2.18 (a) $X = \operatorname{Spec} A$, $f \in A$. f nilpotent $\Leftrightarrow D(f) = \emptyset$.

Pf) From Atiyah–Macdonald we know that the nilradical N equals

$N = \bigcap_{\text{prime}} \mathfrak{p}$. Hence the equivalence follows

(b) $A \xrightarrow{\phi} B$ ring homomorphism. Let

$Y = \operatorname{Spec} B \xrightarrow{f} \operatorname{Spec} A = X$ be induced

morphism of schemes. ϕ injective $\Leftrightarrow$
$f^{\#}: \mathcal{O}_X \to f_*(\mathcal{O}_Y)$ injective.
Furthermore $f(Y)$ is dense in X.

Pf) $\Leftarrow$ If $f^{\#}: \mathcal{O}_X \to f_* \mathcal{O}_Y$ injective then,

$A = \mathcal{O}_X(X) \longrightarrow \mathcal{O}_Y(f^{-1}(X)) = \mathcal{O}_Y(Y) = B$

is injective.

Conversely if $g: A \to B$ injective then the induced map $A_g \to B_{\phi(g)}$ is injective.
But

$$\begin{array}{ccc} \mathcal{O}_{\operatorname{Spec} A}(D(g)) & \longrightarrow & f_* \mathcal{O}_{\operatorname{Spec} B}(D(g)) \\ \| & & \\ A_g & \longrightarrow & B_{\phi(g)} \end{array}$$

is this map,
hence $f^{\#}$ injective on all elements of the base,
hence injective on the stalks, so it is injective.

To show $f(Y)$ closed in X, look at the stalk:

$$f_p^{\#} : \mathcal{O}_{X,p} \hookrightarrow (f_* \mathcal{O}_Y)_p$$
$$\cong$$
$$A_{\mathfrak{p}} \neq 0$$

If $f(Y)$ not closed we would have for some $\mathfrak{p}$ in a nbh of the complement of $f(Y)$.

$$(f_* \mathcal{O}_Y)_{\mathfrak{p}} = \varinjlim \mathcal{O}_Y(f^{-1}(U)) = 0$$

contradicting injectivity of $f_{\mathfrak{p}}^{\#}$

(c) Show that ϕ surjective $\Rightarrow$ f homeomorphism onto a closed subset of X and $f^{\#} : \mathcal{O}_X \to f_* \mathcal{O}_Y$ surjective.

Pf) ϕ surjective $\Rightarrow$ $A_{\mathfrak{q}} \longrightarrow B_{\phi(\mathfrak{q})}$ surjective, hence by same argument as in (b) $f^{\#}$ is surjective.

For $\mathfrak{q}_1, \mathfrak{q}_2 \in \operatorname{Spec} B = Y$, let $f(\mathfrak{q}_1) = f(\mathfrak{q}_2) \Rightarrow$

$\phi^{-1}(\mathfrak{q}_1) = \phi^{-1}(\mathfrak{q}_2)$. Since ϕ surjective $\Rightarrow$ $\mathfrak{q}_1 = \mathfrak{q}_2$ so f is injective. Furthermore $\mathfrak{q}_1 \subset \mathfrak{q}_2 \iff$ $\phi^{-1}(\mathfrak{q}_1) \subset \phi^{-1}(\mathfrak{q}_2) \iff f(\mathfrak{q}_1) \subset f(\mathfrak{q}_2)$.

Hence f is homeomorphism onto its image.

2.18 (c) Finally we show $X \setminus f(Y)$ is open:
Fix $\mathfrak{p} \in X \setminus f(Y)$. from (b) this gives surjective map on stalks.

$$f_{\mathfrak{p}}^{\#} : \mathcal{O}_{X,\mathfrak{p}} \to \varinjlim_{\mathfrak{p} \in U} \mathcal{O}_Y(f^{-1}(U))$$

By 2.2 this is equivalent to

$$\phi_{\mathfrak{p}} : A_{\mathfrak{p}} \to Im(\phi_{\mathfrak{p}})$$

Since $\mathfrak{p} \notin f(\operatorname{Spec} Y) \Rightarrow Im(\phi_{\mathfrak{p}}) = 0$, hence $\mathfrak{p}$ is contained in open nbh not meeting $f(Y)$.

(d) Prove the converse of (c)

(⇐)

$$A \to A/\ker\varphi \xrightarrow{\tilde{\phi}} B$$

This induces

$$\operatorname{Spec} B \xrightarrow{\alpha} \operatorname{Spec}(A/\ker\phi) \xrightarrow{j} \operatorname{Spec} A$$

and morphisms of sheaves

$$\mathcal{O}_{\operatorname{Spec} A} \to j_*(\mathcal{O}_{\operatorname{Spec} A/\ker\varphi}) \xrightarrow{\alpha^{\#}} f_*(\mathcal{O}_{\operatorname{Spec} B})$$

Note that $\alpha^{\#} = \tilde{\phi}$

Fix $\mathfrak{p} \in \operatorname{Spec} A$ and write $\bar{\mathfrak{p}}$ for its image ~~ie~~ in $\operatorname{Spec}(A/\ker\varphi)$. Fix $\mathfrak{p} \in f(\mathfrak{q})$ and write $\mathfrak{q} = f^{-1}(\mathfrak{p})$. Then at stalks:

$$(f_*(\mathcal{O}_{\operatorname{Spec} B}))_{\mathfrak{p}} = \varinjlim_{\mathfrak{p}\in U}(f_*\mathcal{O}_{\operatorname{Spec} B}(U)) = \lim_{p\in U} \mathcal{O}_{\operatorname{Spec} B}(f^{-1}(U)) = (\mathcal{O}_{\operatorname{Spec} B})_{\mathfrak{q}}$$

Where the last equality holds since $f(\mathfrak{q})$ is closed.

Similarly $j_*(\mathcal{O}_{\operatorname{Spec}(A/\ker\varphi)})_{\mathfrak{p}} = (\mathcal{O}_{\operatorname{Spec}(A/\ker\varphi)})_{\bar{\mathfrak{p}}}$

Hence

$$\alpha^{\#}_{\bar{\mathfrak{p}}} : (\mathcal{O}_{\operatorname{Spec}(A/\ker\varphi)})_{\bar{\mathfrak{p}}} \to (\mathcal{O}_{\operatorname{Spec} B})_{\mathfrak{q}}$$

is surjective. By prop 2.2 this corresponds to

$$\phi^{\#}_{\bar{\mathfrak{p}}} : (A/\ker\varphi)_{\bar{\mathfrak{p}}} \to B_{\mathfrak{q}}.$$

We know from A-M that ~~ring~~ R-module ring hom $g : M \to N$ surjective $\Leftrightarrow$

$g_p : M_p \to N_p$ surjective for all primes.

Using this we are done.