

(1.8)

$U \subseteq X$ open subset, show that the functor $T(U, \cdot)$ from sheaves on X to ab groups is a left exact functor.

That is if $0 \rightarrow \mathcal{F}' \rightarrow \mathcal{F} \rightarrow \mathcal{F}''$ is exact, then $0 \rightarrow T(U, \mathcal{F}') \rightarrow T(U, \mathcal{F}) \rightarrow T(U, \mathcal{F}'')$ is exact.
 $0 \rightarrow \mathcal{F}'(U) \rightarrow \mathcal{F}(U) \rightarrow \mathcal{F}''(U)$

Solution $0 \rightarrow \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ exact seq. of sheaves

φ injective, $\ker \varphi = 0$

\Downarrow

$\varphi(U)$ injective, $\ker(\varphi(U)) = 0$ for each open $U \subseteq X$

Want to show that

$0 \rightarrow \mathcal{F}'(U) \xrightarrow{\varphi(U)} \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U)$ is an exact seq. of ab. groups.

Have that $\varphi(U)$ is injective, want to show that $\ker(\psi(U)) = \text{Im}(\varphi(U))$ to prove the claim.

$$\begin{array}{ccc} \mathcal{F}' & \xrightarrow{\varphi} & \mathcal{F} \xrightarrow{\psi} \mathcal{F}'' \text{ exact} \\ \mathcal{F}'(U) & \xrightarrow{\varphi(U)} & \mathcal{F}(U) \xrightarrow{\psi(U)} \mathcal{F}''(U) \end{array}$$

Since φ is injective, the image of φ is isomorphic to a subsheaf of \mathcal{F} : $\text{Im } \varphi \cong \mathcal{G}$ subsheaf of \mathcal{F} .
 (from the definition of a subsheaf and morphisms of sheaves)

Since \mathcal{G} is a sheaf, it's equal to its sheafification, giving $\text{im}(\varphi) = \text{lim}(\varphi)^+ = \ker(\psi)$.

Now, with $U \subseteq X$ open, and sheaves being defined on the open sets, we have $\text{im}(\varphi(U)) = \ker(\psi(U))$ which gives exactness of the sequence of abelian groups.

Ex 1.10

Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X . We define the direct limit of the system $\{\mathcal{F}_i\}$ denoted $\varinjlim \mathcal{F}_i$ to be the sheaf associated the presheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$.

Show that this is a direct limit in the category of sheaves on X i.e. that it has the following universal property:

given a sheaf \mathcal{G} and a collection of morphisms $\mathcal{F}_i \rightarrow \mathcal{G}$ compatible with the maps of the direct system, then $\exists!$ map $\varinjlim \mathcal{F}_i \rightarrow \mathcal{G}$ s.t. $\forall i \in I$ the original map $\mathcal{F}_i \rightarrow \mathcal{G}$ is obtained by composing the maps $\mathcal{F}_i \rightarrow \varinjlim \mathcal{F}_i \rightarrow \mathcal{G}$.

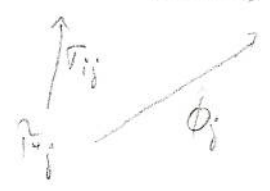
Solution

~~To simplify we consider the \mathcal{F}_i 's as sheaves of modules on X~~

Let \mathcal{F} the presheaf given by $\forall U \subseteq X$ open $U \mapsto \varinjlim \mathcal{F}_i(U)$

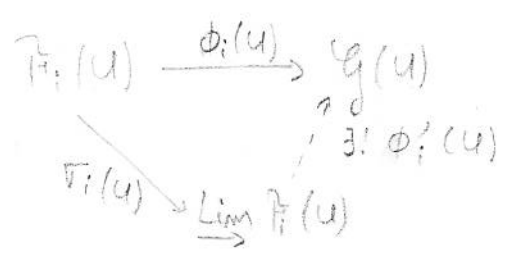
Let \mathcal{G} a sheaf on X and a collection of morphisms

$\phi_i : \mathcal{F}_i \rightarrow \mathcal{G}$ s.t. $\forall i, j \in I$ the diagram $\mathcal{F}_i \xrightarrow{\phi_i} \mathcal{G}$ commutes

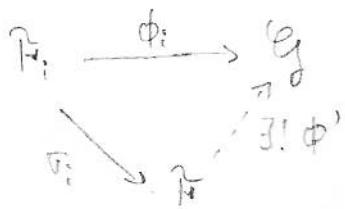


Let $U \subseteq X$ open

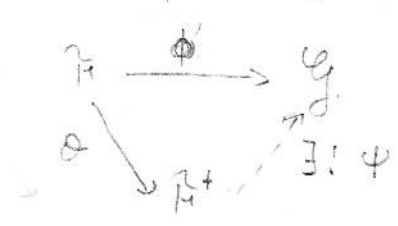
$\forall i \in I$ $\varinjlim \mathcal{F}_i(U)$ sat the universal ppty:



