1.15 For $\varphi, \tilde{\varphi} \in \mathrm{Hom}(\mathcal{F}_{|\mathcal{U}}, \mathcal{G}_{|\mathcal{U}})$, define $\varphi + \tilde{\varphi}$ by $(\varphi+\tilde{\varphi})(V) = \varphi(V) + \tilde{\varphi}(V)$ for all $V \subset \mathcal{U}$ open. For $\tilde{V} \subset V$ an inclusion of open sets, there is a commutative diagram

$$\begin{tikzcd} \mathcal{F}_{|\mathcal{U}}(V) \arrow[r, "\varphi+\tilde{\varphi}"] \arrow[d, "\rho^{\mathcal{F}_{|\mathcal{U}}}_{V,\tilde{V}}"'] & \mathcal{G}_{|\mathcal{U}}(V) \arrow[d, "\rho^{\mathcal{G}_{|\mathcal{U}}}_{V,\tilde{V}}"] \\ \mathcal{F}_{|\mathcal{U}}(\tilde{V}) \arrow[r, "\varphi+\tilde{\varphi}"] & \mathcal{G}_{|\mathcal{U}}(\tilde{V}) \end{tikzcd}$$

b/c composition of maps distributes addition (for abelian groups). The identity elt. wrt. the addition defined above is 0 where $0(V)$ is trivial for all $V \subset \mathcal{U}$ open, and the inverse of φ is $-\varphi$ given by $-\varphi(V)$. Addition is clearly commutative, so $\mathrm{Hom}(\mathcal{F}_{|\mathcal{U}}, \mathcal{G}_{|\mathcal{U}})$ is an abelian group with addition induced by the abelian group structures on $\mathrm{Hom}(\mathcal{F}(V), \mathcal{G}(V))$ (for $V \subset \mathcal{U}$ open).

The internal hom (in presheaves) $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ between $\mathcal{F}$ and $\mathcal{G}$ is defined by

$$\mathcal{U} \longmapsto \mathrm{Hom}(\mathcal{F}_{|\mathcal{U}}, \mathcal{G}_{|\mathcal{U}}).$$

Define restriction maps

$$\rho_{\mathcal{U},V} : \mathcal{H}om(\mathcal{F},\mathcal{G})(\mathcal{U}) \to \mathcal{H}om(\mathcal{F},\mathcal{G})(V)$$

as follows: For $\tilde{\mathcal{U}} \subset \mathcal{U}$ open, $\mathcal{F}_{|\mathcal{U}}(\tilde{\mathcal{U}}) = \mathcal{F}(\tilde{\mathcal{U}})$ and $\mathcal{G}_{|\mathcal{U}}(\tilde{\mathcal{U}}) = \mathcal{G}(\tilde{\mathcal{U}})$, and $\mathcal{F}_{|V}(\tilde{\mathcal{U}}) = \mathcal{F}(\tilde{\mathcal{U}} \cap V)$ & ditto for $\mathcal{G}_{|V}$. Hence the restriction maps $\mathcal{F}(\tilde{\mathcal{U}}) \to \mathcal{F}(\tilde{\mathcal{U}} \cap V)$ and $\mathcal{G}(\tilde{\mathcal{U}}) \to \mathcal{G}(\tilde{\mathcal{U}} \cap V)$ allow us to define $\rho_{\mathcal{U},V}$.

Suppose $\{\mathcal{U}_i\}_{i \in I}$ is an open covering of $\mathcal{U}$, and choose an element $\varphi \in \mathcal{H}om(\mathcal{F},\mathcal{G})(\mathcal{U})$ s.t. $\varphi_{|\mathcal{U}_i} = 0$ for all $i \in I$. Now if $\tilde{\mathcal{U}} \subset \mathcal{U}$ is open, then $\varphi(\tilde{\mathcal{U}} \cap \mathcal{U}_i)$ is trivial. Hence for all $x \in \mathcal{F}(\tilde{\mathcal{U}})$, $\varphi(x_{|\tilde{\mathcal{U}} \cap \mathcal{U}_i}) = 0$ b/c $\{\tilde{\mathcal{U}} \cap \mathcal{U}_i\}_{i \in I}$ is a covering of $\tilde{\mathcal{U}}$ and $\varphi(x) = 0$ b/c $\mathcal{G}$ is a sheaf. I.e. $\varphi = 0$ since $\varphi(\tilde{\mathcal{U}}) = 0$ for all open subsets $\tilde{\mathcal{U}} \subset \mathcal{U}$. This verifies the first sheaf condition.

Now suppose $\varphi_i \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(\mathcal{U}_i)$ s.t.

$$\varphi_i \mid_{\mathcal{U}_i \cap \mathcal{U}_j} = \varphi_j \mid_{\mathcal{U}_i \cap \mathcal{U}_j}$$

for all i, j, and $\tilde{\mathcal{U}} \subset \mathcal{U}$ is open. Then there exists an element $\varphi \in \mathcal{H}om(\mathcal{F}, \mathcal{G})(\mathcal{U}_i)$ s.t. $\varphi|_{\mathcal{U}_i} = \varphi_i$ for all i b/c $\mathcal{G}$ is a sheaf and the φ_i's are compatible on intersections. This verifies the second sheaf condition.

1.18 Recall that $f^{-1}\mathcal{G}$ is the sheaf associated to the presheaf

$$\mathcal{H}(U) = \operatorname*{colim}_{V \supset f(U)} \mathcal{G}(V).$$

Hence there is a canonical bijection between $\mathrm{Hom}_X(\mathcal{H},\mathcal{F})$ and $\mathrm{Hom}_X(f^{-1}\mathcal{G},\mathcal{F})$ for every sheaf $\mathcal{F}$ on X. In what follows we consider $\mathcal{H}$ in order to define natural maps $\Theta : f^{-1}f_*\mathcal{F} \to \mathcal{F}$ and $\eta : \mathcal{G} \to f_*f^{-1}\mathcal{G}$.

For $U \subset X$ open, it suffices to construct

$$\Theta(U) : \operatorname*{colim}_{V \supset f(U)} f_*\mathcal{F}(V) = \operatorname*{colim}_{V \supset f(U)} \mathcal{F}(f^{-1}(V)) \to \mathcal{F}(U).$$

Since $V \supset f(U)$, $f^{-1}(V) \supset f^{-1}(f(U)) \supset U$ implies there exists a unique map by the universality of the colimit; it is compatible with the restriction maps.

For $V \subset Y$ open,

$$f_*f^{-1}\mathcal{G}(V) = f^{-1}\mathcal{G}(f^{-1}(V)) = \operatorname*{colim}_{W \supset f(f^{-1}(V))} \mathcal{G}(W).$$

Since $V \supset f(f^{-1}(V))$, there is a natural map

$$\eta(V) : \mathcal{G}(V) \to f_*f^{-1}\mathcal{G}(V)$$

compatible with the restriction maps.

Next we employ

$$\Theta_{\mathcal{F}} : f^{-1}f_*\mathcal{F} \to \mathcal{F}$$

and

$$\eta_{\mathcal{G}} : \mathcal{G} \to f_*f^{-1}\mathcal{G}$$

to show there is a bijection:

$$\mathrm{Hom}_X(f^{-1}\mathcal{G},\mathcal{F}) \approx \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F})$$

$$\varphi \longmapsto f_*\varphi \circ \eta_{\mathcal{G}}$$

$$\Theta_{\mathcal{F}} \circ f^{-1}\psi \longleftarrow \psi$$

That is, f^{-1} is left adjoint to f_* (often written $f^{-1} \dashv f_*$). To begin with, we show that

$$\psi = f_*\theta_{\mathcal{F}} \circ f_* f^{-1}\psi \circ \eta_{\mathcal{G}}$$

for every $\psi : \mathcal{G} \to f_*\mathcal{F}$. First we find a simpler expression for $f_* f^{-1}\varphi \circ \eta_{\mathcal{G}}$ where $\varphi : \mathcal{G} \to \mathcal{H}$ is a map between sheaves on Y.

If $V \subset Y$ is open, there is a commutative diagram:

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r, "\varphi(V)"] \arrow[d, "\eta_{\mathcal{G}}(V)"'] & \mathcal{H}(V) \arrow[d, "\eta_{\mathcal{H}}(V)"] \\
\operatorname*{colim}_{W \supset f(f^{-1}(V))} \mathcal{G}(W) \arrow[r, "f_* f^{-1}\varphi(V)"] & \operatorname*{colim}_{W \supset f(f^{-1}(V))} \mathcal{H}(W)
\end{tikzcd}$$

(The map $f_* f^{-1}\varphi(V)$ is induced by the maps $\mathcal{G}(W) \to \mathcal{H}(W)$ for $W \supset f(f^{-1}(V))$.)

Hence it suffices to show the equality

$$\psi = f_*\theta_{\mathcal{F}} \circ \eta_{f_*\mathcal{F}} \circ \psi,$$

i.e.

$$f_*\theta_{\mathcal{F}} \circ \eta_{f_*\mathcal{F}} = \mathrm{id}_{f_*\mathcal{F}}.$$

Again, if $V \subset Y$ is open, we are left with showing that $f_*\theta_{\mathcal{F}}(V) \circ \eta_{f_*\mathcal{F}}(V)$ is the identity map on $f_*\mathcal{F}(V) = \mathcal{F}(f^{-1}(V))$. Recall that

$$\eta_{f_*\mathcal{F}}(V) : \mathcal{F}(f^{-1}(V)) \to \operatorname*{colim}_{W \supset f(f^{-1}(V))} \mathcal{F}(f^{-1}(W))$$

is the canonical map into the colimit, while

$$f_*\theta_{\mathcal{F}}(V) : \operatorname*{colim}_{W \supset f(f^{-1}(V))} \mathcal{F}(f^{-1}(W)) \to \mathcal{F}(f^{-1}(V))$$

is induced by the restriction maps $\mathcal{F}(f^{-1}(W)) \to \mathcal{F}(f^{-1}(V))$.

