}^h|_U = 0$

Note that $(f_*\mathcal{G})^{an} \underset{(\text{thm. (1)})}{\cong} f_*^{an} \mathcal{G}^{an} \underset{(\text{def.})}{\cong} f_*^{an} f^{an*} \mathcal{F}^h$, so that the map $f_*^{an} f^{an*}\mathcal{F}^h \to \mathcal{L}^h$ (by the fact that the functor has already been proved to be fully faithful) comes from an algebraic map $f_*\mathcal{G} \to \mathcal{L}$. If we set $\mathcal{J} = \ker(f_*\mathcal{G} \to \mathcal{L})$, then we see (by exactness) that $\mathcal{F}^h$ is an extension of $\mathcal{J}^{an}$ by $\mathcal{K}^{an}$.

Thus it will suffice to prove that the map $\mathrm{Ext}^q_{\mathcal{O}_X}(I, K) \longrightarrow \mathrm{Ext}^q_{\mathcal{O}_{X^{an}}}(I^{an}, K^{an})$ given concretely by mapping a q-extension $\mathcal{H}_\bullet$ to $\mathcal{H}_\bullet^{an}$ is an isomorphism ?

(also cf. EGA 0_{III} §12.3)

Because the map of Ext sheaves $\mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{J}, \mathcal{K})^{an} \to \mathcal{E}xt^q_{\mathcal{O}_{X^{an}}}(\mathcal{J}^{an}, \mathcal{K}^{an})$ is an isom. (since $\phi: X^{an} \to X$ is flat), the morphism of spectral sequences

$$\begin{array}{ccc} H^p(X, \mathcal{E}xt^q_{\mathcal{O}_X}(\mathcal{J}, \mathcal{K})) & \Longrightarrow & \mathrm{Ext}^{p+q}_{\mathcal{O}_X}(\mathcal{J}, \mathcal{K}) \\ & & \downarrow \\ H^p(X, \mathcal{E}xt^q_{\mathcal{O}_{X^{an}}}(\mathcal{J}^{an}, \mathcal{K}^{an})) & \Rightarrow & \mathrm{Ext}^{p+q}_{\mathcal{O}_{X^{an}}}(\mathcal{J}^{an}, \mathcal{K}^{an}) \end{array}$$

is an isom on E_2-terms and hence an isomorphism. ◻

<u>Cor.</u> If $X, Y/\mathbb{C}$ proper, then $\mathrm{Hom}_{\mathbb{C}}(X, Y) \cong \mathrm{Hom}(X^{an}, Y^{an})$

Pf: Look at the graph. ◻