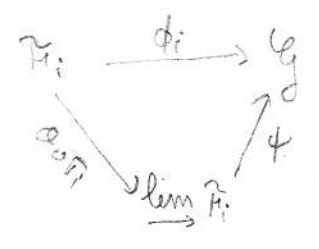
Hence we obtain the following commuting diagram $\forall i \in I$:



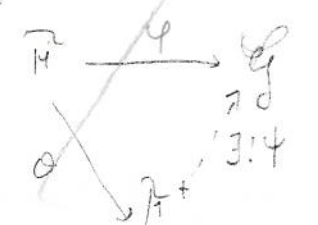
But by the prop 1.2 H $\exists!$ sheaf \mathcal{F}^+ and a morphism of sheaves $\mathcal{O}: \mathcal{F} \rightarrow \mathcal{F}^+$ s.t. the following diag commutes



ence put $\varinjlim \mathcal{F}_i = \mathcal{F}^+$ it sat the universal ppty: $\forall i \in I$ we have:



~~by the prop 1.2 H $\exists!$ sheaf \mathcal{F}^+ and morphism of sheaves $\mathcal{O}: \mathcal{F} \rightarrow \mathcal{F}^+$ s.t. \forall sheaf \mathcal{G} and morphism of presheaves $\psi: \mathcal{F} \rightarrow \mathcal{G}$ $\exists!$ morphism of sheaves ψ s.t. the following commutes~~



~~put $\varinjlim \mathcal{F}_i = \mathcal{F}^+$~~

1.11)

U åpen, $\{V_j\}$ åpen overdekning av U .

Velg $s \in \lim_i \mathcal{F}_i(U)$ s.d. $s|_{V_j} = 0 \forall j$

X Noetherisk \Rightarrow Vi kan følge $n: U \subseteq \bigcup_{j=1}^n V_j$, da

vi kan konstruere $W_n = \bigcup_{j=1}^n V_j$, og da må følgen

$W_1 \subset W_2 \subset W_3 \subset \dots \subset W_n \subset W_{n+1} \subset \dots$ terminere, dvs. $W_n = W_{n+1} = W_{n+2} = \dots$

og siden da $W_n \supseteq U$, har vi $U \subseteq \bigcup_{j=1}^n V_j$.

Siden $\{\mathcal{F}_i\}$ er ettet, $\exists N \gg 0: s \in \mathcal{F}_N(U)$ og da

$s|_{V_j} = 0 \in \mathcal{F}_N(V_j)$. Og da har vi $s = 0 \in \mathcal{F}_N(U)$, siden

\mathcal{F}_N er et knippe, så får derfor at $s = 0 \in \lim_i \mathcal{F}_i(U)$

Velg nå $s_j \in \lim_i \mathcal{F}_i(V_j)$, hvor vi fremdeles har $\{V_j\}_{j=1}^n$ åpen overdekning av U , slik at $s_j|_{V_j \cap V_k} = s_k|_{V_j \cap V_k}, \forall j, k$.

Velg igjen en $N \gg 0$ s.d. $s_j \in \mathcal{F}_N(V_j) \forall j$. Siden \mathcal{F}_N er et knippe finnes da en $s \in \mathcal{F}_N(U)$ s.d.

$s|_{V_j} = s_j \forall j \in \mathcal{F}_N(V_j)$. Tar så kognensjon og får

$s|_{V_j} = s_j \forall j \in \lim_i \mathcal{F}_i(V_j)$.

