Glueing

Warning: Very, very preliminary version. Version prone to errors.
Version 0.2— last update: 4/23/15 11:06:13 AM
Have added the existence proof for products we did yesterday (April 22)

Glueing sheaves

The setting is a scheme with an open covering $\{U_i\}_{i\in I}$ with a sheaf F_i on each open subset U_i. As usual the sheaves can take values in any category, but the principal situation we have in mind is when the sheaves are sheaves of abelian groups. The intersections $U_i \cap U_j$ are denoted by U_{ij}, and triple intersections $U_i \cap U_j \cap U_k$ are written as U_{ijk}.

The glueing data in this case consist of isomorphism $\tau_{ji}\colon F_i|_{U_{ij}} \to F_j|_{U_{ij}}$. The idea is to identify sections of $F_i|_{U_{ij}}$ with $F_j|_{U_{ij}}$ using the isomorphisms τ_{ij}. For the glueing process to be feasible, the τ_{ij}'s must satisfy the three conditions

- $t_{ii} = \mathrm{id}_{F_i}$,
- $\tau_{ji} = \tau_{ij}^{-1}$,
- $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$,

where the last identity takes place where it can; that is, on the triple intersection U_{ijk}. Observe that the three conditions parallel the three requirements for a relation being an equivalence relation; the first reflects reflectivity, the second symmetry and the third transitivity.

The third requirement is obviously necessary for glueing to be possible. A section s_i of $F_i|_{U_{ijk}}$ will be identified with its image $s_j = \tau_{ij}(s_i)$ in $F_j|_{U_{ijk}}$, and in its turn, s_j is going to be equal to $s_k = \tau_{kj}(s_j)$. Then, of course, s_i and s_k are enforced to be equal, which means that $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$.

Proposition 0.1 *In the setting as above, there exists a unique sheaf F on X such that there are isomorphisms $\upsilon_i\colon F|_{U_i} \to F_i$ satisfying $\upsilon_j = \tau_{ij} \circ \upsilon_j$ over the intersections U_{ij}.*

PROOF: Let $V \subseteq X$ be an open set and let $V_i = U_i \cap V$ and $V_{ij} = U_{ij} \cap V$. We are going to sections of F over U, and they are to be obtained by glueing sections of the F_i's along V_i using the isomorphisms τ_{ij}. We define

$$\Gamma(V, F) = \{\, (s_i)_{i\in I} \mid \tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}} \,\} \subseteq \prod_{i\in I} \Gamma(V_i, F_i). \tag{1}$$

The τ_{ij}'s are maps of sheaves and therefore are compatible with all restriction maps, so if $W \subseteq V$ is another open set, one has $\tau_{ij}(s_i|_{W_{ij}}) = s_j|_{W_{ij}}$ if $\tau_{ji}(s_i|_{V_{ij}}) = s_j|_{V_{ij}}$. By this, the defining condition in (1) is compatible with componentwise restrictions, which hence can used for restriction in F. We have thus defined a presheaf.

The first step in the remaining proof, is to establish the isomorphisms $\upsilon_i\colon F|_{U_i} \to F_i$, and to avoid getting confused by the names of the indices, we work with a fixed index $\alpha \in I$. Suppose $V \subseteq U_\alpha$ is an open set. Then naturally one has $V = V_\alpha$, and projecting from the product $\prod_i \Gamma(V_i, F_i)$ onto the component $\Gamma(V_\alpha, F_\alpha) = \Gamma(V, F_\alpha)$ gives us a map of presheaves[1] $\upsilon_\alpha\colon F|_{V_\alpha} \to F_\alpha$. This map sends $(s_i)_{i\in I}$ to s_α. We treat lovers of diagrams with a display:

$$\begin{tikzcd} \Gamma(V,F) \arrow[r] \arrow[dr, "\upsilon_\alpha"'] & \prod_{i\in I}\Gamma(V_i,F_i) \arrow[d, "p_\alpha"] \\ & \Gamma(V,F_\alpha), \end{tikzcd}$$

and proceed to verify that the map just defined gives us the searched for isomorphism:

To begin with, on the intersections $V_{\alpha\beta}$ the requirement in the proposition that $\upsilon_\beta = \tau_{\beta\alpha} \circ \upsilon_\alpha$ is fulfilled. This follows immediately since by the second property of the glue, one has $s_\beta = \tau_{\beta\alpha}(s_\alpha)$

It is surjective: Take a section σ of F_α over some $V \subseteq U_\alpha$ and define $s = (\tau_{i\alpha}(\sigma|_{V_{i\alpha}}))_{i\in I}$. Then s satisfies the condition in (1), and is a legitimate element of $\Gamma(V, F)$; indeed, by the third glueing condition we obtain

$$\tau_{ji}(\tau_{i\alpha}(\sigma|_{V_{ji\alpha}})) = \tau_{j\alpha}(\sigma|_{V_{ji\alpha}})$$

for every $i, j \in I$, and that is just the condition in (1). As $\tau_{\alpha\alpha}(\sigma|_{V_{\alpha\alpha}}) = \sigma$ by the first property of the glue, the element s projects to the section σ of F_α.

It is injective: This is clear, since if $s_\alpha = 0$ if follows that $s_i|_{V_{i\alpha}} = \tau_{i\alpha}(s_\alpha) = 0$ for all $i \in I$. Now F_α is a sheaf and the $V_{i\alpha}$ constitute an open covering of V_α. We conclude that $s_\alpha = 0$ by the locality axiom for sheaves.

The next step is to show that F is a sheaf, and we start withe patching axiom: So assume that $\{V_\alpha\}$ is an open covering of $V \subseteq X$ and that $s_\alpha \in \Gamma(V_\alpha, F)$ is a bunch of sections matching on the intersections $V_{\alpha\beta}$. Since $F|_{U_i\cap V}$ is a sheaf—we just checked that $F|_{U_i}$ is isomorphic to F_i—the sections $s_\alpha|_{V_\alpha\cap U_i}$ patch together to give sections s_i in $\Gamma(U_i \cap V, F)$ matching on $U_{ij} \cap V$. This last condition means that $\tau_{ij}(s_i) = s_j$. By definition $(s_i)_{i\in I}$ then this is a section in $\Gamma(V, F)$ restricting to s_i, and we are done. The locality axiom is easy to verify and is left to reader to verify (do it!). ❑

PROBLEM 0.1. Show the uniqueness statement in the proposition. ✷

Glueing maps of sheaves

This is may be the easiest glueing situation we encounter in this course. The setting is as follows. We are given two sheaves F and G on the scheme X and an open covering

[1] Restrictions operating componentwise, it is straight forward to verify this map being compatible with restrictions.

