5.5 (a) Let $f: \mathbb{A}^1_k \to \mathrm{Spec}(k)$. Then $\mathcal{O}_{\mathbb{A}^1_k}$ is coherent, but its pushforward is not coherent b/c $k[X]$ is not a finitely generated k-module.

(b) Let $i: Z \hookrightarrow X$ be a closed immersion, and $\{U_i\}$ an affine cover of X. The restricted maps $i^{-1}(U_i) \to U_i$ are closed embeddings $\Rightarrow$ of the form (Cor. 5.10) $\mathrm{Spec}(A_i/I_i) \to \mathrm{Spec}(A_i)$ for some ideal $I_i \subset A_i$. Now A_i/I_i is a fin. gen. A_i-module (gen. by $1+I_i$) $\Rightarrow$ i is a finite map.

(c) $\{\mathrm{Spec}(B_i)\}$ open affine cover of Y $\Rightarrow$ $f^{-1}(\mathrm{Spec}(B_i)) = \mathrm{Spec}(A_i)$ where A_i is a fin.-gen. B_i-module (b/c f is finite). Since $\mathcal{F}$ is coherent and X is noetherian, $\mathcal{F}|_{\mathrm{Spec}(A_i)} \approx \widetilde{M}_i$ for M_i some fin. gen. A_i-module. Next we compute that

$$f_*\mathcal{F}|_{\mathrm{Spec}(B_i)} = (f|_{\mathrm{Spec}(A_i)})_* \mathcal{F}|_{\mathrm{Spec}(A_i)} = (f|_{\mathrm{Spec}(A_i)})_*(\widetilde{M}_i) \overset{5.2(d)}{\approx} \widetilde{{}_{B_i}M_i}$$

(where ${}_{B_i}M_i$ denotes M_i considered as a B_i-module). By the fin. gen. above, ${}_{B_i}M_i$ is a fin. gen. B_i-module $\Rightarrow$ $f_*\mathcal{F}$ is coherent.

6.4 We show the more general result: Let A be UFD in which 2 is a unit, $a \in A$ some square free element (p a prime element in A, then $a \notin p^2A$) which is not a unit. Then $A[T]/(T^2-a)$ is normal.

proof: If K is the fraction field of A, the fraction field of $A[T]/(T^2-a)$ is $L = K[t]$ (where t is the residue class of T in $A[T]/(T^2-a)$). Suppose $r+st \in L$ is integral over A so that its minimal polynomial $X^2 - 2rX + (r^2-as^2)$ has coefficients in A, provided $s \neq 0$. Hence $r+st$ is integral over $A \Leftrightarrow -2r \in A$ and $r^2 - as^2 \in A$. By the assumption $2 \in A^*$, it follows that $r \in A$. In order to show that $s \in A$, we observe that $s = \frac{s_1}{s_2}$ with s_1 & s_2 relatively prime elt's of A implies s_2 is a unit. Since $-as^2 \in A$, write $as^2 = a' \Rightarrow$ $as_1^2 = a's_2^2$. If the prime element p divides s_2, then $as_1^2 \in p^2A$. And since s_1 & s_2 are relatively prime, $a \in p^2A$ $\lightning$ Hence s_2 cannot have any prime factors $\Rightarrow$ it is a unit $\Rightarrow s \in A$. Thus the integral closure of A in L coincides with $A[t]$.

6.5 Quadratic hypersurfaces

$$X \equiv \operatorname{Spec}\left(A = \frac{k[x_0,\dots,x_n]}{(x_0^2+\dots+x_r^2)}\right) \; ; \; k \text{ field of characteristic } \neq 2.$$

(a) By 6.4 it suffices to show that $f = x_0^2+\dots+x_r^2$ is $\square$-free. Now $\deg(f) = 2 \Rightarrow f$ is a product of at most 2 nonconstant linear polynomials. If $f = (\sum a_i x_i)^2$, clearly $a_i^2 = 1$, $0 \le i \le r$, and $2a_ia_j = 0$ for $i \neq j$. But then $2 = 2a_i^2a_j^2 = 0$ in $k \Rightarrow$ char$(k) = 2$, violating the assumption.