Oliver E. Anderson

where the objects

We have a category ~~of~~ ^{one} functors

Ex. 1.12

$\mathcal{F}: \text{Top}(X) \rightarrow \text{Mod}_R$ and the

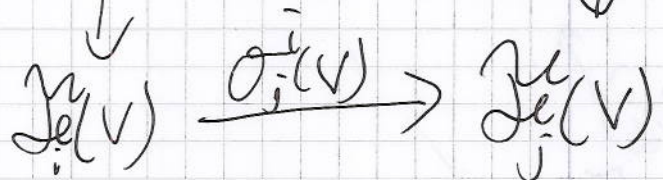
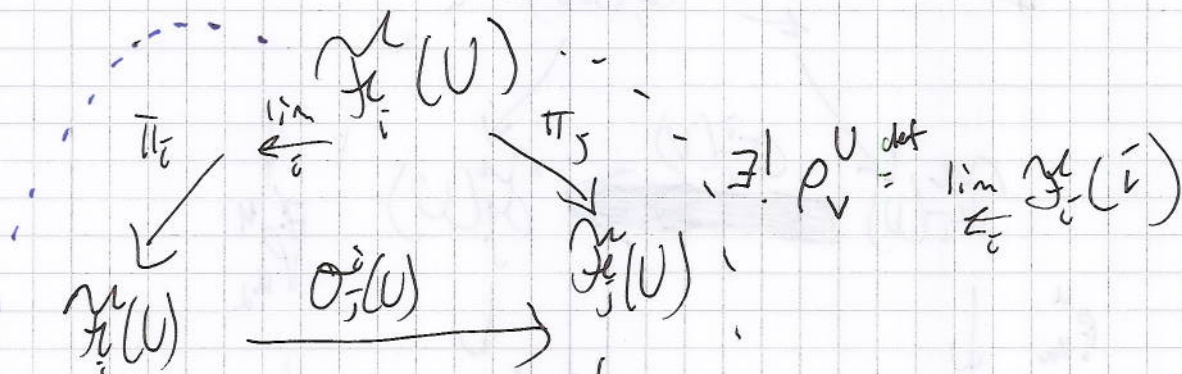
morphisms and natural transformations between the functors. We start by showing that

$U \mapsto \varprojlim_i \mathcal{F}_i(U)$ is a presheaf, i.e. it is a contravariant functor from $\text{top}(X)$ to $R\text{-mod}$.

Since the inverse limit exists in the category of R -modules we must have

that for each open set $U \subseteq X$, we have $\varprojlim_i \mathcal{F}_i(U) \in \text{Obj}(R\text{-mod})$. If $V \subseteq U$, that is

$V \subseteq U \subseteq X$ open sets, then we have the diagram



If $W \subseteq V \subseteq U$ then

so $\rho_W^U = \rho_W^V \circ \rho_V^U$
 or $\varprojlim_i \mathcal{F}_i(i_1 \circ i_2) = \varprojlim_i \mathcal{F}_i(i_2) \circ \varprojlim_i \mathcal{F}_i(i_1)$

$\exists! \rho$

$\exists! \rho_W^V$

So we do indeed get a contravariant functor, hence we have a presheaf. Hence it only remains to show that we have a sheaf. We must show that

$$\lim_{\leftarrow i} \mathcal{F}_i(U) \rightarrow \prod_{\alpha} \lim_{\leftarrow i} \mathcal{F}_i(U_{\alpha}) \rightarrow \prod_{\alpha, \beta} \lim_{\leftarrow i} \mathcal{F}_i(U_{\alpha} \cap U_{\beta})$$

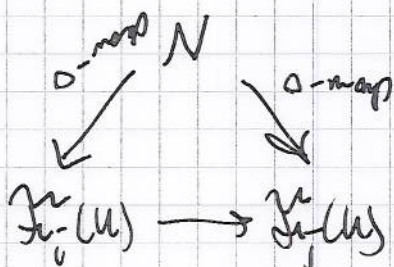
is an equalizer for any open cover $\{U_{\alpha}\}_{\alpha}$ of U and any open set V . This is equivalent to showing that $\lim_{\leftarrow i} \mathcal{F}_i$ satisfies the sheaf axioms in Hartshorne, which is what we will do. So fix U , and open cover $\{U_{\alpha}\}_{\alpha}$ of U . Then we have

$$\begin{array}{ccc}
 \lim_{\leftarrow i} \mathcal{F}_i(U) & & \\
 \swarrow & \xrightarrow{\sigma_i^U(U)} & \searrow \\
 \mathcal{F}_i(U) & & \mathcal{F}_i(U) \\
 \downarrow \rho_{i, U_{\alpha}}^U & & \downarrow \rho_{i, U_{\beta}}^U \\
 \mathcal{F}_i(U_{\alpha}) & \longrightarrow & \mathcal{F}_i(U_{\beta}) \\
 \swarrow & \xrightarrow{\sigma_i^U(U_{\alpha})} & \searrow \\
 \lim_{\leftarrow i} \mathcal{F}_i(U_{\alpha}) & &
 \end{array}$$

By assumption $\rho_{U_{\alpha}}^U(s) = 0$, by commutativity we must have $\rho_{i, U_{\alpha}}^U(\pi_i(s)) = 0$ that is $\pi_i(s)|_{U_{\alpha}} = 0$ and so $\pi_i(s) = 0$ since $\lim_{\leftarrow i} U_{\alpha}$ is a scheme, hence we have $\pi_i(s) = 0$.