$\{U_i\}_i$ of X. On each open set U_i we are given a map $\phi_i\colon F|_{U_i} \to G|_{U_i}$ of sheaves, and the glueing conditions

- $\phi_i|_{U_{ij}} = \phi_j|_{U_{ij}}$

are assumed to be satisfied for all $i, j \in I$. In this context we have

Proposition 0.2 *There exists a unique map of sheaves $\phi\colon F \to G$ such that $\phi|_{U_i} = \phi_i$.*

PROOF: The salient point is this: Take any $s \in \Gamma(V, F)$ where $V \subseteq X$ is open, and let $V_i = U_i \cap V$. Then $\phi_i(s|_{V_i})$ is a well defined element in $\Gamma(V_i, G)$, and it holds true that $\phi_i(s|_{V_{ij}}) = \phi_j(s|_{V_{ij}})$ by the glueing condition. Hence the sections $\phi_i(s|_{V_i})$'s of $G|_{V_{ij}}$ patch together to a section of G over V which we define to be $\phi(s)$. The checking of all remaining details is hassle-free and left to the zealous student. ❑

The glueing of schemes

The possibility of glueing different schemes together along open subschemes, gives rise many new scheme. The most prominent one being the projective spaces. The glueing process is also an important part in many general existence proofs, like in the construction of the fiber-product, which exists without restrictions in the category of schemes.

In the present context of scheme-glueing we are given a family $\{X_i\}_{i\in I}$ of schemes indexed by the set I. In each of the schemes we are given a collection of open subschemes X_{ij} which are the glue lines in the process; the contacting surfaces that are to be glued together; *i.e.*, in the glued scheme they will be identified and will be equal to the intersections of X_i and X_j. The identifications of the different pairs of the X_{ij}'s are encoded by a family of scheme-isomorphisms $\tau_{ji}\colon X_{ij} \to X_{ji}$. Furthermore, we let $X_{ijk} = X_{ik} \cap X_{ij}$— this are different parts of the triple intersection before the glueing has been done—and we have to assume that $\tau_{ij}(X_{ijk}) = X_{jik}$. Notice that X_{ijk} is an open subscheme of X_i.

The three following glueing condition, very much alike the ones we saw for sheaves, must be satisfied for the gluing to be doable:

- $\tau_{ii} = \mathrm{id}_{X_i}$.
- $\tau_{ij} = \tau_{ji}^{-1}$.
- The isomorphism τ_{ij} takes X_{ijk} into X_{jik} and one has $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$.

Proposition 0.3 *Given glueing data as above. Then there exists a scheme X with open immersions $\psi_i\colon X_i \to X$ such that $\psi_i|_{X_{ij}} = \psi_j|_{X_{ji}} \circ \tau_{ij}$, and such that the images $\psi_i(X_i)$ form an open covering of X. Furthermore one has $\psi_i(X_{ij}) = \psi_i(X_i) \cap \psi_j(X_j)$. The scheme X is uniquely characterized by these properties*

PROOF: To build the scheme X we first build the underlying topological space X and subsequently equip it with a sheaf of rings. For the latter, we rely on the patching technic for sheaves presented in proposition 0.1. And finally, we need to show that X is locally affine, but this follows immediately once the immersions ψ_i are there—the X_i's are schemes and are locally affine.

On the level of topological spaces, we start out with the disjoint union $\coprod_i X_i$ and proceed by introducing an equivalence relation on it. We require that to points $x \in X_{ij}$ and $x' \in X_{ji}$ be equivalent if $x' = \tau_{ij}(x)$—observe that if the point x does not lie in any X_{ij} with $i \neq j$, we leave it alone, and it will not be equivalent to any other point.

The three glueing conditions imply readily that we obtain an equivalence relation in this way . The first requirement entails that the relation is reflexive, the second that it is symmetric and the third ensures it is transitive. The topological space X is then the defined to be the quotient of $\coprod_i X_i$ by this relation, and we declare the topology on X to be the quotient topology. If π denotes the quotient map, a subset U of X is open if and only if $\pi^{-1}(U)$ is open.

Topologically the maps $\psi_i\colon X_i \to X$ are just the maps induced by the open inclusions of the X_i's in the disjoint union $\coprod_i X_i$. They are clearly injective since a point $x \in X_i$ never is equivalent to another point in X_i. Now, X has the quotient topology so a subset U of X is open if and only if $\pi^{-1}(U)$ is ope, and this holds if and only if $\psi_i^{-1}(U) = X_i \cap \pi^{-1}(U)$ is open for all i. In view of the formula

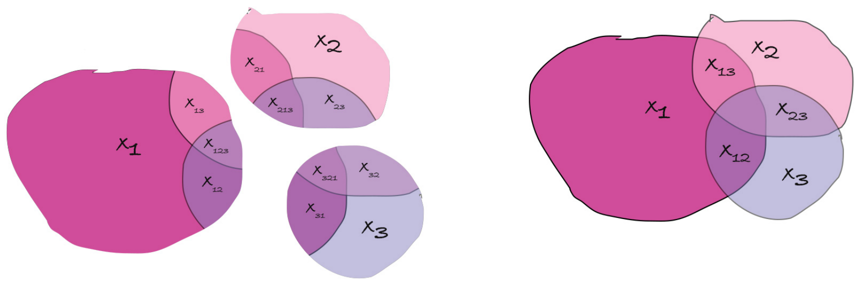
$$\pi^{-1}(\psi_i(U)) = \bigcup_j \tau_{ji}(U \cap X_{ij})$$

we may conclude that each ψ_i is and open immersion.

To simplify notation we now identify X_i and $\psi_i(X_i)$, which is in concordance with our intuitive picture of X as being the union of the X_i's with points in the X_{ij}'s identified according to the τ_{ij}'s. Then X_{ij} becomes $X_i \cap X_j$ ant X_{ijk} becomes the triple intersection $X_i \cap X_j \cap X_k$.