(b) Assuming k contains a primitive 4^{th} root of unity, say i, the linear change of variables $x_0 \mapsto \frac{x_0 - x_1}{2}$, $x_1 \mapsto \frac{x_0+x_1}{2i}$ allows us to write $-x_0x_1$ for $x_0^2+x_1^2$.

check the conditions

(1) $A = \frac{k[x_0,\dots,x_n]}{x_0x_1 - x_2^2}$, with prime divisor $Y = \operatorname{Spec}(A/(x_1,x_2))$. Note that, in A, $x_1 = 0 \Rightarrow x_2 = 0$ so it follows that Y is set-theoretically cut out by x_1. Moreover, $A/(x_1,x_2)$ is a domain so its generic point is (0), which in this case is (x_1,x_2). By localizing A at (x_1,x_2), x_0 becomes a unit and we can write $x_1 = x_0^{-1}x_2^2 \in (x_2) \Rightarrow$
"not contained in the maximal ideal"
x_2 generates the maximal ideal in the localized ring. In particular, $v_Y(x_1) = 2$ (b/c of the equation $x_1 = x_0^{-1}x_2^2$ & units have valuation zero). (And since Y is cut out by x_1 there cannot exists any other prime divisors Z with $v_Z(x_1) \neq 0$.) Next we claim the equality $X \setminus Y = \operatorname{Spec}(A_{x_1})$. LHS consists of prime ideals $\mathfrak{p}$ of $k[x_0,\dots,x_n]$ s.t. $(x_0x_1 - x_2^2) \subset \mathfrak{p}$ and $(x_1,x_2) \not\subset \mathfrak{p}$, while RHS consists of $\mathfrak{p}$ for which $(x_0x_1 - x_2^2) \subset \mathfrak{p}$ and $x_1 \notin \mathfrak{p}$. Clearly $x_1 \notin \mathfrak{p}$ implies $(x_1,x_2) \not\subset \mathfrak{p}$. Conversely, suppose that $\mathfrak{p} \in X \setminus Y$. Then if $x_1 \notin \mathfrak{p}$ we are done, so assume $x_2 \notin \mathfrak{p}$ and $x_1 \in \mathfrak{p}$. With this assumption, $x_0x_1 - x_2^2 \in \mathfrak{p} \Rightarrow x_0x_1 - (x_0x_1 - x_2^2) = x_2^2 \in \mathfrak{p} \Rightarrow x_2 \in \mathfrak{p}$ ↯ and the equality holds.

As rings we have

$$A_{x_1} = \frac{k[x_0,\dots,x_n][\frac{1}{x_1}]}{(x_0x_1 - x_2^2)} \underset{x_0 = \frac{x_2^2}{x_1}}{\approx} \overbrace{k[x_1, \tfrac{1}{x_1}, x_2, \dots, x_n]}^{\text{UFD}}$$

H: Proposition 6.5 shows there is an exact sequence

$$(*) \quad \mathbb{Z} \to Cl(X) \to Cl(X \setminus Y) \to 0.$$

In (*) the map $\mathbb{Z} \to Cl(X)$ sends 1 to the class of Y in the divisor class group of X. This map is surjective b/c $X \setminus Y$ is the Zariski prime spectrum of a UFD ($\Rightarrow Cl(X \setminus Y) = 0$). Moreover, by the above, $(x_1) = 2 \cdot Y$ is a principal divisor. Hence Y has order at most 2 in $Cl(X)$. It remains to show that Y is not principal.

Fact: A UFD $\Leftrightarrow$ $Cl(X) = 0$ (H: Proposition 6.2)

It is well-known that A UFD $\Rightarrow$ every height 1 prime ideal is principal (cf. H: pg. 132, proof of Proposition 6.2). But it is clear that $(x_1, x_2) \in \mathrm{Spec}(A)$ – which defines Y – is not principal: If $\mathfrak{m} = (x_0, \dots, x_n) \in \mathrm{Max}(A)$, then $\mathfrak{m}/\mathfrak{m}^2$ is a k-vector space of dimension $n+1$ (basis $\{\bar{x}_i\}_{i=0}^{n}$). We have $\mathfrak{p} \subset \mathfrak{m}$ and the image of $\mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$ contains $\bar{x}_1$ and $\bar{x}_2$.
(The above is really Example 6.5.2 in H.)

(2) By a similar change of variables as in (1) we may rewrite $x_0^2 + \dots + x_3^2$ as $x_0 x_1 - x_2 x_3$. If $Q \overset{\approx \mathbb{P}^1_k \times \mathbb{P}^1_k}{}$ is the projective variety in $\mathbb{P}^3_k$ defined by this equation, then $Cl(Q) \approx \mathbb{Z} \oplus \mathbb{Z}$ according to Example 6.6.1. Let X' be the affine cone of Q in $\mathbb{A}^4_k$. Exercise 6.3 (b) implies $Cl(X') \approx \mathbb{Z}$. Now $X = X' \times \mathbb{A}^{n-4}_k$, so by Proposition 6.6 we get

$$Cl(X) \approx Cl(X') \approx \mathbb{Z}.$$

(3) Again, let Y be cut out by the equation $x_1 = 0$. There is an exact sequence $\mathbb{Z} \to Cl(X) \to Cl(X \setminus Y) \to 0$, where $Cl(X \setminus Y) = 0$ since $X \setminus Y = \mathrm{Spec}(k[x_1, \frac{1}{x_1}, x_2, \dots, x_n])$ (eliminate x_0 as in part (1)).