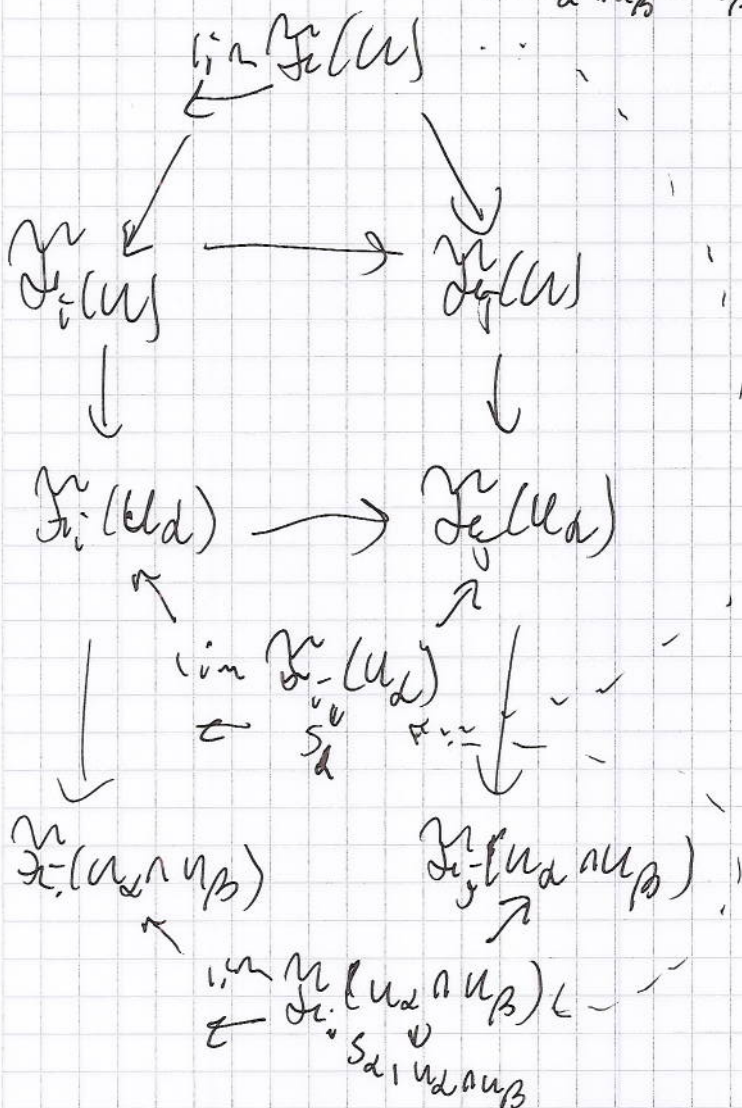
Assume $S \neq 0$ in $\varprojlim \mathcal{F}_i(U)$. Let

N be the R -module consisting of ~~the~~ $\{0, RS\} = RS = \{rs \mid r \in R\}$. Then



is commutative, but the 0-map does not factor uniquely (take the identity map and the 0-map) through $\varprojlim \mathcal{F}_i(U)$ contradicting the universal property of inverse limits. Thus

we conclude that $S=0 \in \varprojlim \mathcal{F}_i(U)$. Now assume that $S_d u_d \cap u_\beta = S_d u_d \cap u_\beta$ where $S_d \in \varprojlim \mathcal{F}_i(U_d)$.