On X_{ij} we have the isomorphisms $\tau_{ji}^\sharp\colon \mathcal{O}_{X_j}|_{X_{ij}} \to \mathcal{O}_{X_i}|_{X_{ij}}$; the sheaf-part of the scheme-isomorphisms $\tau_{ji}\colon X_{ij} \to X_{ji}$. In view of the third glueing condition $\tau_{ki} = \tau_{kj} \circ \tau_{ji}$ above valid on X_{ijk}, we obviously have $\tau_{ki}^\sharp = \tau_{ji}^\sharp \circ \tau_{kj}^\sharp$. The two first glueing conditions translate into $\tau_{ii}^\sharp = \mathrm{id}$ and $\tau_{ji}^\sharp = (\tau_{ji}^\sharp)^{-1}$. The end of the story is that the glueing properties needed to apply proposition 0.1 are satisfied, and we are enabled to glue the different $\mathcal{O}_{X_i}$'s together and thus to equip X with a sheaf of rings. This sheaf of rings restricts to $\mathcal{O}_{X_i}$ on each of the open subsets X_i, and therefore it is a sheaf local rings. So $(X, \mathcal{O}_X)$ is a locally ringed space that is locally affine, hence a scheme.

The uniqueness property is, as usual, left to the industrious student. ❑



Before and after glueing

Global sections of glued schemes

The standard exact sequence for computing global sections from an open covering is valuable tool in the settimng of glued schemes. If X is obtained by blueing the open subschemes X_i along $\tau_{ji}\colon X_{ij} \to X_{ji}$ it reads:

$$0 \longrightarrow \Gamma(X, \mathcal{O}_X) \xrightarrow{\alpha} \bigoplus_i \Gamma(X_i, \mathcal{O}_{X_i}) \xrightarrow{\rho} \bigoplus_{i,j} \Gamma(X_{ij}, \mathcal{O}_{X_{ij}})$$

where $\rho(s_i)_{i\in I} = (s_i|_{X_{ij}} - \tau_{ij}(s_j|_{ji}))$ and $\alpha s = (\psi_i^\sharp(s))_{i\in I}$

The glueing of morphisms

Assume given schemes X and Y and an open covering $\{U_i\}_{i\in I}$ of X. Assume further that there is given a family of morphisms $\phi\colon U_i \to Y$ which mach on the intersections $U_{ij} = U_i \cap U_j$. The aim of this paragraph is show that they can be glued together to give a morphism $X \to Y$:

Proposition 0.4 *In the situation just described, there exists a unique map of schemes $\phi\colon X \to Y$ such that $\phi|_{U_i} = \phi_i$*

PROOF: Clearly the underlying topological map is well defined, so if $U \subseteq Y$ is an open set, we have to define $\phi^\sharp\colon \Gamma(U, \mathcal{O}_Y) \to \Gamma(U, \phi_*\mathcal{O}_X) = \Gamma(\phi^{-1}U, \mathcal{O}_X)$. So take section σ of $\mathcal{O}_Y$ over U. This gives sections $t_i = \phi_i^\sharp(s)$ of $\mathcal{O}_X|_{U_i}$, but since $\phi_i^\sharp$ and $\phi_j^\sharp$ restrict to the same map on U_{ij}, one has $t_i|_{U_{ij}} = t_j|^{U_{ij}}$. The t_i therefore patch together to a section of $\mathcal{O}_X$ over U, which is the section we are aiming at: We define $\phi^\sharp(s)$ to be t. The checking of the remaining details is left to student (as usual). os ❑

A property of the global sections

For a general scheme one may consider $\operatorname{Spec}\Gamma(X, \mathcal{O}_X)$, which in general is different from X—in many cases $\operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ is just point—but there is a canonical map $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ enjoying the following universal property:

Proposition 0.5 *Let X be any scheme. Then there is a canonical (unique) map of schemes $X \to \operatorname{Spec}\Gamma(X, \mathcal{O}_X)$ inducing the identity on global sections of the structure sheaves.*

PROOF: Let $\{U_i\}$ be an affine open covering of X. The restriction maps $\Gamma(X, \mathcal{O}_X) \to \Gamma(U_i, \mathcal{O}_X)$ are ring homomorphisms and therefore they induce maps between the prime spectraSpec $\Gamma(U_i, \mathcal{O}_X)$ Spec $\Gamma(X, \mathcal{O}_X)$. Now the restriction map from sections of $\mathcal{O}_X$ over X to sections of $\mathcal{O}_X$ over $U_i \cap U_j$ is equal to the one from X to U_i followed by the one from U_i to $U_i \cap U_j$; or expressed with a formula $\rho^X_{U_i \cap U_j} = \rho^{U_i}_{U_i \cap U_j} \circ \rho^X_{U_i}$, or for the diagrammoholics, with a diagram like the one below, where all four maps are restrictions; in geometric terms the vertical maps correspond to the inclusions $U_i \subseteq X$ and $U_j \subseteq X$, the skew ones are ψ_i and ψ_j. The two composition are $\psi_i|_{U_i \cap U_j}$ and $\psi_j|_{U_i \cap U_j}$, and that the diagram commutes, means they are equal.

$$\begin{tikzcd}
\Gamma(U_i, \mathcal{O}_{U_i}) \arrow[dr] & \\
\Gamma(U_i \cap U_j, \mathcal{O}_{U_i \cap U_j}) \arrow[u] \arrow[r] \arrow[d] & \Gamma(X, \mathcal{O}_X) \\
\Gamma(U_j, \mathcal{O}_{U_j}) \arrow[ur] &
\end{tikzcd}$$

So the restrictions of the maps ψ_i and ψ_j to the intersection $U_i \cap U_j$ coincide, and hence, by the gluing lemma for maps, they patch together to a map $X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$. By construction $\psi^\sharp$ induces the identity on global sections of the structure sheaves. ❑

Corollary 0.1 *The canonical map $\psi \colon X \to \operatorname{Spec} \Gamma(X, \mathcal{O}_X)$ is universal among the maps from X to affine schemes; i.e., Any map $\alpha \colon X \to \operatorname{Spec} A$ factors as $\alpha = \alpha' \circ \psi$ for a unique map $\alpha' \colon \operatorname{Spec} \Gamma(X, \mathcal{O}_X) \to \operatorname{Spec} A$.*

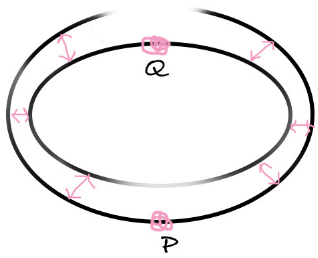
PROOF: The map α is induced by the ring map $\alpha^\sharp \colon A \to \Gamma(X, \mathcal{O}_X)$. ❑

Examples

PROJECTIVE LINE In elementary courses on complex function theory one learns about the Riemann sphere. That is the Wessel plane with one point added, the point at infinity. If z is the complex coordinate centered at the origin, the inverse $1/z$ is the coordinate centered at infinity. Another name for the Riemann sphere is the complex projective line, denoted $\mathbb{P}^1_{\mathbb{C}}$.