Hence the composite map equals the restriction map on $\mathcal{F}(f^{-1}(V))$, i.e. the identity on $\mathcal{F}(f^{-1}(V))$.

For the second composite map, let $\varphi : \mathcal{H} \to \mathcal{F}$ be a map between sheaves on X. Then

$$\varphi \circ \Theta_{\mathcal{H}} = \Theta_{\mathcal{F}} \circ f^{-1} f_* \varphi .$$

In particular, for $\varphi : f^{-1}\mathcal{G} \to \mathcal{F}$ we want to show

$$\varphi = \Theta_{\mathcal{F}} \circ f^{-1} f_* \varphi \circ f^{-1} \eta_{\mathcal{G}} = \varphi \circ \Theta_{f^{-1}\mathcal{G}} \circ f^{-1} \eta_{G},$$

i.e.

$$\Theta_{f^{-1}\mathcal{G}} \circ f^{-1} \eta_{\mathcal{G}} = \mathrm{id}_{f^{-1}\mathcal{G}} .$$

If $\mathcal{U} \subset X$ is open, let $V \supset f(\mathcal{U})$ and denote by

$$\alpha_V : \mathcal{G}(V) \to f^{-1}\mathcal{G}(\mathcal{U}) = \operatorname*{colim}_{W \supset f(\mathcal{U})} \mathcal{G}(W)$$

$$\beta_V : f_* f^{-1}\mathcal{G}(V) \to f^{-1} f_* f^{-1}\mathcal{G}(\mathcal{U}) = \operatorname*{colim}_{W \supset f(\mathcal{U})} f_* f^{-1}\mathcal{G}(W)$$

the canonical maps. Then, by definition, the diagram

$$\begin{tikzcd}
\mathcal{G}(V) \arrow[r, "\eta_{\mathcal{G}}(V)"] \arrow[d, "\alpha_V"'] \arrow[dr, phantom, "\circlearrowleft"] & f_* f^{-1}\mathcal{G}(V) \arrow[d, "\beta_V"] \\
\operatorname*{colim}_{W \supset f(\mathcal{U})} \mathcal{G}(W) \arrow[r, "f^{-1}\eta_{\mathcal{G}}(\mathcal{U})"] & \operatorname*{colim}_{W \supset f(\mathcal{U})} f_* f^{-1}\mathcal{G}(W)
\end{tikzcd} \qquad (1)$$

commutes. Since $V \supset f(\mathcal{U}) \Rightarrow f^{-1}(V) \supset \mathcal{U}$ there is a restriction map

$$\rho_{f^{-1}(V),\mathcal{U}} : f_* f^{-1}\mathcal{G}(V) = f^{-1}\mathcal{G}(f^{-1}(V)) \to f^{-1}\mathcal{G}(\mathcal{U}),$$

so that by definition of $\Theta_{f^{-1}\mathcal{G}}(\mathcal{U})$ we have

$$\Theta_{f^{-1}\mathcal{G}}(U) \circ \beta_V = \rho_{f^{-1}(V),U}. \qquad (2)$$

Moreover, according to the definitions of $\eta_{\mathcal{G}}(V)$ and $f^{-1}\mathcal{G}$,

$$\rho_{f^{-1}(V),U} \circ \eta_{\mathcal{G}}(V) = \alpha_V. \qquad (3)$$

Assembling (1) - (3), we deduce

$$\begin{aligned} & \Theta_{f^{-1}\mathcal{G}}(U) \circ f^{-1}\eta_{\mathcal{G}}(U) \circ \alpha_V \\ = \; & \Theta_{f^{-1}\mathcal{G}}(U) \circ \beta_V \circ \eta_{\mathcal{G}}(V) \\ = \; & \rho_{f^{-1}(V),U} \circ \eta_{\mathcal{G}}(V) \\ = \; & \alpha_V \\ = \; & \mathrm{id} \circ \alpha_V. \end{aligned}$$

This implies the desired equality.

The bijection is natural in the following sense: If $\mathcal{G} \to \mathcal{H}$ is a map of sheaves on Y, there is an induced map of Hom-sets for every sheaf $\mathcal{F}$ on X

$$\mathrm{Hom}_Y(\mathcal{H}, f_*\mathcal{F}) \to \mathrm{Hom}_Y(\mathcal{G}, f_*\mathcal{F}).$$

By pulling back to X, we get a map $f^{-1}\mathcal{G} \to f^{-1}\mathcal{H}$ and an induced map between Hom-sets

$$\mathrm{Hom}_X(f^{-1}\mathcal{H}, \mathcal{F}) \to \mathrm{Hom}_X(f^{-1}\mathcal{G}, \mathcal{F}).$$

Naturality means that these maps are compatible with the bijections between the Hom-sets, i.e. the diagram

$$\begin{tikzcd} \mathrm{Hom}_X(f^{-1}\mathcal{H},\mathcal{F}) \arrow[r] \arrow[d] & \mathrm{Hom}_Y(\mathcal{H},f_*\mathcal{F}) \arrow[d] \\ \mathrm{Hom}_X(f^{-1}\mathcal{G},\mathcal{F}) \arrow[r] & \mathrm{Hom}_Y(\mathcal{G},f_*\mathcal{F}) \end{tikzcd}$$

commutes. And for maps of sheaves $\mathcal{E}\to\mathcal{F}$ on X, naturality means the diagram

$$\begin{tikzcd} \mathrm{Hom}_X(f^{-1}\mathcal{G},\mathcal{E}) \arrow[r] \arrow[d] & \mathrm{Hom}_Y(\mathcal{G},f_*\mathcal{E}) \arrow[d] \\ \mathrm{Hom}_X(f^{-1}\mathcal{G},\mathcal{F}) \arrow[r] & \mathrm{Hom}_Y(\mathcal{G},f_*\mathcal{F}) \end{tikzcd}$$

commutes for every sheaf $\mathcal{G}$ on Y.

1.19 (a) For $p \in X \setminus Z$ there exists an open neighborhood V of p disjoint from Z. Thus

$$(i_*\mathcal{F})(V) = \mathcal{F}(i^{-1}(V)) = \mathcal{F}(V \cap Z) = \mathcal{F}(\emptyset) = 0$$

implies the stalk $\mathcal{F}_p$ is trivial. With the assumptions $p \in Z$ and V an open neighborhood of p, $(i_*\mathcal{F})(V) = \mathcal{F}(V \cap Z)$ implies that $(i_*\mathcal{F})_p \approx \mathcal{F}_p$.

(b) If $p \in \mathcal{U}$, there exists an open neighborhood V of p contained in $\mathcal{U}$. The presheaf $V \mapsto \mathcal{F}(V)$ has stalk $(j_!\mathcal{F})_p = \mathcal{F}_p$ with this assumption. If $p \in \mathcal{U}$, no open neighborhood of p can be contained in $\mathcal{U}$, so the presheaf $V \mapsto 0$ implies that $(j_!\mathcal{F})_p = 0$ in this case. For $V \subset X$ open, the sections of $(j_!\mathcal{F})(V)$ are maps taking values in $\coprod_{p \in V} (j_!\mathcal{F})_p \Rightarrow j_!\mathcal{F}$ is the only sheaf on X satisfying the given conditions, cf. 1.2.

(c) For $p \in \mathcal{U}$, the sequence simplifies to $0 \to \mathcal{F}_p \to \mathcal{F}_p \to 0 \to 0$ at p. For $p \in Z$, the sequence simplifies to $0 \to 0 \to \mathcal{F}_p \to \mathcal{F}_p \to 0$ at p. It follows that

$$0 \to j_!(\mathcal{F}|_{\mathcal{U}}) \to \mathcal{F} \to i_*(\mathcal{F}|_Z) \to 0$$

is exact.

1.21 (a) Suppose $U \subset X$ open is covered by $\{U_i\}_{i\in I}$, and consider a section $s \in \mathcal{I}_Y(U)$ s.t. $s_{|U_i} = 0$ for all $i \in I$. By viewing $s \in \mathcal{O}_X(U)$ it follows that $s=0$. If there exist elt's $s_i \in \mathcal{I}_Y(U_i)$ s.t. $s_{i|U_i \cap U_j} = s_{j|U_i\cap U_j}$ for all i, j, then there exists an element $s \in \mathcal{O}_X(U_i)$ s.t. $s_{|U_i} = s_i$. Since each s_i vanishes on Y, s also vanishes on Y and therefore $s \in \mathcal{I}_Y(U)$. Hence $\mathcal{I}_Y$ is a sheaf.

(b) By restricting a function $U \to k$ that is locally a quotient of polynomials to $U \cap Y$ gives a section of $i_* \mathcal{O}_Y(U)$. It is clear that the sequence

$$0 \to \mathcal{I}_Y \to \mathcal{O}_X \to i_* \mathcal{O}_Y$$

is exact. Next we show that $\mathcal{O}_X \to i_*\mathcal{O}_Y$ is surjective. Let $p \in X$. If $p \notin Y$, then $(i_*\mathcal{O}_Y)_p = 0$ (so the map on stalks is trivially surjective). If $p \in Y$, every element of $(i_*\mathcal{O}_Y)_p$ is represented by a rational function which also represents an element of $(\mathcal{O}_X)_p$, i.e. $(\mathcal{O}_X)_p \to (i_*\mathcal{O}_Y)_p$ is onto since it is given by restriction.

(c) The sequence takes the form

$$0 \to \mathcal{I}_Y \to \mathcal{O}_{\mathbb{P}^1} \to i_*\mathcal{O}_p \oplus i_*\mathcal{O}_q \to 0.$$

(This is exact by the argument for (a).) Now $\mathcal{O}_{\mathbb{P}^1}(\mathbb{P}^1) = k$ – global regular functions on $\mathbb{P}^1$ are constant – and restriction til Y gives a function whose value at p and q are the same. However, there are clearly functions on Y with different values at p and q.