Return to \mathbb{R}^n via $\mathbb{R}^n \rightarrow \mathbb{R}^n$
The map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map
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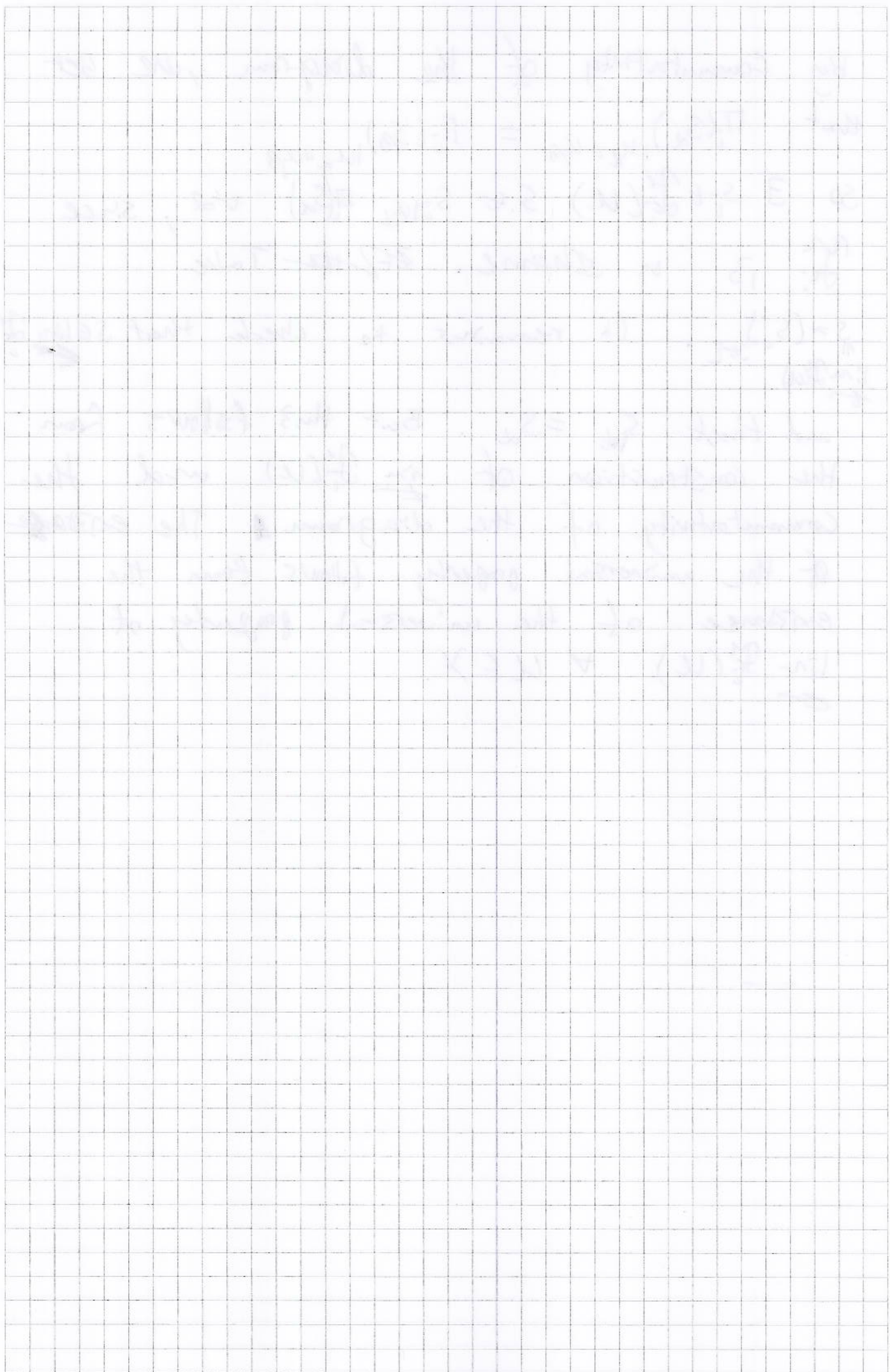


The map $\mathbb{R}^n \rightarrow \mathbb{R}^n$ is a linear map
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By commutativity of the diagram, we get that $\pi_i(S_\alpha)_{u_\alpha \cap u_\beta} = \pi_i(S_\beta)_{u_\alpha \cap u_\beta}$

so $\exists s_i \in \varinjlim \mathcal{F}_i(U)$ s.t. $s_i|_{u_\alpha} = \pi_i(S_\alpha) \forall \alpha$, since $\varinjlim \mathcal{F}_i$ is a scheme. ~~EF2~~ Take

$S = (S_i)_{i \in \mathbb{Z}}$. It remains to check that $S \in \varinjlim \mathcal{F}_i$ and that $S_{u_\alpha} = S_\alpha$. But this follows from the construction of $\varinjlim \mathcal{F}_i(U)$ and the commutativity of the diagram. The existence of the universal property follows from the existence of the universal property of $\varinjlim \mathcal{F}_i(U) \forall U \subseteq X$.



$$1.1.14 \quad S \in \mathcal{F}(U)$$

$$\text{supp } S = \{P \in U \mid S|_P \neq 0\} = \text{closed}$$

\Leftrightarrow

$$(\text{supp } S)^c = \{P \in U \mid S|_P = 0\} = \text{open}$$

$$P \in (\text{supp } S)^c \Rightarrow S|_P = 0 \Rightarrow \exists W_P \subseteq U$$

$$\text{s.t. } P \stackrel{U}{W_P} (S) = 0, \text{ ~~where } W_P \subseteq U~~$$

$$(\text{supp } S)^c, \text{ since } \text{for } Q \in W_P, S|_Q = 0.$$

$$\Rightarrow (\text{supp } S)^c = \text{open}, \text{supp } S = \text{closed}$$

Define sheaf \mathcal{F} on \mathbb{R} with the usual topology.