The construction of $\mathbb{P}^1_{\mathbb{C}}$ can be vastly generalized, and works in fact over any ring A. Let u be a variable ("the coordinate function at the origin"). and let $U_1 = \operatorname{Spec} A[u]$. The inverse u^{-1} is a variable as good as u ("the coordinate at infinity"), and we let $U_2 = \operatorname{Spec} A[u^{-1}]$. Both are copies of the affine line $\mathbb{A}^1_A$ over A.

Inside U_1 we have the open set $U_{12} = D(u)$ which is canonically equal to the prime spectrum $\operatorname{Spec} A[u, u^{-1}]$, the isomorphism coming from the inclusion $A[u] \subseteq A[u, u^{-1}]$. In the same way, inside U_2 there is the open set $U_{21} = D(u^{-1})$. This is also canonical isomorphic to the spectrum $\operatorname{Spec} A[u^{-1}, u]$, the isomorphism being induced by the inclusion $A[u^{-1}] \subseteq A[u^{-1}, u]$. Hence U_{12} and U_{21} are isomorphic schemes (even canonically), and we may glue U_1 to U_2 along U_{12}. The result is called the *projective line over A* and is denoted by $\mathbb{P}^1_A$.



Glueing two affine lines to get $\mathbb{P}^1$

Proposition 0.6 *One has $\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_A}) = A$.*

PROOF: The standard exact sequence gives us

$$\begin{tikzcd}
\Gamma(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1_A}) \arrow[r] & \Gamma(U_1, \mathcal{O}_X) \oplus \Gamma(U, \mathcal{O}_X) \arrow[r] \arrow[d, "\simeq"'] & \Gamma(U_{12}, \mathcal{O}_X) \arrow[d, "\simeq"] \\
& A[u] \oplus A[u^{-1}] \arrow[r, "\rho"'] & A[u, u^{-1}]
\end{tikzcd}$$

and the map ρ sends a pair $(f(u), g(u^{-1}))$ of polynomials with coefficients in A, one in the variable u and one in u^{-1}, to their difference. We claim that the kernel of ρ equals A; *i.e.,* the polynomials f and g must both be constants.

So assume that $f(u) = g(u)$, and let $f(u) = au^n +$ lower terms in u , and in a similar way, let $g(u^{-1}) = bu^{-m} +$ lower terms in u^{-1} , where both $a \neq 0$ and $b \neq 0$, and without loss of generality we may assume that $m \geq n$. Now, assume that $m \geq 1$. Upon multiplication by u^m we obtain $b + u^m h(u) = u^m f(u)$ for some polynomial $h(u)$, and putting $u = 0$ we get $b = 0$, which is a contradiction. Hence $m = n = 0$ and we are done. ❑

A MORE FANCY EXAMPLE The rings $\mathbb{Z}_{(2)}$ and $\mathbb{Z}_{(3)}$ are both DVR's with maximal ideal being (2) and (3) respectively. Their fraction field are both equal to $\mathbb{Q}$. Let $X_1 = \operatorname{Spec} \mathbb{Z}_{(2)}$ and $A_2 = \operatorname{Spec} \mathbb{Z}_3$. Both have a generic point that is open, so there is a canonical open immersion $\operatorname{Spec} \mathbb{Q} \to X_i$ for $i = 1, 2$. Hence we can glue. We obtain a scheme with one point η to closed points. Let us compute the global sections using the classical restriction sequence for the open covering $\{X_1, X_2\}$

$$\begin{tikzcd}
\Gamma(X, \mathcal{O}_X) \arrow[r] & \Gamma(X_1, \mathcal{O}_X) \oplus \Gamma(X_2, \mathcal{O}_X) \arrow[r] \arrow[d, "="'] & \Gamma(X_1 \cap X_2, \mathcal{O}_X) \arrow[d, "="] \\
& \mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(3)} \arrow[r, "\rho"'] & \mathbb{Q}
\end{tikzcd}$$

the map ρ send a pair (an^{-1}, bm^{-1}) to the difference $an^{-1} - bm^{-1}$, hence the kernel consists of diagonal, so to speak, in $\mathbb{Z}_{(2)} \oplus \mathbb{Z}_{(3)}$, that is the intersection $\mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$. This is a semi local ring with two maximal ideals (2) and (3). Hence there is a map $X \to \operatorname{Spec} \mathbb{Z}_{(2)} \cap \mathbb{Z}_{(3)}$ and it is left as an exercise to show this is an isomorphism.

More generally, if $P = \{p_1, \dots, p_r\}$ is a finite set different primes one may let $X_p = \operatorname{Spec} \mathbb{Z}_{(p)}$ for $p \in P$. There is, as in the previous case, canonical open embedding $\operatorname{Spec} \mathbb{Q} \to X_p$. Let the image be $\{\eta_p\}$. Obviously the glueing conditions are all satisfied (the transition maps are all equal to $\mathrm{id}_{\operatorname{Spec}\mathbb{Q}}$ and $X_{pq} = \{\eta_p\}$ fora all p), and then we do the glueing and obtain an X. Again, to compute the global sections, we get the sequence

$$\begin{tikzcd} \Gamma(X, \mathcal{O}_X) \arrow[r] & \bigoplus_{p\in P} \Gamma(X_p, \mathcal{O}_X) \arrow[r] \arrow[d, "="'] & \bigoplus_{p,q\in P} \Gamma(X_1 \cap X_2, \mathcal{O}_X) \arrow[d, "="] \\ & \bigoplus_{p\in P} \mathbb{Z}_{(p)} \arrow[r, "\rho"'] & \bigoplus_{p,q\in P} \mathbb{Q}. \end{tikzcd}$$

The map r sends the sequence $(a_p)_{p\in P}$ to the sequence $(a_p - a_q)_{p,q\in P}$ and the kernel of ρ is the intersection $A_P = \bigcap_{p\in P} \mathbb{Z}_{(p)}$, which is a semilocal ring whose maximal ideals are the $(p)A_p$'s for $p \in P$. There is a canonical morphism $X \to \operatorname{Spec} A_p$, and again we leave it to the industrious student to verify that this is an isomorphism.