(d) We show there is a short exact sequence

$$0 \to \mathcal{O} \to \mathcal{K} \xrightarrow{\varphi} \bigoplus_{p\in X} i_p(\mathcal{I}_p) \to 0.$$

For $q \in X$ and f a germ of the stalk $\mathcal{K}_q$, $\varphi_q(f) = 0$ if and only if f is regular at q. Hence it remains to show that φ_q is surjective. An element of $\mathcal{I}_q$ is an equivalence class of h/z^n where h is a rational function of z regular at q. We can write $h/z^n = c_n/z^n + h_1/z^m h_2$ for some $m<n$, where $h_2(q) \neq 0$ and $c_n = h(q)$. Proceeding inductively there exist constants $c_n, \ldots, c_1$ and a polynomial h_0 s.t.

$$h/z^n = c_n/z^n + \ldots + c_1/z + h_0/h_2 .$$

Here h_0/h_2 is regular at q, so that h/z^n and $g = c_n/z^n + \ldots + c_1/z$ define the same element in I_q. Note that the latter has only a pole at q. Thus φ_q is surjective.

<u>Remark:</u> The stalk of $\bigoplus_{p \in X} i_p(I_p)$ at q is $\mathcal{K}/\mathcal{O}_p$ so exactness is clear, but will use the more detailed analysis in the solution of (e).

(e) The global section functor is left exact by <u>1.8</u> so we focus on surjectivity of the map $\Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O})$. By (d) we can consider $f \in \Gamma(X, \mathcal{K}/\mathcal{O})$ is an element of $\Gamma(X, \bigoplus_{p \in X} i_p(I_p))$, i.e. $f = f_1 + \ldots + f_r$ where f_i is equivalent to $c_n/z^n + \ldots + c_1/z$. For each such "principal part" there exists a rational function g_i as in (d) (regular everywhere except at q_i and differs from f by a regular function at q_i). Thus $g_1 + \ldots + g_r$ is an element of $\Gamma(X, \mathcal{K})$ which maps to f, and surjectivity follows.

<u>Cousin problem:</u> Given a finite set of points on $\mathbb{P}^1$ with corresponding principal parts, there exists a rational function regular everywhere except at the given points which has the same "principal parts" at those points.

2.7 A continuous map between the underlying topological spaces $f: \mathrm{Spec}(K) \to X$ corresponds to choosing some point $x = f((0)) \in X$. For f to be a scheme map, we need to have a sheaf map $f^{\#}: \mathcal{O}_X \to f_* \mathcal{O}_{\mathrm{Spec}(K)}$. Suppose $\mathcal{U} \subset X$ open. If $x \notin \mathcal{U}$, then $f^{-1}(\mathcal{U}) = \phi$, so that $f^{\#}(\mathcal{U}) = 0$. On the other hand, if $x \in \mathcal{U}$ we need maps compatible with restrictions (for all open $\mathcal{U}$ containing x)

$$f^{\#}(\mathcal{U}) : \mathcal{O}_X(\mathcal{U}) \to K.$$

By the defn of "colimit" this amounts to the choice of a map

$$f^{\#}_x : \mathcal{O}_{X,x} = \operatorname*{colim}_{x \in \mathcal{U}} \mathcal{O}_X(\mathcal{U}) \to K.$$

In addition, we require that $f^{\#}_x(m_x) = (0)$. This gives the desired bijection with the (necessarily injective) field maps $\mathcal{O}_x / m_x = k(x) \hookrightarrow K$.

Remark: $f_* \mathcal{O}_{\mathrm{Spec}(k)}$ is the "skyscraper" sheaf with ring of sections K, cf. H: 1.17.

2.16 (a) $x \in U \cap X_f \iff x \in U$ and $f_x \notin \mathfrak{m}_x \iff x \in U$ and $\bar{f}_x \notin \mathfrak{m}_x \iff$ $x \in U$ and $\bar{f} \notin \mathfrak{m}_x \iff x \in D_{\bar{f}}$. Thus X_f is a union of open subsets, i.e. open in X ($\{U_i\}$ affine open covering of $X \Rightarrow X_f = \bigcup_i U_\alpha \cap X_f$).

(b) Take a finite cover of X by open affines $U_i = \mathrm{Spec}(A_i)$, $1 \le i \le n$, and let $a_i = a_{|U_i}$, $f_i \equiv f_{|U_i}$ (elts. of A_i). Then $a_{|X_f} = 0 \Rightarrow a_i$ maps to 0 in the localization $(A_i)_{f_i}$ for all $i \Rightarrow \exists\, n_i \in \mathbb{Z}$ s.t. $f_i^{n_i} a_i = 0$ in A_i for all i. Let $n = \sup\{n_i\}$. Then $f^n a$ vanishes on U_i for all $i \Rightarrow$ it is zero as an elt. of $A = \Gamma(X, \mathcal{O}_X)$.

(c) Write $U_i = \mathrm{Spec}(A_i)$, $f_i \equiv f_{|U_i}$, $b_i \equiv b_{|U_i \cap X_f}$. By (a), $U_i \cap X_f = D_{f_i}$, so that $b_i = c_i / f_i^{n_i}$ for $c_i \in A_i$, $n_i \in \mathbb{Z}$. We may increase n_i for all i if need be so that they are the same, equal to n', i.e. $b_i = c_i / f_i^{n'}$ for $1 \le i \le n$. Then for all $i \ne j$, $c_{i|U_i \cap U_j} - c_{j|U_i \cap U_j}$ vanishes on $U_i \cap U_j \cap X_f \Rightarrow$ by part (b) applied to the quasi-compact set $U_i \cap U_j$, there exists an integer n_{ij} s.t. $f_i^{n_{ij}} c_i$ and $f_j^{n_{ij}} c_j$ agree on $U_i \cap U_j$. Let $n'' = \sup\{n_{ij}\}$. Then the sections $f_i^{n''} c_i$ can be glued to give a section of $A = \Gamma(X, \mathcal{O}_X)$, which agrees with $f_i^{n'+n''} b_i = (f^{n'+n''} b)_{|D_{f_i}}$ on $U_i \cap X_f$ for all $i \Rightarrow$ its restriction to X_f equals $f^{n'+n''} b$, as desired ($n = n' + n''$).

(d) Define $\varphi: A_f \to \Gamma(X_f, \mathcal{O}_{X_f})$ by $a/f^n \mapsto (a/f^n)_{|X_f}$ (homomorphism b/c restriction is so). It is well-defined b/c the stalk f_x is invertible in the local ring $\mathcal{O}_x$ if $x \in X_f$. If $(a/f^n)_{|X_f} = 0$, then $a_{|X_f} = 0 \Rightarrow$ by part (b) $\exists\, n > 0$ s.t. $f^n a = 0 \Rightarrow a = 0 \in A_f \Rightarrow \varphi$ is injective. If $b \in \Gamma(X_f, \mathcal{O}_{X_f})$, part (c) $\Rightarrow$ $\exists\, n > 0$ s.t. $f^n b$ is the restriction of an element $a \in \Gamma(X, \mathcal{O}_X)$ to X_f. We have $\varphi(a/f^n) = a_{|X_f} / f^n_{|X_f} = f^n b / f^n_{|X_f} = b \Rightarrow \varphi$ is surjective.

2.17 (a) Define g_i to be the inverse of $f_{|f^{-1}(U_i)} : f^{-1}(U_i) \to U_i$. Then on intersections $U_i \cap U_j$, the maps g_i and g_j agree $\Rightarrow$ can glue the maps g_i into $g: Y \to X$ s.t. $g_{|U_i} = g_i$, and clearly g is an inverse of f (topologically). Moreover, since g_i is an isomorphism for all i, the induced maps of stalks are isomorphisms $\Rightarrow f^\#$ is an isomorphism.

(b) If $X = \mathrm{Spec}(A)$, the identity generates the unit ideal and we can take $r=1, f=1$ (1_x is the mult. identity for all points $x \in X \Rightarrow$ not in the maximal ideal m_x of $\mathcal{O}_x$).

Conversely, suppose $f_1, \dots, f_r \in A = \Gamma(X, \mathcal{O}_X)$ generate the unit ideal of A and $X_{f_1}, \dots, X_{f_r}$ are affine. Then for every $x \in X$, $(f_1)_x, \dots, (f_r)_x$ generate the unit ideal of $\mathcal{O}_x \Rightarrow \exists$ i s.t. $(f_i)_x \notin m_x \Rightarrow x \in X_{f_i} \Rightarrow$ the X_{f_i}'s cover X. According to 2.16 (d), $\Gamma(X_{f_i}, \mathcal{O}_{X_{f_i}}) \approx A_{f_i} \Rightarrow X_{f_i} \approx \mathrm{Spec}(A_{f_i})$ since X_{f_i} is affine. And by 2.4 the identity map on $A = \Gamma(X, \mathcal{O}_X)$ induces a scheme map $\varphi : X \to \mathrm{Spec}(A)$. Thus the maps $\varphi_{f_i} : X_{f_i} \to \mathrm{Spec}(A_{f_i})$ induced by φ are isomorphisms for all $i \Rightarrow \varphi$ isomorphism by part (a), i.e. X is affine.

3.13 Properties of morphisms of finite type

(a) $i: X \hookrightarrow Y = \bigcup_{i\in I} U_i = \mathrm{Spec}(A_i)$. On each U_i there is a closed immersion $i^{-1}(i(X)\cap U_i) \to U_i$ induced by $A_i \to A_i/I_i$ for some ideal $I_i \subset A_i$ (cf. 3.11(b) and Corollary 5.10). Since A_i/I_i is a fin.gen. A_i-algebra, i is of finite type.

(b) $i: U \hookrightarrow X$ quasi-compact (3.2) open immersion. By 3.3(a) it suffices to show that i is locally of finite type. Let $Y=\bigcup_{i\in I}\mathrm{Spec}(A_i)$ be an affine open covering. Then i restricts to open immersions $U_i \hookrightarrow \mathrm{Spec}(A_i)$, and the source acquires a covering by basic open sets $D_f \simeq \mathrm{Spec}((A_i)_f)$. Note that $(A_i)_f$ is a fin.gen. A_i-algebra. And since i is quasi-compact, finitely many of the D_f's cover U_i.