In this case we claim that Y is principal: Consider the ideal $(x_1) \subset A$ corresponding to the closed subset Y. If (x_1) is a prime ideal, then Y will be the principal divisor associated to the rational function x_1.

We'll prove this by showing that $A/(x_1)$ is a domain $\Leftrightarrow \dfrac{k[x_0,\dots,x_n]}{(x_1, x_2^2+\dots+x_r^2)}$ domain b/c $(x_1, x_0x_1 - (x_2^2+\dots+x_r^2)) = (x_1, x_2^2+\dots+x_r^2) \Leftrightarrow \dfrac{k[x_0, x_2, \dots, x_n]}{(x_2^2+\dots+x_r^2)}$ domain (x_1 missing) $\Leftrightarrow f = x_2^2+\dots+x_r^2$ is irreducible for $r \geq 4$

<u>Exercise:</u> $f_n = a_0x_0^2 + \dots + a_nx_n^2$ where $a_0 \cdots a_n \neq 0$ is irreducible in $k[x_0,\dots,x_n]$ for $n \geq 2$.

<u>Solution:</u> Induction shows that it suffices to establish that f_2 is irreducible (since otherwise, letting $x_3 = \dots = x_r = 0$ would give a factorization of f_2).

Suppose for contradiction that

$$f_2 = (\alpha_0x_0 + \alpha_1x_1 + \alpha_2x_2)(\beta_0x_0 + \beta_1x_1 + \beta_2x_2)$$
$$= \sum_{i=0}^{2} \alpha_i\beta_i x_i^2 + \sum_{0 \leq i < j \leq 2} (\alpha_i\beta_j + \alpha_j\beta_i)x_ix_j.$$

Now contemplate the linear equations

$$\begin{pmatrix} \alpha_0 & 0 & 0 \\ 0 & \alpha_1 & 0 \\ 0 & 0 & \alpha_2 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \overset{(*)}{=} \begin{pmatrix} a_0 \\ a_1 \\ a_2 \end{pmatrix}, \quad \begin{pmatrix} \alpha_1 & \alpha_0 & 0 \\ \alpha_2 & 0 & \alpha_0 \\ 0 & \alpha_2 & \alpha_1 \end{pmatrix} \begin{pmatrix} \beta_0 \\ \beta_1 \\ \beta_2 \end{pmatrix} \overset{(**)}{=} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

$(*) \Rightarrow \alpha_0\alpha_1\alpha_2 \neq 0$ and $\beta_0\beta_1\beta_2 \neq 0$

$(**) \Rightarrow \alpha_0\alpha_1\alpha_2 = 0$ (for $(**)$ to have a nonzero solution, the determinant of the coefficient matrix must be zero).

(c) Exercise 6.3 $\Rightarrow$ exact sequence

$$(*)\quad 0 \to \mathbb{Z} \to Cl(Q) \to Cl(X) \to 0 .$$

In $(*)$, $\mathbb{Z} \to Cl(Q)$ maps 1 to $Q \cdot H$ (H any hyperplane of $\mathbb{P}^n_k$ not containing Y).

(1) When $r = 2$, $Cl(X) \approx \mathbb{Z}/2$. As in Example 6.5.2, $Cl(Q)$ is cyclic with generator the ruling of the projective quadric cone $\Rightarrow$ the class of a hyperplane section $Q \cdot H$ is twice the generator of $Cl(Q)$ (since it maps to zero in $Cl(X)$ by exactness).

(2) When $r = 3$, $Cl(X) \simeq \mathbb{Z} \Rightarrow Cl(Q) \simeq \mathbb{Z} \oplus \mathbb{Z}$.

(3) When $r \geq 4$, $Cl(X)$ is trivial $\Rightarrow Cl(Q) \simeq \mathbb{Z}\{Q \cdot H\}$.

(d) The homogeneous coordinate ring $S(Q) = k[x_0, \dots, x_n] / (x_0^2 + \dots + x_r^2)$ is also the affine coordinate ring of X, and $Cl(X) = 0$ by part (3) of (b). Hence it is a UFD according to Proposition 6.2 (and (a)). It follows that every prime ideal of height 1 is principal (in the same ring) $\Rightarrow$ the prime ideal corresponding to Y is principal. Its generator gives V.