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(1.8) $\mathcal{U} \subseteq X$ open subset, show that the functor $\Gamma(\mathcal{U}, \cdot)$ from sheaves on X to ab groups is a left exact functor.

That is if $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}''$ is exact,

then $0 \to \Gamma(\mathcal{U}, \mathcal{F}') \to \Gamma(\mathcal{U}, \mathcal{F}) \to \Gamma(\mathcal{U}, \mathcal{F}'')$ is exact

$0 \to \mathcal{F}'(\mathcal{U}) \to \mathcal{F}(\mathcal{U}) \to \mathcal{F}''(\mathcal{U})$

<u>Solution</u> $0 \to \mathcal{F}' \xrightarrow{\varphi} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ exact seq. of sheaves

φ injective, $\ker \varphi = 0$

$\Updownarrow$

$\varphi(\mathcal{U})$ injective, $\ker(\varphi(\mathcal{U})) = 0$ for each open $\mathcal{U} \subseteq X$

Want to show that

$$0 \to \mathcal{F}'(\mathcal{U}) \xrightarrow{\varphi(\mathcal{U})} \mathcal{F}(\mathcal{U}) \xrightarrow{\psi(\mathcal{U})} \mathcal{F}''(\mathcal{U})$$ is an exact seq. of ab. groups

Have that $\varphi(\mathcal{U})$ is injective, want to show that $\ker(\psi(\mathcal{U})) = \operatorname{Im}(\varphi(\mathcal{U}))$ to prove the claim

$\mathcal{F}' \overset{\varphi}{\hookrightarrow} \mathcal{F} \xrightarrow{\psi} \mathcal{F}''$ exact

$\mathcal{F}'(\mathcal{U}) \overset{\varphi(\mathcal{U})}{\hookrightarrow} \mathcal{F}(\mathcal{U}) \xrightarrow{\psi(\mathcal{U})} \mathcal{F}''(\mathcal{U})$

Since φ is injective, the image of φ is isomorphic to a subsheaf of $\mathcal{F}$: $\operatorname{Im}\varphi \cong \mathcal{G}$ subsheaf of $\mathcal{F}$ (from the definition of a subsheaf and morphisms of sheaves)

Since $\mathcal{G}$ is a sheaf, it's equal to its sheafification, giving $\operatorname{im}(\varphi) = (\operatorname{im}(\varphi))^+ = \ker(\psi)$.

Now, with $\mathcal{U} \subseteq X$ open, and sheaves being defined on the open sets, we have $\operatorname{im}(\varphi(\mathcal{U})) = \ker(\psi(\mathcal{U}))$ which gives exactness of the sequence of abelian groups

Ex 1.10

Let $\{\mathcal{F}_i\}$ be a direct system of sheaves and morphisms on X. We define the direct limit of the system $\{\mathcal{F}_i\}$ denoted $\varinjlim \mathcal{F}_i$ to be the sheaf associated the preasheaf $U \mapsto \varinjlim \mathcal{F}_i(U)$.

Show that this is a direct limit in the category of sheaves on X i.e that it has the following universal property:
given a sheaf $\mathcal{G}$ and a collection of morphisms $\mathcal{F}_i \longrightarrow \mathcal{G}$ compatible with the maps of the direct system, then $\exists!$ map $\varinjlim \mathcal{F}_i \longrightarrow \mathcal{G}$ s.t. $\forall i \in I$ the original map $\mathcal{F}_i \longrightarrow \mathcal{G}$ is obtained by composing the maps $\mathcal{F}_i \longrightarrow \varinjlim \mathcal{F}_i \longrightarrow \mathcal{G}$.