$$1. \quad U \text{ open}, \quad 0 \notin U$$

$$\mathcal{F}(U) = \{f: U \rightarrow \mathbb{R}, f = \text{const.}\}$$

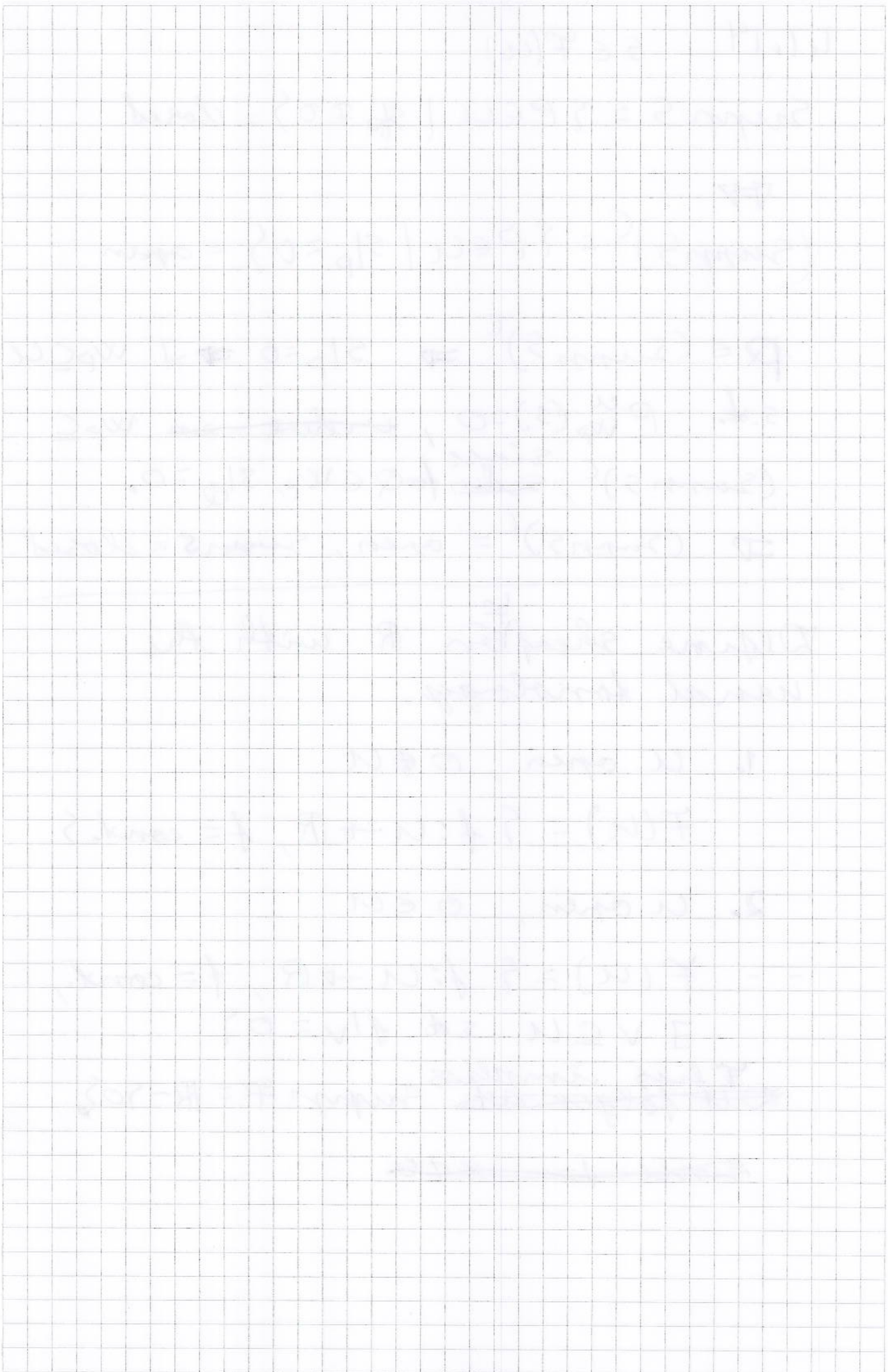
$$2. \quad U \text{ open}, \quad 0 \in U$$

$$\mathcal{F}(U) = \{f: U \rightarrow \mathbb{R}, f = \text{const.},$$

$$\exists V \supseteq U \text{ s.t. } f|_V = 0\}$$

~~This implies~~ ~~set of all~~ $\text{supp } \mathcal{F} = \mathbb{R} - \{0\}$

~~since for~~ ~~x to~~



$U \subseteq X$ $\text{Hom}(F/U, G/U)$: Ex. 1.15
is an abelian group.