AN EVEN MORE FANCY EXAMPLE! In the previous example we worked with a finite set of primes, but the hypothesis of the glueing theorem impose no restriction on the number of schemes to be glued together, and we are free to take P infinite, for example we can use the set $\mathbb{P}$ of all primes! And the glued scheme $X_{\mathbb{P}}$ is a peculiar animal: It is not affine and not noetherian, but locally noetherian. There is a map $\phi\colon X_{\mathbb{P}} \to \operatorname{Spec} \mathbb{Z}$ which is bijective but not a homeomorphism, but the property that for all open subsets $U \subseteq \operatorname{Spec} \mathbb{Z}$ the map induced on sections $\phi^\sharp\colon \Gamma(U, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}}) \to \Gamma(\phi^{-1}U, \mathcal{O}_{X_{\mathbb{P}}})$ is an isomorphism!

We obtain as before a scheme $X_{\mathbb{P}}$ by glueing the different $\operatorname{Spec} \mathbb{Z}_{(p)}$'s together along the generic points. However, when computing the global sections things, we see things change. The kernel of ρ is still $\bigcap_{p\in\mathbb{P}} \mathbb{Z}_{(p)}$, but now, this intersection equals $\mathbb{Z}$! Indeed a rational number $\alpha = a/b$ lies in $\mathbb{Z}_{(p)}$ precisely when the denominator b does not have p as factor, so lying in all $\mathbb{Z}_{(p)}$, means that b has no non-trivial prime-factor. That is, $b = \pm 1$, and $\alpha \in \mathbb{Z}$.

There is a morphism $X_{\mathbb{P}} \to \operatorname{Spec} \mathbb{Z}$ which one may think about as follows. Each of the schemes $\operatorname{Spec} \mathbb{Z}_{(p)}$ maps in a natural way into $\operatorname{Spec} \mathbb{Z}$, the mapping being induced by the inclusions $\mathbb{Z} \subseteq \mathbb{Z}_{(p)}$. The generic points of the $\operatorname{Spec} \mathbb{Z}_p$'s are all being mapped to the generic point of $\operatorname{Spec} \mathbb{Z}$. Hence they patch together to give a map $X_{\mathbb{P}} \to \operatorname{Spec} \mathbb{Z}$. This is a continuous bijection by construction, but it is not a homeomorphism! Indeed, the subsets $\operatorname{Spec} \mathbb{Z}_{(p)}$ are open in $X_{\mathbb{P}}$ by the glueing construction, but they are not open in $\operatorname{Spec} \mathbb{Z}$.

The scheme $X_{\mathbb{P}}$ is locally noetherian but not noetherian. It is not affine. The sets $U_p = X_{\mathbb{P}} \setminus \{(p)\}$ map bijectively to $D(p) \subseteq \operatorname{Spec} \mathbb{Z}$ and $\Gamma(U_p, \mathcal{O}_{X_{\mathbb{P}}}) = \mathbb{Z}_p$, but U_p and $D(p)$ are not isomorphic.

PROBLEM 0.2. Show that the scheme $X_{\mathbb{P}}$ is locally noetherian but not noetherian. Show that for any $U \subseteq \operatorname{Spec}\mathbb{Z}$, one has $\Gamma(\phi^{-1}(U), \mathcal{O}_{X_{\mathbb{P}}}) = \Gamma(U, \mathcal{O}_{\operatorname{Spec}\mathbb{Z}})$. Conclude that $\phi^{-1}(U)$ is not affine.

✷

PROBLEM 0.3. Glue X_2 to itself along the generic point to obtain a scheme X. Show that X is not affine. HINT: Show that $\Gamma(X, \mathcal{O}_X) = \mathbb{Z}_{(2)}$. ✷

Products

One of the most fundamental properties of schemes is the unrestricted existent of fiber products Citing. It is extremely useful in many situations and takes on astonishingly versatile roles. We begin the paragraph with recalling the definition of the fibre product of sets, then slide into a very general situation to discuss fibre product in general categories, for then to return to the present context of schemes. We prove the existence theorem, and finish up by discussing a series of examples.

Recall the fibre product in the category Sets of sets. The departure is from two sets X_1 and X_2 both with a map to a third set S; *i.e.,* we are given a diagram

$$\begin{tikzcd} X_1 \arrow[dr, "\phi_1"'] & & X_2 \arrow[dl, "\phi_2"] \\ & S & \end{tikzcd}$$

The fibre product $X_1 \times_S X_2$ is a defined as $X_1 \times_S X_2 = \{\, (x, y) \mid \phi(x) = \psi(y) \,\}$. Clearly the diagram, where π_1 and π_2 denote the restrictions of two projections to the fiber product, is commutative,

$$\begin{tikzcd} & X_1 \times_S X_2 \arrow[dl, "\pi_{X_1}"'] \arrow[dr, "\pi_2"] & \\ X_1 \arrow[dr, "\phi_{X_1}"'] & & X_2 \arrow[dl, "\phi_2"] \\ & S & \end{tikzcd}$$

And more is true; the fibre product enjoys a universal property. Given any two maps $f_1 \colon Z \to X_1$ and $f_2 \colon Z \to X_2$ such that $\phi_1 \circ f_{X_1} = \phi_2 \circ f_2$ there is a unique map $f \colon Z \to X_1 \times_S X_2$ such that $\pi_1 \circ f = f_1$ and $\pi_2 \circ f = f_2$; just use the map whose two components are f_1 and f_2 and observe that it takes values in $X_1 \times_S X_2$ since the relation $\phi_1 \circ f_1 = \phi_2 \circ f_2$ holds.

The name fiber product stems from the fact that the fiber of the map $\phi = \phi_i \circ \pi_i$ from $X \times_S Y$ to S over a point $s \in S$, is just the direct product of the fibers of ϕ_1 and ϕ_2 over s, that is $\phi^{-1}(s) = \phi_1^{-1}(s) \times \phi_2^{-1}(s)$.