Solution

~~To simplify we consider the $\mathcal{F}_i$'s as sheaves of moduls on X~~

Let $\mathcal{F}$ the presheaf given by $\forall U \subseteq X$ open $U \mapsto \varinjlim \mathcal{F}_i(U)$

Let $\mathcal{G}$ a sheaf on X and a collection of morphisms $\phi_i : \mathcal{F}_i \longrightarrow \mathcal{G}$ s.t $\forall i, j \in I$ the diagram

$$\begin{tikzcd} \mathcal{F}_i \arrow[r, "\phi_i"] & \mathcal{G} \\ \mathcal{F}_j \arrow[u, "\psi_{ij}"] \arrow[ur, "\phi_j"'] & \end{tikzcd}$$

commutes

Let $U \subset X$ open

$\forall i \in I$ $\varinjlim \mathcal{F}_i(U)$ sat the universal ppty:

$$\begin{tikzcd} \mathcal{F}_i(U) \arrow[r, "\phi_i(U)"] \arrow[dr, "\psi_i(U)"'] & \mathcal{G}(U) \\ & \varinjlim \mathcal{F}_i(U) \arrow[u, dashed, "\exists!\, \phi'_i(U)"'] \end{tikzcd}$$

Hence we obtain the following commuting diagram $\forall i \in I$:

$$\begin{tikzcd} \tilde{F}_i \arrow[r, "\phi_i"] \arrow[dr, "\pi_i"'] & \mathcal{G} \\ & \tilde{F} \arrow[u, dashed, "\exists!\,\phi'"'] \end{tikzcd}$$

But by the prop 1.2 H $\exists!$ sheaf F^+ and a morphism of sheaves $\theta : \tilde{F} \to \tilde{F}^+$ s.t. the following diag commutes

$$\begin{tikzcd} \tilde{F} \arrow[r, "\phi'"] \arrow[dr, "\theta"'] & \mathcal{G} \\ & \tilde{F}^+ \arrow[u, dashed, "\exists!\,\psi"'] \end{tikzcd}$$

hence put $\varinjlim \tilde{F}_i = \tilde{F}^+$ it sat the universal ppty : $\forall i \in I$ we have

$$\begin{tikzcd} \tilde{F}_i \arrow[r, "\phi_i"] \arrow[dr, "\theta\circ\pi_i"'] & \mathcal{G} \\ & \varinjlim \tilde{F}_i \arrow[u, "\psi"'] \end{tikzcd}$$

~~by the prop 1.2 H $\exists!$ sheaf $\tilde{F}^+$ and morphism of sheaves $\theta : \tilde{F} \to \tilde{F}^+$ s.t $\forall$ sheaf $\mathcal{G}$ and morphism of presheaves $\varphi : \tilde{F} \to \mathcal{G}$ $\exists!$ morphism of sheaves ψ s.t~~ $\cancel{\begin{tikzcd} \tilde{F} \arrow[r, "\varphi"] \arrow[dr, "\theta"'] & \mathcal{G} \\ & \tilde{F}^+ \arrow[u, dashed, "\exists!\psi"'] \end{tikzcd}}$ ~~commutes put~~ $\cancel{\varinjlim \tilde{F}_i = \tilde{F}^+}$

1.11 La U åpen, $\{V_j\}$ åpen overdekning av U.

Velg $s \in \lim_i \mathcal{F}_i(U)$ s.a. $s|_{V_j} = 0 \;\forall j$

X Noethersk $\Rightarrow$ Vi kan velge $n : U \subseteq \bigcup_{j=1}^{n} V_j$, da vi kan konstruere $W_k = \bigcup_{j=1}^{k} V_j$, og da må følgen $W_1 \subset W_2 \subset W_3 \subset \dots \subset W_n \subset W_{n+1} \subset \dots$ terminere, dvs $W_n = W_{n+1} = W_{n+2} \dots$, og siden da $W_n \supseteq U$, har vi $U \subseteq \bigcup_{j=1}^{n} V_j$.

Siden $\{\mathcal{F}_i\}_i$ er nettet, $\exists N >> 0 : s \in \mathcal{F}_N(U)$ og da

$\forall j = 1, \dots, n$ $s|_{V_j} = 0$ i $\mathcal{F}_N(U)$. Og da har vi $s = 0$ i $\mathcal{F}_N(U)$, siden $\mathcal{F}_N$ er et knippe, så får da to at $s = 0$ i $\lim_i \mathcal{F}_i(U)$