(c)
$$\begin{tikzcd} X \arrow[r, "f"] \arrow[rr, bend right, "h"'] & Y \arrow[r, "g"] & Z \end{tikzcd}$$
f, g finite type, $h = g\circ f$

If $U=\mathrm{Spec}(C)$ is an open affine subscheme of Z, $g^{-1}(U) = \bigcup_{i\in I, |I|<\infty} \mathrm{Spec}(B_i)$ where B_i is a fin.gen. C-algebra according to 3.3(b). Likewise, $f^{-1}(\mathrm{Spec}(B_i))$ acquires a finite covering $\{\mathrm{Spec}(A_{ij})\}_{j\in J, |J|<\infty}$ where A_{ij} is a fin.gen. B_i-algebra. Hence A_{ij} is a fin.gen. C-algebra ($B_i[x_1,\dots,x_m] \twoheadrightarrow A_{ij}$, $C[x_1',\dots,x_n'] \twoheadrightarrow B_i$ give a surjection $C[x_1,\dots,x_m,x_1',\dots,x_n'] \twoheadrightarrow A_{ij}$). Done b/c $h^{-1}(U) = \bigcup \mathrm{Spec}(A_{ij})$.

(d)
$$\begin{tikzcd} X\times_Y Z \arrow[r, "f'"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Z \arrow[d, "g"] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$
f finite type

Suppose $U=\mathrm{Spec}(A) \overset{\text{open}}{\subset} Y$. Then $f^{-1}(U)$ can be covered by finitely many open affines $\mathrm{Spec}(B_i)$ s.t. B_i is a fin. gen. A-algebra. Next choose $\mathrm{Spec}(C)\subset g^{-1}(U)$. By the construction of the pullback, $f'^{-1}(\mathrm{Spec}(C))$ is covered by $\mathrm{Spec}(B_i\otimes_A C)$ – clearly a finite affine covering.

If $\{b_i\}_{i=1}^n$ generates B_i as an A-algebra, then $\{b_i\otimes 1\}_{i=1}^n$ generates $B_i\otimes_A C$ as a C-algebra $\Rightarrow$ $B_i\otimes_A C$ fin.gen. C-algebra. In order to conclude, cover Y by open affines $\{U_i\}$ and note that $\{g^{-1}(U_i)\}$ is a covering of Z. By the above, every $g^{-1}(U_i)$ can be covered by open affines, each of which inverse image in $X\times_Y Z$ can be covered by finitely many open affines satisfying the required finiteness criterion.

(e) The map factors as $X \times_S Y \to X \to S$. Now combine the assumptions with parts (c) and (d).

(f) It suffices to show that f is locally of finite type: $Z = \bigcup_{i \in I} \mathrm{Spec}(C_i)$ and choose a covering $\{\mathrm{Spec}(B_{ij})\}_{\substack{i \in I \\ j \in J}}$ of Y s.t. $\mathrm{Spec}(B_{ij}) \subset \mathrm{Spec}(C_i)$. Need to show $f^{-1}(\mathrm{Spec}(B_{ij}))$ has a finite covering $\{\mathrm{Spec}(A_{ijk})\}$ where A_{ijk} is a fin.gen. B_{ij}-algebra. In the solution we dodge the indices and let $\mathrm{Spec}(C) \subset Z$, $\mathrm{Spec}(B) \subset g^{-1}(\mathrm{Spec}(C))$ and $\mathrm{Spec}(A) \subset f^{-1}(\mathrm{Spec}(B))$. Then $\mathrm{Spec}(A) \subset h^{-1}(\mathrm{Spec}(C))$, for $h = g \circ f$, so that A is a fin.gen. C-algebra (cf. 3.3 (c)). By the setup there exists ring maps $C \to B \to A$. Now suppose $C[x_1, \dots, x_n] \to A$ is surjective (given by $C \to A$ and by picking out algebra generators $x_i \mapsto a_i$). Using the factorization of $C \to A$ through B, we get the factorization

$$C[x_1, \dots, x_n] \to B[x_1, \dots, x_n] \to A.$$

Hence A is a fin.gen. B-algebra.

(g) Let $\{\mathrm{Spec}(B_i)\}$ be a finite covering of Y. Then $f^{-1}(\mathrm{Spec}(B_i))$ has a finite covering $\{\mathrm{Spec}(A_{ij})\}$ and $X = \bigcup f^{-1}(\mathrm{Spec}(B_i))$. This displays X as a finite union of quasi-compact open subsets, so it is quasi-compact. Moreover, A_{ij} is a fin.gen. B_i-algebra and each B_i is noetherian b/c Y is noetherian; in particular, A_{ij} is noetherian (Hilbert's basis thm.). Combined this verifies that X is noetherian.

4.2 We start with the following observation.

Lemma $X \xrightarrow{f} Y$ scheme map, X reduced, $Z \overset{i}{\hookrightarrow} Y$ closed subscheme s.t. $f(X) \subset i(Z)$ – the set-theoretic images – $\Rightarrow$ f factors uniquely as $X \to Z \hookrightarrow Y$.

proof: • Suppose $X = \mathrm{Spec}(A)$, $Y = \mathrm{Spec}(B)$, so that f corresponds to a ring map $\varphi: B \to A$ and Z corresponds to an ideal $\mathfrak{I} \subset B$. By assumption on f, $\mathfrak{I} \subseteq \varphi^{-1}(\mathfrak{p})$ ($\varphi^{-1}(\mathfrak{p})$ lies in $V(\mathfrak{I})$) for all $\mathfrak{p} \in \mathrm{Spec}(A)$. The nilradical of A, say $\mathrm{Nil}(A)$ is the intersection of all prime ideals of A, so we conclude that $\mathfrak{I} \subseteq \varphi^{-1}(\mathrm{Nil}(A))$. Thus $\mathfrak{I} \subset \ker(\varphi)$ since A is reduced, and φ factors uniquely through $B \to B/\mathfrak{I}$ $\Rightarrow$ f factors uniquely through the inclusion $Z \hookrightarrow Y$.

• X reduced scheme: cover Y by open affines $\mathrm{Spec}(B_i)$, take their inverse images in X and perform the construction locally on affine patches. By uniqueness, the factorizations glue. ▨

We are ready to solve the exercise:

$$\begin{tikzcd}
X \arrow[dr, "\exists! h"] \arrow[drr, bend left, "g"] \arrow[ddr, bend right, "f"'] & & \\
& Y \times_S Y \arrow[r, "pr_2"] \arrow[d, "pr_1"'] & Y \arrow[d] \\
& Y \arrow[r] & S
\end{tikzcd}$$

$Y \to S$ separated $\Rightarrow$ $\Delta(Y) \overset{\text{closed}}{\subset} Y \times_S Y$

$\Rightarrow h^{-1}(\Delta(Y)) \subset X$ is closed. Now

$pr_1 h_{|U} = f_{|U} = pr_1 \Delta f_{|U}$ and

$pr_2 h_{|U} = g_{|U} = f_{|U} = pr_2 \Delta f_{|U}$

$\Rightarrow h_{|U} = \Delta f_{|U} = \Delta g_{|U}$

$\Rightarrow h(U) \subset \Delta(Y) \Rightarrow U \subset h^{-1}(\Delta(Y))$.

Since U is dense and $h^{-1}(\Delta(Y)) \subset X$ is closed, $h^{-1}(\Delta(Y)) = X$. According to the lemma, since X is reduced, h factors as $\Delta_Y \circ \tilde{h}$ for some $\tilde{h}: X \to Y$. By the defn of Δ_Y, $pr_1 \circ \Delta_Y = \mathrm{id}_Y = pr_2 \circ \Delta_Y$, and hence

$$f = pr_1 \circ h = pr_1 \circ \Delta_Y \circ \tilde{h} = \tilde{h}$$
$$g = pr_2 \circ h = pr_2 \circ \Delta_Y \circ \tilde{h} = \tilde{h}$$
$$\Rightarrow f = g.$$

Remark: We also have $f^{\#} = g^{\#}$.

Counterexamples (a) $X = Y = \mathrm{Spec}\left(\frac{k[x,y]}{(x^2, xy)}\right)$, $S = \mathrm{Spec}(k) \Rightarrow Y$ is separable over S b/c it is affine and X is not reduced. As maps we take $f = \mathrm{id}$ and g the map induced by the ring map given by $x \mapsto 0$, $y \mapsto y$. As the open subset we take $\mathcal{U} = D(y) \simeq \mathrm{Spec}\left(\left(\frac{k[x,y]}{(x^2,xy)}\right)_y\right) \Rightarrow f|_{\mathcal{U}}$ is the inclusion $\mathcal{U} \hookrightarrow X$ & since

$$\left(\frac{k[x,y]}{(x^2,xy)}\right)_y \approx \frac{k[x,y]_y}{(x^2,x)} \approx \frac{k[x,y]_y}{(x)} \approx k[y]_y,$$

localizing the ring map yields the identity map on $\mathrm{Spec}(k[y,y^{-1}]) \Rightarrow f|_{\mathcal{U}} = g|_{\mathcal{U}}$. It remains to show that $\mathcal{U}$ is dense: The minimal prime (x) of $\frac{k[x,y]}{(x^2,xy)}$ is a member of $\mathcal{U} \subset \mathrm{Spec}\left(\frac{k[x,y]}{(x^2,xy)}\right)$, so we have the inclusion $\overline{\mathcal{U}} \supset V((x)) = \mathrm{Spec}\left(\frac{k[x,y]}{(x^2,xy)}\right)$ (the equality holds b/c x is nilpotent).