THE FIBER PRODUCT IN GENERAL CATEGORIES The notion of a fiber product—expressed as the solution to a universal problem—can *mutatis mutandis* be formulated in any category C. Given any two maps $\psi_i \colon X_i \to S$ in the category C. An object—that we shall denote by $X_1 \times_S X_2$—is said to be the *fiber product* (fiberproduktet) of the objects X_i—or more precisely of the two maps $\psi_i \colon X_i \to S$—if the following two conditions are fulfilled:

- There are two maps $\pi_i \colon X_1 \times_S X_2 \to X_i$ in C such that $\psi_1 \circ \pi_1 = \psi_2 \circ \pi_2$ (called the projections).
- For any two maps $\phi_i \colon X \to X_i$ in C such that $\phi \circ \psi_1 = \phi \circ \psi_2$, there is a unique map $\phi \colon X \to X_1 \times_S X_2$ such that $\pi_i \circ \phi = \phi_i$ for $i = 1, 2$.

If the fiber product exists, it is unique up to a unique isomorphism as is true for any solution to a universal problem. However, it is a good exercise to check this in detail in this specific situation.

PROBLEM 0.4. Show that if the fiber product exists, it is unique up to a unique isomorphism. ✷

Recall the functor $h_X \colon \mathsf{C} \to \mathsf{Sets}$ that to any object T in C associates the set $h_X(T) = \mathrm{Hom}_C(T, X)$ and to any map $\alpha \colon T' \to T$ in C associates the map $h_x(\alpha) \colon h_X(T') \to h_X(T)$ sending f to $f \circ \alpha$. The given maps ψ_i gives rise to maps of functors[2] $h_{\psi_i} \colon h_{X_i} \to h_S$ sending a map $f \in h_{X_i}(T)$ to the composition $\psi_i \circ f$. The universal property of the fiber product translates into the following. For any T in C, one has the equivalence of functors from C (or isomorphism if you want):

$$h_{X_1 \times_S X_2} = h_{X_1} \times_S h_{X_2} \tag{2}$$

where the map send $\psi \in h_{X_1 \times_S X_2}(T)$ to the pair $\pi_i \circ \psi$. This formulation says that for any object T in the category C, the set $\mathrm{Hom}_C(T, X_1 \times_S X_2)$ of maps into the fiber product is the fiber product of the two sets $\mathrm{Hom}_C(T, X_i)$ over $\mathrm{Hom}_C(T, S)$, which sometimes is useful.

There is a shorter notation based on the category of *objects over S* where S an object in C. The objects in this new category are maps $\psi \colon X \to S$ in C and maps from $\psi \colon X \to S$ to $\psi' \colon Y \to S$ are maps $f \colon X \to Y$ rendering the following diagram commutative

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, "\psi"'] & & Y \arrow[dl, "\psi'"] \\ & S & \end{tikzcd}$$

This category is denoted C/S. If $\psi \colon X \to S$ is an object from C/S one uses the shorthand notation X/S fort $\psi \colon X \to s$; the map ψ is understood. One furthermore

[2] Normally these are called natural transformstiond.

$h_{X/S}(Y/S) = \mathrm{Hom}_{\mathsf{C}/S}((,X)/S, Y/S)$, and with these conventions the relation (2) takes the form

$$h_{X\times_S Y/S} = h_{X/S} \times h_{Y/S}.$$

Products of affine schemes

The affine category is equivalent to the category of rings and the category of rings we have the tensor product. It has the universal property *dual* to the one of fibered product that we are interested in. To be precise, assume that A_1 and A_2 are B-algebras, *i.e.,* we have two maps of rings α_i

$$\begin{tikzcd} A_1 & & A_2 \\ & B \arrow[ul, "\alpha_2"] \arrow[ur, "\alpha_1"'] & \end{tikzcd}$$

There maps $\beta_i \colon A_i \to A_1 \otimes_B A_2$ sending a to $a \otimes 1$ respectively $1 \otimes a$. They are B-algebra homomorphism since $aa' \otimes 1 = (a\otimes 1)(a' \otimes 1)$ respectively $1 \otimes aa' = (1 \otimes a)(1 \otimes a')$, and they fit into the following commutative diagram as $\alpha_1(b) \otimes 1 = 1 \otimes \alpha_2(b)$ by the definition of the tensor product $A_1 \otimes_B A_2$.

$$\begin{tikzcd} & A_1 \otimes_B A_2 & \\ A_1 \arrow[ur, "\beta_1"] & & A_2 \arrow[ul, "\beta_2"'] \\ & B \arrow[ul, "\alpha_2"] \arrow[ur, "\alpha_1"'] & \end{tikzcd} \tag{3}$$

Moreover the tensor product is *universal* in this respect. Indeed, assume that $\gamma_i \colon A_i \to C$ are B-algebra homomorphisms, *i.e.,* $\gamma_1 \circ \alpha_1 = \gamma_2 \circ \alpha_2$; or said differently, they fit into the commutative diagram analogous to (3) with the b_i's replaced by the γ_i's. The association $a_1 \otimes a_2 \to \gamma_1(a_1)\gamma(a_2)$ is bi-B-linear, and hence it extends to a B-algebra homomorphism $\gamma \colon A_1 \otimes_B A_2 \to C$, that obviously have the property that $\gamma \circ \beta_i = \gamma_i$.

The map γ is te unique B-algebra homomorphism $\gamma \colon A_1 \otimes_B A_2 \to C$ with $\gamma \circ \beta_i = \gamma_i$.

Applying the Spec-functor to all this, we get the diagram

$$\begin{tikzcd} & \mathrm{Spec}(A_1 \otimes_B A_2) \arrow[dl, "\pi_1"] \arrow[dr, "\pi_2"] & \\ \mathrm{Spec}\, A_1 \arrow[dr] & & \mathrm{Spec}\, A_2 \arrow[dl] \\ & \mathrm{Spec}\, B & \end{tikzcd} \tag{4}$$

and the affine scheme $\mathrm{Spec}(A_1 \otimes_B A_2)$ enjoys the property of being universal among affine schemes sitting in a diagram like 4. Hence $\operatorname{Spec}(A_1 \otimes_B A_2)$ equipped with the two projections π_1 and π_2 is the fibered product in the category AffSch of affine schemes. One even has the stronger statement that is the fuber product in the bigger category Sch of schemes.