Take $\varphi, \psi \in \text{Hom}(F/U, G/U)$. Then
for $V \subseteq U$ we have

$$(\varphi + \psi)(s) = \varphi(s) + \psi(s)$$

$$\begin{array}{ccc} F/U(V) & \xrightarrow{\varphi_V} & G/U(V) \\ & \xrightarrow{\psi_V} & \end{array}$$

Define $(\varphi + \psi)_V(s) = \varphi_V(s) + \psi_V(s)$

$W \subseteq V \subseteq U$. Then $\varphi_W(s|_W) = (\varphi_V(s))|_W$.

$$\begin{array}{ccc} F/U(V) & \xrightarrow{\text{restr.}} & F/U(W) & \text{Since} \\ & & & \text{diagram} \\ \varphi_V \downarrow & & \downarrow \varphi_W & \text{commutes} \\ G/U(V) & \xrightarrow{\text{restr.}} & G/U(W) & \end{array}$$

~~Prop 1.11 $q: F \rightarrow G$ morphism of sheaves.
 q iso $\Leftrightarrow q_p: F_p \rightarrow G_p$ iso $\forall p \in X$.~~

~~Def~~

~~Def $q: F \rightarrow G$ morphism of presheaves
 the presheaf kernel, cokernel, image to be~~

~~$U \mapsto \ker(q(U))$~~

~~$U \mapsto \text{coker}(q(U))$~~

~~$U \mapsto \text{im}(q(U))$~~

~~← Not nec. a sheaf.~~

~~Sheafification~~

~~Def A subsheaf of sheaf \mathcal{F} is a sheaf \mathcal{F}'
 s.t. $U \in X \Rightarrow \mathcal{F}'(U) \subseteq \mathcal{F}(U)$~~

Hence: $(\varphi + \tau)_W(S/W) = \varphi_W(S/W) + \tau_W(S/W)$

G/W module gr.
 $= (\varphi_V(S))_W + (\tau_V(S))_W$

$\cong (\varphi_V(S) + \tau_V(S))_W$

$= ((\varphi + \tau)_V(S))_W$

Hence:

$$F_W(W) \xrightarrow{(\varphi + \tau)_W} G_W(W)$$

next \downarrow

\circ

restr. \downarrow

$$F_W(W) \xrightarrow{(\varphi + \tau)_W} G_W(W)$$

$\varphi + \tau : F_W \rightarrow G_W$ is morphism of sheaves.

Letting $\tau \in \text{Hom}(F_W, G_W)$ induces a morphism of sheaves $-\tau : F_W \rightarrow G_W$

$f\tau_W = -\tau_W$

$(-T_V + T_V)(s) = 0$, hence $0 \in \text{Hom}(F_U, G_U)$

G_U abelian $\Rightarrow \text{Hom}(F_U, G_U)$ abelian.

Hence $\text{Hom}(\)$ is a presheaf.

① Let $U = \cup U_i$ be open subset w/ covering

Let $\varphi \in \text{Hom}(F_U, G_U)$ s.t. $\varphi|_{U_i} = 0 \ \forall i$.

$\varphi|_{U_i} = 0 \Rightarrow \varphi_{V \cap U_i} = (\varphi|_{U_i})|_{V \cap U_i} = 0 \ \forall V \subseteq U$

$V \subseteq U$, $s \in F_U(V) = F(V)$

$(\hat{\varphi}_V(s))|_{V \cap U_i} = \varphi_{V \cap U_i}(s|_{V \cap U_i}) = 0 \ \forall i \in I$

$V = \cup_i U_i \cap V$ and G_U a sheaf

$\Rightarrow \hat{\varphi}_V(s) = 0$. Axiom I ok.

② Let $\varphi_i \in \text{Hom}(F_{U_i}, G_{U_i})$ be s.t.

$\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$.

Let $V \subseteq U$ open and $s \in \mathcal{F}_U(V)$.

For $i \in I$ put $t_i = \varphi_{i, V \cap U_i}(s|_{V \cap U_i}) \in \mathcal{G}_U(V \cap U_i)$

Then K_{ij}

$$t_i|_{U_i \cap U_j} = \varphi_{i, V \cap U_i}(s|_{V \cap U_i})|_{U_i \cap U_j}$$

$$= \varphi_{i, V \cap U_j}(s|_{V \cap U_j})$$

$$= \varphi_{j, V \cap U_j}(s|_{V \cap U_j})$$

$$= \varphi_{j, V \cap U_j}(t_j|_{V \cap U_j})$$

Since \mathcal{G}_U is a sheaf and $V = \bigcup_i U_i \cap V$,

there is $t \in \mathcal{G}_U(V)$ s.t.

$$t|_{V \cap U_i} = t_i = \varphi_{i, V \cap U_i}(s|_{V \cap U_i}) \quad \forall i$$

Put $\varphi_V(s) = t$.