Velg nå $s_j \in \lim_i \mathcal{F}_i(V_j)$, hvor vi fremdeles har $\{V_j\}_{j=1}^{n}$ åpen overdekning av U, slik at $s_j|_{V_j \cap V_k} = s_k|_{V_j \cap V_k}$, $\forall j, k$. Velger igjen en $N >> 0$ s.a. $s_j \in \mathcal{F}_N(V_j) \;\forall j$. Siden $\mathcal{F}_N$ er et knippe finnes da en $s \in \mathcal{F}_N(U)$ s.a. $s|_{V_j} = s_j \;\forall j$ i $\mathcal{F}_N(V_j)$. Tar så kogrensen og får

$s|_{V_j} = s_j \;\forall j$ i $\lim_i \mathcal{F}_i(V_j)$.

Oliver E. Anderson

Ex. 1.12

We have a category where the objects are functors $\mathcal{F}: \mathbf{Top}(X) \longrightarrow R\text{-Mod}$, and the morphisms are natural transformations between the functors. We start by showing that

$U \longmapsto \varprojlim_i \mathcal{F}_i(U)$ is a presheaf, i.e. it is a contravariant functor from $top(X)$ to R-mod. Since the inverse limit exists in the category of R-modules we must have that for each open set $U \subseteq X$, we have $\varprojlim_i \mathcal{F}_i(U) \in Obj(R\text{-mod})$. If $V \xhookrightarrow{i} U$, that is $V \subseteq U \subseteq X$ open sets, then we have the diagram

$$
\begin{tikzcd}
 & \varprojlim_i \mathcal{F}_i(U) \arrow[dl, "\pi_i"'] \arrow[dr, "\pi_j"] \arrow[dd, dashed, "\exists!\, \rho^U_V \overset{def}{=} \varprojlim_i \mathcal{F}_i(i)"] & \\
\mathcal{F}_i(U) \arrow[rr, "\theta^i_j(U)"] \arrow[d] & & \mathcal{F}_j(U) \arrow[d] \\
\mathcal{F}_i(V) \arrow[rr, "\theta^i_j(V)"] \arrow[d] & & \mathcal{F}_j(V) \arrow[d] \\
 & \varprojlim_i \mathcal{F}_i(V) \arrow[ul] \arrow[ur] \arrow[d, dashed, "\exists!\, \rho^V_W"] & \\
\mathcal{F}_i(W) \arrow[rr, "\theta^i_j(W)"] & & \mathcal{F}_j(W) \\
 & \varprojlim_i \mathcal{F}_i(W) \arrow[ul] \arrow[ur] &
\end{tikzcd}
$$

$\exists!\, \rho^U_W$

If $W \subseteq V \subseteq U$ then

$W \xhookrightarrow{i_2} V \xhookrightarrow{i_1} U$

so $\rho^U_W = \rho^V_W \circ \rho^U_V$

or $\varprojlim_i \mathcal{F}_i(i_1 \circ i_2) = \varprojlim_i \mathcal{F}_i(i_2) \circ \varprojlim_i \mathcal{F}_i(i_1)$

So we do indeed get a contravariant functor, hence we have a presheaf. Hence it only remains to show that we have a sheaf. We must show that

$$\varprojlim_i \mathcal{F}_i(U) \longrightarrow \prod_\alpha \varprojlim_i \mathcal{F}_i(U_\alpha) \rightrightarrows \prod_{\alpha,\beta} \varprojlim_i \mathcal{F}_i(U_\alpha \cap U_\beta)$$

is an equalizer for any open cover $\{U_\alpha\}_\alpha$ of U and any open set U. This is equivalent to showing that $\varprojlim \mathcal{F}_i$ satisfies the sheaf axioms in Hartshorne, which is what we will do. So fix U, and open cover $\{U_\alpha\}_\alpha$ of U. Then we have

$$\begin{tikzcd}
 & \varprojlim \mathcal{F}_i(U) \arrow[dl] \arrow[dr] \arrow[dddr, dashed, bend left, "\mathcal{F}\rho^U_{U_\alpha}"] & \\
\mathcal{F}_i(U) \arrow[rr, "\sigma^i_j(U)"] \arrow[d, "\rho^U_{iU_\alpha}"'] & & \mathcal{F}_j(U) \arrow[d] \\
\mathcal{F}_i(U_\alpha) \arrow[rr] & & \mathcal{F}_j(U_\alpha) \\
 & \varprojlim \mathcal{F}_i(U_\alpha) \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

By assumption $\rho^U_{U_\alpha}(S)=0$, by commutativity we must have $\rho^U_{iU_\alpha}(\pi_i(S))=0$ that is $\pi_i(S)|_{U_\alpha}=0$ $\forall \alpha$ so $\pi_i(S)=0$ since $\mathcal{F}_i$ is a Scheme, hence $\forall i$ we have $\pi_i(S)=0$.