Proposition 0.7 *Given $\phi_i\colon \operatorname{Spec} A_i \to \operatorname{Spec} B$. Then $\operatorname{Spec}(A_1 \otimes_B A_2)$ with the two projection π_1 and π_2 defined as above is the fiber product of the $\operatorname{Spec} A_i$'s in the category of schemes. That is, if Z is a scheme and $\psi_i\colon Z \to \operatorname{Spec} A_i$ are morphisms with $\psi_1 \circ \pi_1 = \psi_2 \circ \phi_2$, there exists a unique morphism $\psi\colon Z \to \operatorname{Spec}(A_1 \otimes_B A_2)$ such that $\pi_i \circ \psi = \psi_i$ for $i = 1, 2$.*

Proof: We know that the proposition is true whenever Z is an affine scheme; so the salient point is that Z is not necessarily affine. For short we let $X = \operatorname{Spec}(A_1 \otimes_B A_2)$. The proof is just an application of the glueing lemma for morphisms. One covers Z by open affine U_α such that the intersections $U_{\alpha\beta} = U_\alpha \cap U_\beta$ are affine as well (this can be done, *e.g.,* by refining any open affine cover using distinguished open sets). By the affine case of the proposition, for each U_α we get a map $\psi_\alpha\colon U_\alpha \to X$, such that $\psi_\alpha \circ \pi_i = \psi_i|_{U_\alpha}$, and by the uniqueness part of the affine case, these maps coincide on the intersections $U_{\alpha\beta}$, and the can be patched together to a map $\psi\colon Z \to X$. It is unique since the ψ_α's are unique. ❑

A useful lemma

Lemma 0.1 *If $X \times_S Y$ exists and $U \subseteq X$ is an open subscheme, then $U \times_S Y$ exists and is (canonicaly isomorphic to) an open subset of $X \times_S Y$ and projections restrict to projections. Indeed $\pi_X^{-1}(U)$ with the two restrictions $\pi_Y|_{\pi_X^{-1}(U)}$ and $\pi_X|_{\pi_X^{-1}(U)}$ as projections is a product.*

Proof: The situation is displayed as follows:

$$\begin{tikzcd}
 & Y \\
\pi_X^{-1}(U) \arrow[r, hook] \arrow[ur] \arrow[d] & X \times_s Y \arrow[u, "\pi_Y"'] \arrow[d, "\pi_X"] \\
U \arrow[r, hook] & X,
\end{tikzcd}$$

and we shall verify that $\pi_X^{-1}(U)$ and two projections restricted to $\pi_X^{-1}(U)$ satisfy the universal property. If Z is a scheme and $\phi_X\colon Z \to U$ and $\phi_Y\colon Z \to Y$ are two morphisms over S we may consider ϕ_U as a map into X, and therefore they induce a map of schemes $\phi\colon Z \to X \times_S Y$ whist $\phi_X = \pi_X \circ \phi$ and $\phi_Y = \pi_Y \circ \phi$. Clearly $\pi_X \circ \phi = \phi_U$ takes values in U and therefore ϕ takes values in $\pi_X^{-1}(U)$. It follows immediately that ϕ is unique (see the exercise below), and we are through ❑

PROBLEM 0.5. Assume that $U \subseteq X$ is an open subscheme and let $\iota\colon U \to X$ be the inclusion map. Let ϕ_1 and ϕ_2 be two maps of a schemes from a scheme Z to U and assume that $\iota \circ \phi_1 = \iota \circ \phi_2$. Then $\phi_1 = \phi_2$. ✷

Lemma 0.2 *Assume that $U \subseteq S$ is an open set and that the two structure maps $\psi_X\colon X \to S$ and $\psi_Y\colon Y \to S$ both takes values in U. Then $X \times_U Y = X \times_S Y$, with the interpretation that if one of them exists, the other does as well.*

PROOF: Clear ❑

The glueing process

The following proposition will be basis for all glueing necessay fro the construction:

Proposition 0.8 *Let $\psi_X\colon X \to S$ and $\psi_Y\colon Y \to S$ be two maps of schemes, and assume that there is an open covering $\{U_i\}_{i\in I}$ of X such that $U_i \times_S Y$ exist for all $i \in I$. Then $X \times_S Y$ exists. The products $U_i \times_S Y$ form an open covering of $X \times_S Y$ and projections restrict to projections.*

PROOF: We need some notation. Let $U_{ij} = U_i \cap U_j$ be the intersections of the U_i's,and let $\pi_i\colon U_i \times_S Y \to U_i$ denote the projections. By lemma 0.1 there are isomorphisms $\theta_{ji}\colon \pi_i^{-1}(U_{ij}) \to U_{ij} \times_S Y$, and glueing functions we shal use $\tau_{ji} = \theta_{ij}^{-1} \circ \theta_{ji}$ that identifies $\pi_i^{-1}(U_{ij})$ with $\pi_j^{-1}(U_{ij})$. The picture is like this

$$U_i \times_S Y \supseteq \pi_i^{-1}(U_{ij}) \xrightarrow[\simeq]{\theta_{ji}} U_{ij} \times_S Y \xrightarrow[\simeq]{\theta_{ij}^{-1}} \pi_j^{-1}(U_{ji}) \subseteq U_j \times_S Y.$$

The glueing maps τ_{ij} clearly satisfy the glueing conditions being compositions of that the particular form, and the scheme emerging from glueing process is $X \times_S Y$.

The two projections are essential parts of product. The projection onto Y is there all the time since we never touch Y during the construction. The projection onto X is obtained by glueing the projections π_i along the $\pi_i^{-1}(U_{ij})$. By lemma 0.1 we know that the when we identify $\pi_i^{-1}(U_{ij})$ as the product $U_{ij} \times_S Y$ the projection π_{ij} onto U_{ij} corresponds to the restriction $\pi_i|_{\pi_i^{-1}(U_{ij})}$. This means that $\pi_i|_{\pi_i^{-1}(U_{ij})} = \pi_{ij} \circ \theta_{ji}$. To say that $\pi_i|_{\pi_i^{-1}(U_{ij})}$ and $\pi_j|_{\pi_j^{-1}(U_{ij})}$ becomes equal after glueing is to say that $\pi_i|_{\pi_i^{-1}(U_{ij})} = \pi_j|_{\pi_j^{-1}(U_{ij})} \circ \tau_{ji}$ (remember that in the glueing process we identify points x and $\tau_{ji}(x)$), but this holds true since

$$\pi_j|_{\pi_j^{-1}(U_{ij})} \circ \tau_{ji} = \pi_{ij} \circ \theta_{ij} \circ \tau_{ji} = \pi_{ij} \circ \theta_{ij} \circ \theta_{ij}^{-1} \circ \theta_{ji} = \pi_{ij} \circ \theta_{ji} = \pi_i|_{\pi_i^{-1}(U_{ij})},$$

and we can glue the π_i's together to obtain π_X.