(b) Let Y be $\mathbb{A}^1_k$ with a double origin, and let f & g map $X \equiv \mathbb{A}^1_k$ to the two copies of $\mathbb{A}^1_k$ used in the construction of Y. Then f & g coincide on $\mathbb{A}^1_k \setminus \{0\}$, but they are not equal.

4.4 3.13 (g) $\Rightarrow X, Y, Z$ noetherian (S-schemes).

$$\begin{tikzcd} Z \arrow[rr, "f|_Z"] \arrow[dr, "\text{proper}"'] & \circledcirc & Y \arrow[dl, "\text{separated}"] \\ & S & \end{tikzcd} \quad \overset{4.8(e)}{\Longrightarrow} f|_Z \text{ proper} \Rightarrow f(Z) \subset Y \text{ closed.}$$

$f(Z) \hookrightarrow Y \to S$; composition of finite type separated maps, so it remains to show it is universally closed.

$Z \to S$ closed due to properness & $Z \to f(Z)$ closed $\Rightarrow f(Z) \to S$ closed.

$$\begin{tikzcd} & W \arrow[d, "\text{any map}"] \\ f(Z) \arrow[r] & S \end{tikzcd} \overset{?}{\Longrightarrow} \begin{array}{c} f(Z)\times_S W \to W \\ \text{closed.} \end{array}$$

Have the commutative diagram:

$$\begin{tikzcd} Z\times_S W \arrow[d] \arrow[dr] & \\ f(Z)\times_S W \arrow[r] & W \end{tikzcd}$$

Now given that $Z\times_S W \to f(Z)\times_S W$ is surjective, it follows that $f(Z)\times_S W \to W$ is closed b/c $Z\times_S W \to W$ is closed ($Z \to S$ proper) — C closed in $f(Z)\times_S W$ $\Rightarrow (Z\times_S W \to f(Z)\times_S W)^{-1}(C)$ closed & $(f(Z)\times_S W \to W)(C) = (W \to Z\times_S W)$ applied to $(Z\times_S W \to f(Z)\times_S W)^{-1}(C)$ by surjectivity — We'll prove surjectivity in two steps:

<u>1st step</u> $f: X \to Y$ surjective S-map $\Rightarrow f\times \mathrm{id}_W : X\times_S W \to Y\times_S W$ is surjective. :

$$\begin{tikzcd} X\times_S W \arrow[r, "f\times \mathrm{id}_W"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & Y\times_S W \arrow[d] \\ X \arrow[r, "f"] & Y \end{tikzcd}$$

This is a pullback $X\times_Y Y\times_S W \simeq X\times_S W$

For surjectivity of $f\times \mathrm{id}_W$ we employ the second step.

<u>2nd step</u> $x \in X \to Y \ni y$, $X \to S$, $Y \to S$, $\circledcirc$, $x \mapsto s \in S$, $y \mapsto s$ $\overset{?}{\Longrightarrow} \exists\, * \in X\times_S Y$ s.t. $pr_X(*) = x$, $pr_Y(*) = y$.

In order to prove the implication, we consider the residue fields at x & y. Have commutative diagrams:

$$\begin{tikzcd} \mathrm{Spec}(k(x)\otimes_{k(s)} k(y)) \arrow[rr] \arrow[dr, dashed, "\exists!"] \arrow[dd] & & \mathrm{Spec}(k(y)) \ni (0) \arrow[dd] \\ & X\times_S Y \arrow[dl] \arrow[dr] & \\ \mathrm{Spec}(k(x)) \arrow[r] & X \arrow[r] \arrow[d] & Y \ni y \arrow[dl] \\ & S & \end{tikzcd}$$

$(0) \mapsto x \mapsto s$, $\quad (0) \mapsto y \mapsto s$, $\quad s \in S$

By the universal property of the pullback, $\exists!\ \mathrm{Spec}(k(x)\otimes_{k(s)} k(y)) \to X\times_S Y$.

Now the tensor product of two fields over a common base field is nonzero, so there exists an element $\tilde{*}$ in $\mathrm{Spec}(k(x)\otimes_{k(s)} k(y))$; its image in the pullback $X\times_S Y$ is the desired point.

According to the above, we have shown:

<u>Lemma</u> $f: X \to Y$ surjective

$g: Y \to Z$ separated & finite type $\quad \Longrightarrow \quad g$ proper.

$g \circ f$ proper

(no noetherian assumption required in this lemma)

5.1 First some background from linear algebra: If M an A-module, its dual is $\check{M} \equiv \mathrm{Hom}_A(M,A)$ (contravariant functor on the category of A-modules). There exists a natural A-linear map $\Theta: M \to \check{\check{M}}$ sending $m \in M$ to the A-linear map $\check{M} \to A$ given by $f \mapsto f(m)$. Now suppose M a free A-module of finite rank, say with basis $m_1, \dots, m_r$. Define $f_i \in \check{M}$ by $f_i(m_j) = \begin{cases} 0 & i \neq j \\ 1 & i = j \end{cases} \Rightarrow \check{M}$ is a free A-module with dual basis $f_1, \dots, f_r$. Moreover, $\Theta(m_j)(f_i) = f_i(m_j)$, so $\Theta(m_1), \dots, \Theta(m_r)$ form a basis for $\check{\check{M}}$ dual to the basis of $\check{M}$. Since Θ takes a basis for M to a basis for its double dual, we get:

<u>Lemma</u> M free A-module of finite rank $\Rightarrow \Theta: M \xrightarrow{\cong} \check{\check{M}}$.

(a) Since $\mathcal{E}$ is locally free and the isomorphism in the lemma is natural, so that the isomorphisms patches, we get $\mathcal{E} \xrightarrow{\cong} \check{\check{\mathcal{E}}}$ by defining $\mathcal{E}(V) \to \check{\check{\mathcal{E}}}(V)$ as above for $V \subset \mathcal{U}$ where $\mathcal{E}_{|\mathcal{U}}$ is a free $\mathcal{O}_X$-module of finite rank.

(b) Working locally we may assume $\mathcal{E}$ is a free $\mathcal{O}_X$-module with basis $\varepsilon_1, \dots, \varepsilon_r$. Let $\check{\varepsilon}_1, \dots, \check{\varepsilon}_r$ denote the corresponding dual basis for $\check{\mathcal{E}}$. Now for $\mathcal{U}$ open in X, define

$$\varphi_{\mathcal{U}}: \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})_{|\mathcal{U}} \to (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})_{|\mathcal{U}} \;;\; f \mapsto \sum_{i=1}^r \check{\varepsilon}_i \otimes f(\varepsilon_i),$$

$$\psi_{\mathcal{U}}: (\mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{O}_X) \otimes_{\mathcal{O}_X} \mathcal{F})_{|\mathcal{U}} \to \mathcal{H}om_{\mathcal{O}_X}(\mathcal{E}, \mathcal{F})_{|\mathcal{U}} \;;\; f \otimes a \mapsto (\varepsilon \mapsto f(\varepsilon)a).$$

When computing the composite maps we get

$$\begin{aligned} f \mapsto \sum_{i=1}^r \check{\varepsilon}_i \otimes f(\varepsilon_i) \mapsto \Big(\varepsilon \mapsto \sum_{i=1}^r \check{\varepsilon}_i(\varepsilon) f(\varepsilon_i)\Big) &= \Big(\varepsilon \mapsto f\Big(\sum_{i=1}^r \check{\varepsilon}_i(\varepsilon)\varepsilon_i\Big)\Big) \\ &= (\varepsilon \mapsto f(\varepsilon)), \end{aligned}$$

$$\begin{aligned} f \otimes a \mapsto (\varepsilon \mapsto f(\varepsilon)a) \mapsto \sum_{i=1}^r \check{\varepsilon}_i \otimes f(\varepsilon_i)a &= \sum_{i=1}^r \check{\varepsilon}_i f(\varepsilon_i) \otimes a \\ &= f \otimes a. \end{aligned}$$

These isomorphisms patches, and the claimed isomorphism follows.

(c) For $U \subset X$ open, $\mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} -$ is left adjoint to $\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U), -)$ and the isomorphisms

$$\mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{F}(U), \mathcal{G}(U)) \approx \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{F}(U), \mathrm{Hom}_{\mathcal{O}_X(U)}(\mathcal{E}(U), \mathcal{G}(U)))$$

patch up. To deduce an isomorphism between sheaves (recall that $\mathcal{E} \otimes \mathcal{F}$ is defined using sheafification) use that sheafification is left adjoint to the forgetful functor from sheaves to presheaves.

<u>Note:</u> In (c), $\mathcal{E}$ can be an arbitrary $\mathcal{O}_X$-module.

(d) For $\mathcal{E} \approx \mathcal{O}_Y^{\oplus n}$, there are natural isomorphisms

$$f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{E} \approx f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y^{\oplus n} \approx \bigoplus_{i=1}^{n} f_*\mathcal{F} \otimes_{\mathcal{O}_Y} \mathcal{O}_Y \approx (f_*\mathcal{F})^{\oplus n},$$

and

$$\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E} \approx \mathcal{F} \otimes_{\mathcal{O}_X} f^*(\mathcal{O}_Y^{\oplus n}) \approx \mathcal{F} \otimes_{\mathcal{O}_X} \left((f^{-1}(\mathcal{O}_Y))^{\oplus n} \otimes_{f^{-1}\mathcal{O}_Y} \mathcal{O}_X \right)$$
$$\approx \mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X^{\oplus n} \approx \mathcal{F}^{\oplus n}.$$

Thus

$$f_*(\mathcal{F} \otimes_{\mathcal{O}_X} f^*\mathcal{E}) \approx f_*(\mathcal{F}^{\oplus n}) \approx (f_*\mathcal{F})^{\oplus n}.$$

Again, these isomorphisms glue.

<u>Note:</u> $f^{-1}(\mathcal{O}_Y^{\oplus n}) \approx (f^{-1}(\mathcal{O}_Y))^{\oplus n}$ since f^{-1} commutes with direct sums, and likewise for f_*.