For $i \in I$, $W \subseteq V \subseteq U$ open, and $s \in \mathcal{F}_U(V)$
putting $t_i = t|_{V \cap U_i}$ and $\varphi_W(s|_W) = \tilde{t}$

$\tilde{t}_i = \tilde{t}|_{W \cap U_i}$ we get.

$$(t/w)|_{w=v_i} = t|_{w=v_i} = t_i|_{w=v_i}$$

$$= \varphi_{i, v_i}(s|v_i)|_{w=v_i}$$

$$= \varphi_{i, v_i}(s|v_i) = \tilde{t}_i$$

Now $G|_u$ is a sheaf and $W = U \cup V_i \cap W$

$$\Rightarrow t|_W = \tilde{t} \text{ i.e. } \varphi_v(s)|_W = \varphi_w(s|_w)$$

Hence OK.

~~S2 part~~

Skyscraper sheaves

X top. space with discrete topology, $P \in X$ a fixed point, A abelian group (with discrete topology)

$P \in U \subseteq X$, define sheaf:
$$i_p(A)(U) = \begin{cases} A, & P \in U \\ 0, & P \notin U \end{cases}$$

to show: (i) the stalk $(i_p(A))_Q = \begin{cases} A, & Q \in \overline{\{P\}} \\ 0, & Q \notin \overline{\{P\}} \end{cases}$

(ii) $i_p(A) = \text{colim}_{Q \in \overline{\{P\}}} i_{i_p^{-1}(Q)}(A)$, with A constant sheaf, and $i_{i_p^{-1}(Q)}$ inclusion map $i_{i_p^{-1}(Q)}: \overline{\{P\}} \rightarrow X$

↑
closed set consisting of P

Proof: (i) ^{1st case}: Fix $Q \in \overline{\{P\}}$, let U be open neighborhood of Q _{top. basis} $\rightarrow P \in U$

consider the stalk:

$$\begin{aligned} (i_p(A))_Q &= \text{colim}_{Q \in U} i_p(A)(U) = \text{colim}_{Q \in U} A \\ &= \text{colim} (A \xrightarrow{\text{id}} A \xrightarrow{\text{id}} A \rightarrow \dots) = A \end{aligned}$$

^{2nd case} case: Fix $Q \notin \overline{\{P\}}$

$\Rightarrow \exists$ open $U_0 = X \setminus \overline{\{P\}}$ (since $\{P\}$ closed) with $Q \in U_0$ but $P \notin U_0$.

consider the stalk:

$$(i_p(A))_Q = \text{colim}_{Q \in U} i_p(A)(U)$$

take element $\langle u, s \rangle \in (i_p(A))_Q$, with $s \in F(U)$

and define further $W := U \cap U_0$

by def. of $U_0 \Rightarrow i_p(A)(W) = 0$

[On page 62: $\langle u, s \rangle, \langle v, t \rangle \in (i_p(A))_Q$
 $\langle u, s \rangle = \langle v, t \rangle$ if $\exists W = U_0 \cap V$,
 s.t. $s|_W = t|_W$

In our case: $U \cap W = U \cap U_0 \subseteq U$,
 $\stackrel{!}{=} W$

$$\text{so } S|_{\text{univ}} = S|_W \rightarrow \langle u, s \rangle = \langle w, s|_w \rangle = 0$$

(so τ restricted to subset of U)

Since we can define such a W for any $u \in Q$:

$$(i_p(A))_Q = \text{colim}_{Q \ni u} \underbrace{i_p(A)(u)}_{=0} = \text{colim}(0 \rightarrow 0 \rightarrow \dots) = 0$$

(ii) $i_{iuc} : \overline{\{P\}} \rightarrow X$ and define $i_{iuc, X}(A)(u) := A(i_{iuc}^{-1}(u))$,
 = set of all continuous functions from $i_{iuc}^{-1}(u) \rightarrow A$
 (with discrete topology)

1st case: open subset $U \subseteq X$ with $P \notin U$

$$\Rightarrow \overline{\{P\}} \cap U = \emptyset \Rightarrow i_{iuc}^{-1}(U) = \emptyset \rightarrow A(i_{iuc}^{-1}(U)) = 0$$

the fiber

2nd case: open $U \subseteq X$, with $P \in U$, then we
 want to show $A(i_{iuc}^{-1}(U)) = A$:

We know (acc. discrete topology) that any

continuous map $f: i_{iuc}^{-1}(U) \rightarrow A$, send

$\overline{\{P\}} \cap U$ into $\{f(P)\}$ (since cont. maps send closed sets to closed sets)

Since A has discrete topology $f(P)$ is only a point $\Rightarrow \{f(P)\} = \overline{\{f(P)\}}$

Hence, any continuous map is determined by where the point P is sent to.

\Rightarrow any continuous map is constant function from $i_{iuc}^{-1}(U) \subseteq \overline{\{P\}} \rightarrow A$

Since we can send P to any point in A

$$\Rightarrow A(i_{iuc}^{-1}(U)) = A$$

$$\Rightarrow i_{iuc, X}(A) = i_p(A) \quad \square$$