Assume $s \neq 0$ in $\varprojlim \tilde{\mathcal{F}}_i(\mathcal{U})$. Let N be the R-module ~~consisting of~~ $\{0, Rs\} = Rs = \{rs \mid r \in R\}$. Then

$$\begin{tikzcd}
 & N \arrow[dl, "0\text{-map}"'] \arrow[dr, "0\text{-map}"] & \\
\tilde{\mathcal{F}}_i(\mathcal{U}) \arrow[rr] & & \tilde{\mathcal{F}}_j(\mathcal{U})
\end{tikzcd}$$

is commutative, but the 0-map does not factor uniquely (take the identity map and the 0-map) through $\varprojlim \tilde{\mathcal{F}}_i(\mathcal{U})$ contradicting the universal property of inverse limits. Thus we conclude that $s = 0 \in \varprojlim \tilde{\mathcal{F}}_i(\mathcal{U})$. Now assume that $s_\alpha|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} = s_\beta|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta}$ where $s_\alpha \in \varprojlim \tilde{\mathcal{F}}_i(\mathcal{U}_\alpha)$.

$$\begin{tikzcd}
 & \varprojlim \tilde{\mathcal{F}}_i(\mathcal{U}) \arrow[dl] \arrow[dr] & \\
\tilde{\mathcal{F}}_i(\mathcal{U}) \arrow[rr] \arrow[d] & & \tilde{\mathcal{F}}_j(\mathcal{U}) \arrow[d] \\
\tilde{\mathcal{F}}_i(\mathcal{U}_\alpha) \arrow[rr] \arrow[dd] & & \tilde{\mathcal{F}}_j(\mathcal{U}_\alpha) \arrow[dd] \\
 & \varprojlim \tilde{\mathcal{F}}_i(\mathcal{U}_\alpha) \ni s_\alpha \arrow[ul] \arrow[ur] & \\
\tilde{\mathcal{F}}_i(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) & & \tilde{\mathcal{F}}_j(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \\
 & \varprojlim \tilde{\mathcal{F}}_i(\mathcal{U}_\alpha \cap \mathcal{U}_\beta) \ni s_\alpha|_{\mathcal{U}_\alpha \cap \mathcal{U}_\beta} \arrow[ul] \arrow[ur] &
\end{tikzcd}$$

By commutativity of the diagram, we get that $\pi_i(s_\alpha)_{|U_\alpha \cap U_\beta} = \pi_i(s_\beta)_{|U_\alpha \cap U_\beta}$

so $\exists\ s_i \in \mathcal{F}_i^{\mathcal{U}}(\mathcal{U})$ s.t. $s_{i|U_\alpha}$ $\pi_i(s_\alpha)$ $\forall \alpha$, since $\mathcal{F}_i^{\mathcal{U}}$ is a scheme. ~~Ef2 the~~ Take

$s = (s_i)_{i \in I}$. It remains to check that $s \in \varprojlim \mathcal{F}_i^{\mathcal{U}}$ $\varprojlim \mathcal{F}_i(\mathcal{U})$

and that $s_{|U_\alpha} = s_\alpha$. But this follows from the construction of $\varprojlim \mathcal{F}_i^{\mathcal{U}}(\mathcal{U})$ and the commutativity of the diagram. The existence of the universal property follows from the existence of the universal property of $\varprojlim \mathcal{F}_i^{\mathcal{U}}(\mathcal{U})$ $\forall\ \mathcal{U} \subseteq X$.