5.5 (a) Let $f: \mathbb{A}^1_k \to \operatorname{Spec}(k)$. Then $\mathcal{O}_{\mathbb{A}^1_k}$ is coherent, but its pushforward is not coherent b/c $k[X]$ is not a finitely generated k-module.

(b) Let $i: Z \hookrightarrow X$ be a closed immersion, and $\{U_i\}$ an affine cover of X. The restricted maps $i^{-1}(U_i) \to U_i$ are closed embeddings $\Rightarrow$ of the form (Cor. 5.10) $\operatorname{Spec}(A_i/I_i) \to \operatorname{Spec}(A_i)$ for some ideal $I_i \subset A_i$. Now A_i/I_i is a fin. gen. A_i-module (gen. by $1 + I_i$) $\Rightarrow$ i is a finite map.

(c) $\{\operatorname{Spec}(B_i)\}$ open affine cover of Y $\Rightarrow$ $f^{-1}(\operatorname{Spec}(B_i)) = \operatorname{Spec}(A_i)$ where A_i is a fin. gen. B_i-module (b/c f is finite). Since $\mathcal{F}$ is coherent and X is noetherian, $\mathcal{F}|_{\operatorname{Spec}(A_i)} \approx \widetilde{M}_i$ for M_i some fin. gen. A_i-module. Next we compute that

$$f_*\mathcal{F}|_{\operatorname{Spec}(B_i)} = (f|_{\operatorname{Spec}(A_i)})_* \mathcal{F}|_{\operatorname{Spec}(A_i)} = (f|_{\operatorname{Spec}(A_i)})_*(\widetilde{M}_i) \overset{5.2(d)}{\approx} \widetilde{{}_{B_i}M_i}$$

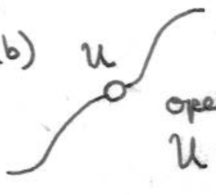
(where ${}_{B_i}M_i$ denotes M_i considered as a B_i-module). By the fin. gen. above, ${}_{B_i}M_i$ is a fin. gen. B_i-module $\Rightarrow$ $f_*\mathcal{F}$ is coherent.

5.15 (a) According to the equivalences of categories in Corollary 5.5, namely

$\{A\text{-mod}\} \approx \{\text{q.c. } \mathcal{O}_{\mathrm{Spec}(A)}\text{-mod}\}$ and $\{\text{fin.gen. } A\text{-mod}\} \approx \{\text{coherent } \mathcal{O}_{\mathrm{Spec}(A)}\text{-mod}\}$

we have: If $\mathcal{F}$ is quasi-coherent, there exists an A-module M s.t. $\mathcal{F} \approx \widetilde{M}$.

Clearly $M = \bigcup_{S \subset M} M_S^A$ where M_S^A is the A-submodule of M generated by some finite subset S of M. Next we show the isomorphism $\mathcal{F} \approx \bigcup_{S\subset M} \widetilde{M_S^A}$. In effect, we check on open subsets $\mathcal{U}$ of X. Note that $\widetilde{M}(D_f) \approx M_f \approx \bigcup_{S\subset M} (M_S^A)_f$. In general, cover $\mathcal{U}$ by finitely many basic open subsets D_{f_i}, $1 \le i \le r$.
Let $m \in \mathcal{F}(\mathcal{U}) \approx \widetilde{M}(\mathcal{U})$, and denote by m_i the image of m in M_{f_i} induced by the inclusion $D_{f_i} \subset \mathcal{U}$. Then there exists a finite subset $S_i \subset M$ s.t. $m_i \in \Gamma(D_{f_i}, \widetilde{M_{S_i}^A})$. Hence $m_i \in \Gamma(D_{f_i}, \widetilde{M^A_{\bigcup_{i=1}^r S_i}})$, so that the sheaf axiom $\overset{\text{uniqueness}}{\Rightarrow}$ $m \in \Gamma(\mathcal{U}, \widetilde{M^A_{\bigcup_{i=1}^r S_i}})$. Since $\bigcup_{i=1}^r S_i$ is a finite set, this shows that every elt. of $\mathcal{F}(\mathcal{U})$ is a member of the $\mathcal{U}$-sections of some coherent sheaf (contained in $\mathcal{F}$). Thus $\mathcal{F}$ is the union of its coherent subsheaves.

(b) 

$i : \mathcal{U} \hookrightarrow X$ open subset $\Rightarrow$ $\mathcal{U}$ noetherian.

By Proposition 5.8, $i_*\mathcal{F}$ is quasi-coherent b/c $\mathcal{U}$ is noetherian and $\mathcal{F}$ is coherent (in particular quasi-coherent). Part (a) $\Rightarrow i_*\mathcal{F} = \bigcup_{i\in I} \mathcal{F}_{\alpha_i}$, i.e. $i_*\mathcal{F}$ is the union of its subsheaves. Hence $i_*\mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{U})$ is the union of the $\mathcal{F}_{\alpha_i}(\mathcal{U})$'s. Now $\mathcal{F}$ coherent $\Rightarrow \mathcal{F}(\mathcal{U})$ noetherian $\Rightarrow \exists$ finitely many α_i s.t. $\mathcal{F}(\mathcal{U}) = \bigcup \mathcal{F}_{\alpha_i}(\mathcal{U})$ – b/c otherwise there would exist an infinite ascending chain of submodules. Since the latter is a finite union, $\exists\, \alpha$ s.t. $\mathcal{F}_\alpha(\mathcal{U}) = \mathcal{F}(\mathcal{U})$. Note that $\mathcal{F}_\alpha|_\mathcal{U}$ and $\mathcal{F}$ are coherent sheaves (in fact $\mathcal{F}_\alpha$ is coherent on X by the construction) on $\mathcal{U}$ with the same group of global sections. An easy check reveals then that $\mathcal{F}_\alpha|_\mathcal{U} = \mathcal{F}$.

(c) Define $\rho : \mathcal{G} \to i_*(\mathcal{G}_{|\mathcal{U}})$ by the restriction maps $\mathcal{G}(V) \to \mathcal{G}(\mathcal{U} \cap V) = i_*(\mathcal{G}_{|\mathcal{U}})(V)$ for $V \subset X$ open. Then $\rho^{-1}(i_*\mathcal{F}) \subset \mathcal{G}$ – the set-theoretic inverse image of $i_*\mathcal{F} \subset i_*(\mathcal{G}_{|\mathcal{U}})$ under ρ – is a subsheaf of a quasi-coherent sheaf $\Rightarrow$ it is quasi-coherent. By part (b), there exists a coherent subsheaf $\mathcal{F}'$ of $\rho^{-1}(i_*\mathcal{F})$ s.t. $\mathcal{F}'_{|\mathcal{U}} \simeq \mathcal{F}$. And $\mathcal{F}'$ is a subsheaf of $\mathcal{G}$.

(d) Let $\{\mathrm{Spec}(A_i)\}_{1 \le i \le r}$ be a finite open affine covering of X. Then the intersection $\mathcal{U} \cap \mathrm{Spec}(A_i)$ is an open subset of $\mathrm{Spec}(A_i)$. Part (c) applied to the coherent sheaf $\mathcal{F}_{|\mathcal{U} \cap \mathrm{Spec}(A_1)}$ and the quasi-coherent sheaf $\mathcal{G}_{|\mathrm{Spec}(A_1)}$ yields a subsheaf $\mathcal{F}_1 \subset \mathcal{G}_{|\mathrm{Spec}(A_1)}$ on $\mathrm{Spec}(A_1)$ s.t. $\mathcal{F}_{1\,|\mathcal{U} \cap \mathrm{Spec}(A_1)} \simeq \mathcal{F}_{|\mathcal{U} \cap \mathrm{Spec}(A_1)}$.

In the next step, consider the open subset $(\mathcal{U} \cup \mathrm{Spec}(A_1)) \cap \mathrm{Spec}(A_2)$ of $\mathrm{Spec}(A_2)$, i.e. $(\mathrm{Spec}(A_2) \cap \mathcal{U}) \cup (\mathrm{Spec}(A_1) \cap \mathcal{U})$, and $\mathcal{G}_{|\mathrm{Spec}(A_1) \cup \mathrm{Spec}(A_2)}$. Note that $\mathcal{G}_{|\mathrm{Spec}(A_2)}$ is a quasi-coherent sheaf on $\mathrm{Spec}(A_2)$ and the restriction of $\mathcal{F}_1$ to $(\mathcal{U} \cup \mathrm{Spec}(A_1)) \cap \mathrm{Spec}(A_2)$ defines a coherent subsheaf (on the latter intersection). Thus the conditions of part (c) are satisfied $\Rightarrow \exists$ coherent subsheaf $\mathcal{F}_2 \subset \mathcal{G}_{|\mathrm{Spec}(A_2)}$ whose restriction to $(\mathcal{U} \cup \mathrm{Spec}(A_1)) \cap \mathrm{Spec}(A_2)$ coincides with the corresponding restriction for $\mathcal{F}$, i.e. $\mathcal{F}_{|(\mathcal{U} \cup \mathrm{Spec}(A_1)) \cap \mathrm{Spec}(A_2)}$. In particular, $\mathcal{F}_1$ and $\mathcal{F}_2$ agree on the intersection $\mathrm{Spec}(A_1) \cap \mathrm{Spec}(A_2)$. Thus $\mathcal{F}_1$ & $\mathcal{F}_2$ glue together, and by iterating we get a coherent subsheaf $\mathcal{F}' \subset \mathcal{G}$ s.t. $\mathcal{F}'_{|\mathcal{U}} \simeq \mathcal{F}$ (this works in a finite number of steps since the covering is finite).

(e)* Following the hint, let $\mathcal{U} \overset{\text{open}}{\subset} X$ and choose a section $s \in \mathcal{F}(\mathcal{U})$. Applying part (d) to the subsheaf of $\mathcal{F}_{|\mathcal{U}}$ generated by $s \Rightarrow$ there exists a coherent subsheaf $\mathcal{F}'$ of $\mathcal{F}$ s.t. $s \in \mathcal{F}'(\mathcal{U})$ for all $\mathcal{U}$ and s as above. Hence $\mathcal{F}$ is the union of these coherent subsheaves.

* This part is not required in the main application <u>6.10.</u>

6.4 We show the more general result: Let A be UFD in which 2 is a unit, $a \in A$ some square free element (p a prime element in A, then $a \notin p^2A$) which is not a unit. Then $A[T]/(T^2-a)$ is normal.

proof: If K is the fraction field of A, the fraction field of $A[T]/(T^2-a)$ is $L = K[t]$ (where t is the residue class of T in $A[T]/(T^2-a)$). Suppose $r+st \in L$ is integral over A so that its minimal polynomial $X^2 - 2rX + (r^2 - as^2)$ has coefficients in A, provided $s \neq 0$. Hence $r+st$ is integral over $A \Leftrightarrow -2r \in A$ and $r^2 - as^2 \in A$. By the assumption $2 \in A^*$, it follows that $r \in A$. In order to show that $s \in A$, we observe that $s = \frac{s_1}{s_2}$ with s_1 & s_2 relatively prime elt's of A implies s_2 is a unit. Since $-as^2 \in A$, write $as^2 = a' \Rightarrow$ $a s_1^2 = a' s_2^2$. If the prime element p divides s_2, then $as_1^2 \in p^2A$. And since s_1 & s_2 are relatively prime, $a \in p^2A$ $\lightning$ Hence s_2 cannot have any prime factors $\Rightarrow$ it is a unit $\Rightarrow s \in A$. Thus the integral closure of A in L coincides with $A[t]$.

6.5 Quadratic hypersurfaces

$$X \equiv \operatorname{Spec}\left(A = \frac{k[x_0,\dots,x_n]}{(x_0^2+\dots+x_r^2)}\right) \;;\; k \text{ field of characteristic } \neq 2.$$

(a) By 6.4 it suffices to show that $f = x_0^2+\dots+x_r^2$ is $\square$-free. Now $\deg(f)=2 \Rightarrow f$ is a product of at most 2 nonconstant linear polynomials. If $f = (\sum a_i x_i)^2$, clearly $a_i^2 = 1$, $0\le i\le r$, and $2a_ia_j = 0$ for $i\neq j$. But then $2 = 2a_i^2a_j^2 = 0$ in $k \Rightarrow \operatorname{char}(k)=2$, violating the assumption.

(b) Assuming k contains a primitive 4^{th} root of unity, say i, the linear change of variables $x_0 \mapsto \frac{x_0-x_1}{2}$, $x_1 \mapsto \frac{x_0+x_1}{2i}$ allows us to write $-x_0x_1$ for $x_0^2+x_1^2$.

check the conditions

(1) $A = \dfrac{k[x_0,\dots,x_n]}{x_0x_1-x_2^2}$, with prime divisor $Y = \operatorname{Spec}(A/(x_1,x_2))$. Note that, in A, $x_1 = 0 \Rightarrow x_2 = 0$ so it follows that Y is set-theoretically cut out by x_1. Moreover, $A/(x_1,x_2)$ is a domain so its generic point is (0), which in this case is (x_1,x_2). By localizing A at (x_1,x_2), "not contained in the maximal ideal!" x_0 becomes a unit and we can write $x_1 = x_0^{-1}x_2^2 \in (x_2) \Rightarrow$ x_2 generates the maximal ideal in the localized ring. In particular, $v_Y(x_1) = 2$ (b/c of the equation $x_1 = x_0^{-1}x_2^2$ & units have valuation zero). (And since Y is cut out by x_1 there cannot exists any other prime divisors Z with $v_Z(x_1)\neq 0$.) Next we claim the equality $X\setminus Y = \operatorname{Spec}(A_{x_1})$. LHS consists of prime ideals $\mathfrak{p}$ of $k[x_0,\dots,x_n]$ s.t. $(x_0x_1-x_2^2)\subset\mathfrak{p}$ and $(x_1,x_2)\not\subset\mathfrak{p}$, while RHS consists of $\mathfrak{p}$ for which $(x_0x_1-x_2^2)\subset\mathfrak{p}$ and $x_1\notin\mathfrak{p}$. Clearly $x_1\notin\mathfrak{p}$ implies $(x_1,x_2)\not\subset\mathfrak{p}$. Conversely, suppose that $\mathfrak{p}\in X\setminus Y$. Then if $x_1\notin\mathfrak{p}$ we are done, so assume $x_2\notin\mathfrak{p}$ and $x_1\in\mathfrak{p}$. With this assumption, $x_0x_1-x_2^2\in\mathfrak{p} \Rightarrow x_0x_1-(x_0x_1-x_2^2) = x_2^2\in\mathfrak{p} \Rightarrow$ $x_2\in\mathfrak{p}$ $\lightning$ and the equality holds.

As rings we have

$$A_{x_1} = \frac{k[x_0,\dots,x_n][\frac{1}{x_1}]}{(x_0x_1-x_2^2)} \underset{x_0=\frac{x_2^2}{x_1}}{\approx} \overbrace{k[x_1,\tfrac{1}{x_1},x_2,\dots,x_n]}^{\text{UFD}}$$

H: Proposition 6.5 shows there is an exact sequence

$$(*) \quad \mathbb{Z} \to Cl(X) \to Cl(X \setminus Y) \to 0.$$

In $(*)$ the map $\mathbb{Z} \to Cl(X)$ sends 1 to the class of Y in the divisor class group of X. This map is surjective b/c $X \setminus Y$ is the Zariski prime spectrum of a UFD ($\Rightarrow Cl(X \setminus Y) = 0$). Moreover, by the above, $(x_1) = 2 \cdot Y$ is a principal divisor. Hence Y has order at most 2 in $Cl(X)$. It remains to show that Y is not principal.

Fact: A UFD $\Leftrightarrow Cl(X) = 0$ (H: Proposition 6.2)

It is well-known that A UFD $\Rightarrow$ every height 1 prime ideal is principal (cf. H: pg. 132, proof of Proposition 6.2). But it is clear that $(x_1, x_2) \in Spec(A)$ – which defines Y – is not principal: If $m = (x_0, \ldots, x_n) \in Max(A)$, then m/m^2 is a k-vector space of dimension $n+1$ (basis $\{\bar{x}_i\}_{i=0}^{n}$). We have $\mathfrak{p} \subset m$ and the image of $\mathfrak{p}$ in m/m^2 contains $\bar{x}_1$ and $\bar{x}_2$.
(The above is really Example 6.5.2 in H.)

(2) By a similar change of variables as in (1) we may rewrite $x_0^2 + \ldots + x_3^2$ as $x_0x_1 - x_2x_3$. If $\underset{\approx \mathbb{P}_k^1 \times \mathbb{P}_k^1}{Q}$ is the projective variety in $\mathbb{P}_k^3$ defined by this equation, then $Cl(Q) \approx \mathbb{Z} \oplus \mathbb{Z}$ according to Example 6.6.1. Let X' be the affine cone of Q in $\mathbb{A}_k^4$. Exercise 6.3 (b) implies $Cl(X') \approx \mathbb{Z}$. Now $X = X' \times \mathbb{A}_k^{n-4}$, so by Proposition 6.6 we get

$$Cl(X) \approx Cl(X') \approx \mathbb{Z}.$$

(3) Again, let Y be cut out by the equation $x_1 = 0$. There is an exact sequence $\mathbb{Z} \to Cl(X) \to Cl(X \setminus Y) \to 0$, where $Cl(X \setminus Y) = 0$ since $X \setminus Y = Spec(k[x_1, \frac{1}{x_1}, x_2, \ldots, x_n])$ (eliminate x_0 as in part (1)).

In this case we claim that Y is principal: Consider the ideal $(x_1) \subset A$ corresponding to the closed subset Y. If (x_1) is a prime ideal, then Y will be the principal divisor associated to the rational function x_1.
We'll prove this by showing that $A/(x_1)$ is a domain $\Leftrightarrow \dfrac{k[x_0,\dots,x_n]}{(x_1, x_2^2+\dots+x_r^2)}$ domain b/c $(x_1, x_0x_1-(x_2^2+\dots+x_r^2)) = (x_1, x_2^2+\dots+x_r^2) \Leftrightarrow \dfrac{k[x_0,x_2,\dots,x_n]}{(x_2^2+\dots+x_r^2)}$ domain (x_1 missing) $\Leftrightarrow f = x_2^2+\dots+x_r^2$ is irreducible for $r \geq 4$

Exercise: $f_n = a_0x_0^2+\dots+a_nx_n^2$ where $a_0\cdots a_n \neq 0$ is irreducible in $k[x_0,\dots,x_n]$ for $n \geq 2$.

solution: Induction shows that it suffices to establish that f_2 is irreducible (since otherwise, letting $x_3=\dots=x_r=0$ would give a factorization of f_2).
Suppose for contradiction that

$$\begin{aligned} f_2 &= (\alpha_0x_0+\alpha_1x_1+\alpha_2x_2)(\beta_0x_0+\beta_1x_1+\beta_2x_2) \\ &= \sum_{i=0}^{2}\alpha_i\beta_ix_i^2 + \sum_{0\leq i<j\leq 2}(\alpha_i\beta_j+\alpha_j\beta_i)x_ix_j. \end{aligned}$$

Now contemplate the linear equations

$$\begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \overset{(*)}{=} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \quad \begin{pmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_2 & 0 & \alpha_0 \\ 0 & \alpha_2 & \alpha_1 \end{pmatrix}\begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \overset{(**)}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$(*) \Rightarrow \alpha_0\alpha_1\alpha_2 \neq 0$ and $\beta_0\beta_1\beta_2 \neq 0$

$(**) \Rightarrow \alpha_0\alpha_1\alpha_2 = 0$ (for $(**)$ to have a nonzero solution, the determinant of the coefficient matrix must be zero).