1.1.14 $\quad s \in \mathcal{F}(U)$

$\text{supp}\, s = \{P \in U \mid s|_P \neq 0\}$ = closed

$\Leftrightarrow$

$(\text{supp}\, s)^c = \{P \in U \mid s|_P = 0\}$ = open

$P \in (\text{supp}\, s)^c \Rightarrow s|_P = 0 \Rightarrow \exists\, W_P \subseteq U$ s.t. $\rho^U_{W_P}(s) = 0$, ~~widere er~~ $W_P \subseteq (\text{supp}\, s)^c$, since ~~siden~~ for $Q \in W_P$, $s|_Q = 0$.

$\Rightarrow (\text{supp}\, s)^c$ = open, $\text{supp}\, s$ = closed

---

Define sheaf $\mathcal{F}$ on $\mathbb{R}$ with the usual topology.

1. U open, $0 \notin U$

   $\mathcal{F}(U) = \{f : U \to \mathbb{R},\ f = \text{cont.}\}$

2. U open, $0 \in U$

   $\mathcal{F}(U) = \{f : U \to \mathbb{R},\ f = \text{cont.},\ \exists\, V \subseteq U \text{ s.t. } f|_V = 0\}$

~~Det følger at~~ This implies $\text{supp}\, \mathcal{F} = \mathbb{R} - \{0\}$.

~~since for $x \neq 0$~~

$U \subseteq X$ $\quad \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ is an abelian group. $\quad$ Ex. 1.15

Take $\varphi, \psi \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$. Then for $V \subseteq U$ we have

$$\cancel{(\varphi+\psi)(s) = \varphi(s)+\psi(s)}$$

$$\begin{tikzcd} \mathcal{F}|_U(V) \arrow[r, shift left, "\varphi_V"] \arrow[r, shift right, "\psi_V"'] & \mathcal{G}|_U(V) \end{tikzcd}$$

Define $(\varphi+\psi)_V(s) = \varphi_V(s) + \psi_V(s)$

$W \subseteq V \subseteq U$. Then $\varphi_W(s|_W) = (\varphi_V(s))|_W$

$$\begin{tikzcd} \mathcal{F}|_U(V) \arrow[r, "\text{restr.}"] \arrow[d, "\varphi_V"'] & \mathcal{F}|_U(W) \arrow[d, "\varphi_W"] \\ \mathcal{G}|_U(V) \arrow[r, "\text{restr}"'] & \mathcal{G}|_U(W) \end{tikzcd}$$

since diagram commutes

Prop B.11 $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of sheaves

φ iso $\iff$ $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ iso $\forall p \in X$.

Def $\varphi : \mathcal{F} \to \mathcal{G}$ morphism of presheaves

Define presheaf kernel, cokernel, image to be

$U \longmapsto \ker(\varphi(U))$

$U \longmapsto \operatorname{coker}(\varphi(U))$ ← Not nec. a sheaf.

$U \longmapsto \operatorname{im}(\varphi(U))$ ←

Sheafification

Def A subsheaf of sheaf $\mathcal{F}$ is a sheaf $\mathcal{F}'$ s.t. $U \subseteq X$ $\mathcal{F}'(U)$ subgroup of $\mathcal{F}(U)$

Hence: $(\varphi+\psi)_W(s|_W) = \varphi_W(s|_W) + \psi_W(s|_W)$

$= (\varphi_V(s))|_W + (\psi_V(s))|_W$

$\mathcal{G}|_U(W)$ abelian gr.

$\overset{\ast}{=} (\varphi_V(s) + \psi_V(s))|_W$

$= ((\varphi+\psi)_V(s))|_W$

Hence:

$$\begin{tikzcd} \mathcal{F}|_U(V) \arrow[r, "(\varphi+\psi)_V"] \arrow[d, "\text{rest}"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathcal{G}|_U(V) \arrow[d, "\text{restr.}"] \\ \mathcal{F}|_U(W) \arrow[r, "(\varphi+\psi)_W"'] & \mathcal{G}|_U(W) \end{tikzcd}$$

$\varphi+\psi : \mathcal{F}|_U \to \mathcal{G}|_U$ is morphism of sheaves.