(c) Exercise 6.3 $\Rightarrow$ exact sequence

$$(*) \quad 0 \to \mathbb{Z} \to Cl(Q) \to Cl(X) \to 0.$$

In (*), $\mathbb{Z} \to Cl(Q)$ maps 1 to $Q \cdot H$ (H any hyperplane of $\mathbb{P}^n_k$ not containing Y).

(1) When $r = 2$, $Cl(X) \approx \mathbb{Z}/2$. As in Example 6.5.2, $Cl(Q)$ is cyclic with generator the ruling of the projective quadric cone $\Rightarrow$ the class of a hyperplane section $Q \cdot H$ is twice the generator of $Cl(Q)$ (since it maps to zero in $Cl(X)$ by exactness).

(2) When $r = 3$, $Cl(X) \approx \mathbb{Z} \Rightarrow Cl(Q) \approx \mathbb{Z} \oplus \mathbb{Z}$.

(3) When $r \geq 4$, $Cl(X)$ is trivial $\Rightarrow Cl(Q) \approx \mathbb{Z}\{Q \cdot H\}$.

(d) The homogeneous coordinate ring $S(Q) = k[x_0, \dots, x_n]/(x_0^2 + \dots + x_r^2)$ is also the affine coordinate ring of X, and $Cl(X) = 0$ by part (3) of (b). Hence it is a UFD according to Proposition 6.2 (and (a)). It follows that every prime ideal of height 1 is principal (in the same ring) $\Rightarrow$ the prime ideal corresponding to Y is principal. Its generator gives V.

6.10 (a) The equivalence of categories (cf. H: Corollary 5.5)

$$\{\text{coherent } \mathcal{O}_{\mathbb{A}^1_k}\text{-modules}\} \simeq \{\text{fin.gen. } k[x]\text{-modules}\}$$

implies $K(\mathbb{A}^1_k) \simeq K(k[x])$ – Grothendieck group of fin.gen $k[x]$-modules –.
Since $k[x]$ is a PID, the latter group is infinite cyclic generated by $k[x]$ as shown in AM: 7.26 iii) (MAT 4200: Commutative Algebra, Fall 2008).

Remark: The Grothendieck group is an example of an $\mathbb{A}^1$-invariant functor.

(b) The rank map is well-defined: A short exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ yields $\dim_k \mathcal{F}_\xi = \dim_k \mathcal{F}'_\xi + \dim_k \mathcal{F}''_\xi$ b/c the stalk functor is exact.
Note that $\gamma(\mathcal{F}) + \gamma(\mathcal{G}) = \gamma(\mathcal{F} \oplus \mathcal{G})$ and $\dim_k (\mathcal{F} \oplus \mathcal{G})_\xi = \dim_k \mathcal{F}_\xi + \dim_k \mathcal{G}_\xi$, so that the rank map is a group homomorphism. It is surjective b/c $\gamma(\mathcal{O}_X)$ maps to 1.

(c) $i : Y \hookrightarrow X$ closed immersion with open complement $U = X \setminus Y$. Define $K(Y) \to K(X)$ by $\gamma(\mathcal{F}) \mapsto \gamma(i_* \mathcal{F})$ & $K(X) \to K(U)$ by $\gamma(\mathcal{F}) \mapsto \gamma(\mathcal{F}_{|U})$ (extend linearly). These maps are well-defined b/c i_* and restriction define exact functors (check on stalks: $(i_*\mathcal{F})_p = \begin{cases} \mathcal{F}_p & \text{if } p \in Y \\ 0 & \text{if } p \notin Y \end{cases}$ by H: 1.19(a)).
The map $K(X) \to K(U)$ is surjective b/c every coherent sheaf $\mathcal{F}$ on U can be extended to a coherent sheaf $\tilde{\mathcal{F}}$ on X s.t. $\tilde{\mathcal{F}}_{|U} = \mathcal{F}$ according to 5.15. It remains to consider exactness in the middle. Clearly

$$K(Y) \to K(X) \to K(U) \to 0$$

is a complex (image of $K(Y) \to K(X)$ contained in the kernel of $K(X) \to K(U)$).
Now suppose $\mathcal{F}$ is a coherent sheaf on X s.t. $\gamma(\mathcal{F}_{|U}) = 0$. We show that $\gamma(\mathcal{F})$ is in the image of $K(Y) \to K(X)$ following the hint.

Assuming the existence of such a filtration, we have

$$\gamma(\mathcal{F}) = \sum_{i=0}^{n-1} \gamma(\mathcal{F}_i/\mathcal{F}_{i+1}) \in K(X),$$

i.e. the class represented by $\mathcal{F}$ is in the image of $K(Y) \to K(X)$. Here we use that $\gamma(\mathcal{F}_i) = \gamma(\mathcal{F}_{i+1}) + \gamma(\mathcal{F}_i/\mathcal{F}_{i+1})$. There is an adjoint functor pair

$$i^* : \mathrm{Coh}_X \rightleftarrows \mathrm{Coh}_Y : i_*$$

between the categories of coherent sheaves on X and Y (cf. H: 5.5).
In particular, there is a natural map

$$\eta_{\mathcal{F}} : \mathcal{F} \to i_* i^* \mathcal{F}.$$

On an affine open subscheme $\mathrm{Spec}(A)$ of X, $\mathcal{F}$ has the form $\widetilde{M}$ for M some finitely generated A-module and $\mathrm{Spec}(A) \cap Y = \mathrm{Spec}(A/I)$ for some ideal $I \subset A$ (bijection between closed subschemes of affine schemes and ideals). Note that $\eta_{\widetilde{M}}$ is surjective since it corresponds to the canonical map $M \to M/IM$.

Define $\mathcal{F}_0 = \mathcal{F}$ and $\mathcal{F}_i$ inductively by $\mathcal{F}_i \equiv \ker(\eta_{\mathcal{F}_{i-1}})$. With this definition it follows that $\mathcal{F}_i/\mathcal{F}_{i+1}$ is a coherent $\mathcal{O}_Y$-module.
Next we verify that the filtration stabilizes. To wit, on an open affine, $\mathcal{F}_i|_{\mathrm{Spec}(A)} = I^i M$ and the support of $\widetilde{M}$ is contained in the closed subscheme $V(I)$ $(= \mathrm{Spec}(A/I))$.
Hence $I \subset \sqrt{I} \subset \sqrt{\mathrm{Ann}(M)}$ according to H: 5.6 (b). Now A is noetherian, so $\exists\, N \gg 0$ s.t. $I^N \subset \mathrm{Ann}(M) \Rightarrow I^N M = 0$.
This shows the filtration stabilizes since X is noetherian.

2.2 Clearly $\mathcal{K}$ and $\bigoplus_{p \in X} i_p(I_p)$ are flasque sheaves, so the first assertion follows from I, 1.21 (d). By part (e) of the same exercise, the sequence

$$\Gamma(X, \mathcal{K}) \to \Gamma(X, \mathcal{K}/\mathcal{O}) \to 0 \to \cdots .$$

is exact, and computes the sheaf cohomology of X with coefficients in $\mathcal{O}$; therefore, the group $H^p(X, \mathcal{O})$ is trivial for all $p > 0$.

4.1 By II Proposition 5.8, $f_* \mathcal{F}$ is a quasi-coherent sheaf on Y. Let $\{\mathcal{U}_i\}$ be an affine covering of Y. Since f is affine, $\{f^{-1}(\mathcal{U}_i)\}$ is an affine covering of X. Hence Theorem 4.5 $\Rightarrow$ there exists natural isomorphisms (for all $p \geq 0$)

$$\check{H}^p(\{f^{-1}(\mathcal{U}_i)\}, \mathcal{F}) \approx H^p(X, \mathcal{F}) \quad \text{and} \quad \check{H}^p(\{\mathcal{U}_i\}, f_* \mathcal{F}) \approx H^p(Y, f_* \mathcal{F}).$$

The Čech complexes computing $\check{H}^p(\{f^{-1}(\mathcal{U}_i)\}, \mathcal{F})$ resp. $\check{H}^p(\{\mathcal{U}_i\}, f_* \mathcal{F})$ are

$$\prod_{i_0 < \cdots < i_p} \mathcal{F}(f^* \mathcal{U}_{i_0 \cdots i_p}) \quad \text{resp.} \quad \prod_{i_0 < \cdots < i_p} \mathcal{F}(f^{-1}(\mathcal{U}_{i_0 \cdots i_p})),$$

where $f^* \mathcal{U}_{i_0 \cdots i_p} = \bigcap_{k=0}^{p} f^{-1}(\mathcal{U}_{i_k}) = f^{-1}(\mathcal{U}_{i_0 \cdots i_p})$. It follows that these complexes are isomorphic, so the sheaf cohomology groups are isomorphic.

1.3 Cover $\mathbb{A}^2_k \setminus \{(0,0)\}$ by the open affines

$$D(x) = \operatorname{Spec}(k[x,\tfrac{1}{x},y]) \quad \text{and} \quad D(y) = \operatorname{Spec}(k[y,\tfrac{1}{y},x]),$$

so that

$$\Gamma(D(x) \cap D(y), \mathcal{O}_X) = k[x^{\pm 1}, y^{\pm 1}].$$

This yields the Čech complex

$$0 \to k[x^{\pm 1}, y] \oplus k[y^{\pm 1}, x] \xrightarrow{(f,g) \mapsto g-f} k[x^{\pm 1}, y^{\pm 1}] \to 0.$$

Hence $\check{H}^1(\mathfrak{U}, \mathcal{O}_X)$ is the quotient of $k[x^{\pm 1}, y^{\pm 1}]$ by the image of d^0, i.e. the k-vector space generated by monomials $x^i y^j$ where at least one of i or j is non-negative. Since $H^1(U, \mathcal{O}_X) \simeq \check{H}^1(\mathfrak{U}, \mathcal{O}_X)$, the assertion follows.