~~Taking the~~ $\psi \in \mathrm{Hom}(\mathcal{F}|_U, \mathcal{G}|_U)$ induces a morphism of sheaves $-\psi : \mathcal{F}|_U \to \mathcal{G}|_U$

$(-\psi)_V = -\psi_V$

$(-\tau_V + \tau_V)(s) = 0$, hence $0 \in Hom(\mathcal{F}|_U, \mathcal{G}|_U)$

$\mathcal{G}|_U$ abelian $\Rightarrow$ $Hom(\mathcal{F}|_U, \mathcal{G}|_U)$ abelian.

Hence $Hom(\ )$ is a presheaf.

① Let $U = \bigcup U_i$ be open subset w/ covering

Let $\varphi \in Hom(\mathcal{F}|_U, \mathcal{G}|_U)$ s.t. $\varphi|_{U_i} = 0 \ \forall i$.

$$\varphi|_{U_i} = 0 \Rightarrow \varphi_{V \cap U_i} = (\varphi|_{U_i})_{V \cap U_i} = 0 \quad \forall V \subseteq U$$

$V \subseteq U$, $s \in \mathcal{F}|_U(V) = \mathcal{F}(V)$

$$(\varphi_V(s))|_{V \cap U_i} = \varphi_{V \cap U_i}(s|_{V \cap U_i}) = 0 \quad \forall i \in I$$

$V = \bigcup_i U_i \cap V$ and $\mathcal{G}|_U$ a sheaf

$\Rightarrow \varphi_V(s) = 0$. Axiom I ok.

② Let $\varphi_i \in Hom(\mathcal{F}|_{U_i}, \mathcal{G}|_{U_i})$ be s.t.

$\varphi_i|_{U_i \cap U_j} = \varphi_j|_{U_i \cap U_j}$.

Let $V \subseteq U$ open and $s \in \mathcal{F}|_U(V)$.
For $i \in I$ put $t_i = \varphi_{i,V\cap U_i}(s|_{V\cap U_i}) \in \mathcal{G}|_U(V\cap U_i)$
Then $\forall i,j$

$$t_i|_{U_i\cap U_j} = \varphi_{i,V\cap U_i}(s|_{V\cap U_i})|_{U_i\cap U_j}$$

$$= \varphi_{i,V\cap U_i\cap U_j}(s|_{V\cap U_i\cap U_j})$$

$$= \varphi_{j,V\cap U_i\cap U_j}(s|_{V\cap U_i\cap U_j})$$

$$= t_j|_{U_i\cap U_j}$$

Since $\mathcal{G}|_U$ is a sheaf and $V = \bigcup_i U_i \cap V$,

there is $t \in \mathcal{G}|_U(V)$ s.t.

$$t|_{V\cap U_i} = t_i = \varphi_{i,V\cap U_i}(s|_{V\cap U_i}) \ \forall i$$

Put $\varphi_V(s) = t$.

For $i \in I$, $W \subseteq V \subseteq U$ open, and $s \in \mathcal{F}|_U(V)$
putting $t_i = t|_{V\cap U_i}$ and $\varphi_W(s|_W) = \tilde{t}$
$\tilde{t}_i = \tilde{t}|_{W\cap U_i}$ we get.

$(t|_W)|_{W\cap V_i} = t|_{W\cap V_i} = t_i|_{W\cap V_i}$

$= \varphi_{i,V\cap V_i}(s|_{V\cap V_i})|_{W\cap V_i}$

$= \varphi_{i,W\cap V_i}(s|_{W\cap V_i}) = \tilde{t}_i$.

Now $G|_U$ sheaf and $W = \bigcup V_i \cap W$

$\Rightarrow t|_W = \tilde{t}$ i.e. $\varphi_V(s)|_W = \varphi_W(s|_W)$

Hence OK.

~~§2 Hard~~

Nina Bindel, Ex. 1.17, 12. 4. 2013

Skyscraper sheaves

X top. space ~~with discrete topology~~, $P \in X$ a fixed point, A abelian group (with discrete topology)

$P \in \mathcal{U} \subseteq X$, define sheaf: $i_P(A)(\mathcal{U}) = \begin{cases} A, & P \in \mathcal{U} \\ 0, & P \notin \mathcal{U} \end{cases}$

to show: (i) the stalk $(i_P(A))_Q = \begin{cases} A, & Q \in \overline{\{P\}} \\ 0, & Q \notin \overline{\{P\}} \end{cases}$

closed set consisting of P

(ii) $i_P(A) = inc_*(A)$ $(= i_*(A))$, with A constant sheaf on $\overline{\{P\}}$, and inc inclusion map $inc: \overline{\{P\}} \to X$

Proof: (i) 1st case: Fix $Q \in \overline{\{P\}}$, let $\mathcal{U}$ be open nbhd of Q $\overset{\text{topol. reasons}}{\Rightarrow}$ $P \in \mathcal{U}$

consider the stalk:

$(i_P(A))_Q = \operatorname{colim}_{Q \in \mathcal{U}} \underbrace{i_P(A)(\mathcal{U})}_{= A} = \operatorname{colim}_{Q \in \mathcal{U}} A$

$= \operatorname{colim}(A \xrightarrow{id} A \xrightarrow{id} A \to \dots) = A$

2nd case: Fix $Q \notin \overline{\{P\}}$

$\Rightarrow \exists$ open $\mathcal{U}_0 = X \setminus \overline{\{P\}}$ (since $\overline{\{P\}}$ closed)

with $Q \in \mathcal{U}_0$ but $P \notin \mathcal{U}_0$.

consider the stalk:

$(i_P(A))_Q = \operatorname{colim}_{Q \in \mathcal{U}} i_P(A)(\mathcal{U})$

take element $\langle \mathcal{U}, s \rangle \in (i_P(A))_Q$, with $s \in \mathcal{F}(\mathcal{U})$

and define further $W := \mathcal{U} \cap \mathcal{U}_0$

by def. of $\mathcal{U}_0 \Rightarrow i_P(A)(W) = 0$

On page 62: $\langle \mathcal{U}, s \rangle, \langle V, t \rangle \in (i_P(A))_Q$

$\langle \mathcal{U}, s \rangle = \langle V, t \rangle$ if $\exists W = \mathcal{U}_0 \cap V$,

s.th. $s|_W = t|_W$

In our case: $\mathcal{U} \cap W = \underbrace{\mathcal{U} \cap \mathcal{U}_0}_{= W} \subseteq \mathcal{U}$, so $S|_{U\cap U_0} = S|_W \quad \Rightarrow \langle U, s\rangle = \langle W, s|_W\rangle = 0$

(so s restricted to subset of U)

Since we can define such a W for ev $U \ni Q$:

$(i_P(A))_Q = \underset{Q\in U}{\operatorname{colim}} \underbrace{i_P(A)(U)}_{=0} = \operatorname{colim}(0 \to 0 \to \dots) = 0$

(ii) $inc : \overline{\{P\}} \to X$ and define $inc_*(A)(U) := A(inc^{-1}(U))$,

= set of all continuous functions from $inc^{-1}(U) \to A$

(with discrete topology)

1st case: open subset $U \subseteq X$ with $P \notin U$

$\Rightarrow \overline{\{P\}} \subseteq X \setminus U \Rightarrow_{\text{the fiber}} inc^{-1}(U) = \emptyset \Rightarrow A(inc^{-1}(U)) = 0$

2nd case: open $U \subseteq X$, with $P \in U$, then we want to show $A(i^{-1}(U)) = A$:

We know (bc. discrete topology) that any continuous map $f: inc^{-1}(U) \to A$, send $\overline{\{P\}}$ into $\overline{\{f(P)\}}$ (since cont. maps send closed sets to closed sets)

Since A has discrete topology $f(P)$ is only a point $\Rightarrow \{f(P)\} = \overline{\{f(P)\}}$

Hence, any continuous map is determined by where the point P is sent to.

$\Rightarrow$ any continuous maps is constant function from $inc^{-1}(U) \subseteq \overline{\{P\}} \to A$

Since we can send P to any point in A

$\Rightarrow A(inc^{-1}(U)) = A$

$\Rightarrow inc_*(A) = i_P(A)$ $\square$