It is a matter of easy verification that the the glued scheme with the two projection has the universal property.

❑

It is worth while commenting that the product $X \times_S Y$ is not defined as a particular scheme, it is just an isomorphism class of schemes (having the fundamental property that there is a unique isomorphism respecting the projections between any two). In the proof above both $\pi_i^{-1}(U_{ij})$ and $\pi_j^{-1}(U_{ij})$ are products of $U_{ij} \times_S Y$, but the are not equal only canonically isomorphic. In the construction we could use any of them, or as we in fact did, we can use any non-specified representative in the isomorphism class. This makes the situation much symmetric in i and j.

An immediate consequence of the glueing proposition 0.8 is the following lemma, that is the case when S is affine.

Lemma 0.3 *Assume that S is affine, then $X \times_S Y$ exists*

PROOF: First if Y as well is affine, we are done. Ideed, cover X by open affine sets U_i. Then $U_i \times_S Y$ exists by the affine case, and we are in the position to apply proposition 0.8 above. We then cover Y by affine open sets V_i. As we just verified, $X \times_S V_i$ all exists and applying proposition 0.8 once more, we can conclude that $X \times_S Y$ exists. apply the glueing ❑

The final reduction

Let $\{S_i\}$ be an open affine covering of S and let $U_i = \psi^{-1}(S_i)$ and $V_i = \psi_Y^{-1}(S_i)$. By lemma 0.3 the products $U_i \times_{S_i} V_i$ all exists. Using the following lemma and for the third time the glueing proposition 0.8 we are trough:

Lemma 0.4 *With current notation, we have the equality $U_i \times_{S_i} V_i = U_i \times_S Y$.*

PROOF: The key diagram is

$$
\begin{tikzcd}
& Z \arrow[dl, "f"'] \arrow[dr, "g"] & & \\
U_i \arrow[dr, "\psi_X|_{U_i}"'] & & Y \arrow[dl, "\psi_Y"] & \supseteq V_i \\
& S & &
\end{tikzcd}
$$

where f and g are given maps. If one follows the left path in the diagram, one ends up in S_i, and hence the same must hold following the right path. But then, V_i being equal the inverse image $\psi_Y^{-1}(S_i)$, it follows that g necessarily factors through V_i, and we are done. ❑

Notation.

If $S = \operatorname{Spec} A$ on often writes $X \times_A Y$ in short for $X \times_{\operatorname{Spec} A} Y$. If $S = \operatorname{Spec} \mathbb{Z}$, one writes $X \times Y$. In case $Y = \operatorname{Spec} B$ the shorthand notation $X \otimes_A B$ is frequently seen as well—it avoids writing Spec twice.

Examples

We start out by some nice examples.

Varieties over an algebraically closed field First in the important case that X and Y are varieties over the algebraically closed field k and $S \operatorname{Spec} k$, *i.e.,* two integral schemes of finite type over k. Then the product $X \times_k Y$ will be a variety (*i.e.,* an integral scheme of finite type over k) and the closed points of the product $X \times_k Y$ will be the direct product of the closed points in X and Y; indeed on level of functors $h_{X \times_k Y}(k)$ is the product $h_X(k) \times h_Y(k)$, and closed points correspond to maps $\operatorname{Spec} k \to X$.

It is not entirely obvious that $A \otimes_k B$ is an integral domain when A and B are, and in fact, in general it is by no means true. But it holds true whenever A and B are of finite type over the field k and k is algebraically closed. The standard reference for this is Zariski and Samuel's book *Commutative algebra I* which is the Old Covenant for algebraists. It is also implicit in Hartshorn's book, exercise **3.15** b) on page 22.

Non algebraically closed field In this case the situation is more complicated. The simple and good example being $\operatorname{Spec} \mathbb{C} \times_{\operatorname{Spec} \mathbb{R}} \operatorname{Spec} \mathbb{C}$. This scheme has *two* distinct closed points and is not integral, it is not even connected! The example also shows that the underlying set of the fiber product is not necessarily equal to the fiber product of the underlying sets (in this case this is just one point). So we issue a warning: The product of integral schemes is in general not necessarily integral!

The tensor product $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C}$ is in fact isomorphic to the direct product $\mathbb{C} \times \mathbb{C}$; indeed, we compute

$$\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C}[t]/(t^2+1) = \mathbb{C}[t]/(t-i)(t+i) = \mathbb{C}[t]/(t-i) \times \mathbb{C}[t]/(t+i) = \mathbb{C} \times \mathbb{C}$$

where we have used the Chinese remainder theorem in the last equation.

This little example can easily be generalize: Assume that L is a simple, separable field extension of K of degree d; that is $L = K(\alpha)$ where the minimal polynomial $f(t)$ of α over K is separable and of degree d. Assume further that Ω is a field extension of K in which $f(t)$ splits completely; *e.g.,* a normal extension of L or any algebraically closed field containing K; then one has $L \otimes_K \Omega = \Omega \times \ldots \times \Omega$ where the product has d factors. And $\operatorname{Spec} L \times \operatorname{Spec} \Omega$ has d connected components and is not integral.

Problem 0.6. With the assumptions of the discussion above, show that it holds rue that $L \otimes_K \Omega = \Omega \times \ldots \times \Omega$. ✷

Another example along same lines shows that $X \times_S Y$ is necessarily reduced even if both X and Y are. let k be a non-perfect field in characteristic p. This means that there is an $a \in k$ that is not a p-th power of any element in k. Let $L = k(b)$ where $b^p = a$, that is $L = k[t]/(t^p - a)$. As $T^p - a$ is irreducible over k, this is a field, but one has

$$L \otimes_k L = L[T^p - a] = L[t]/(t^p - b^p) = L[t]/((t-b)^p)$$

which is not reduced. So we issue a second warning: the fiber product of integral schemes is not in general necessarily reduced!

One can elaborate these example and construct an example of two noetherian schemes X and Y such that $X \times_S Y$ is not noetherian, even if S is the spectrum of a field.

For example one may take L to the sub field of $\overline{\mathbb{Q}}$ generated by all elements such that $\xi^{2^n} = 2$ for some n. Then L is the union of the ascending chain of fields

$$\mathbb{Q} \subseteq \mathbb{Q}(\sqrt{2}) \subseteq \mathbb{Q}(\sqrt[4]{2}) \subseteq \mathbb{Q}(\sqrt[8]{2}) \subseteq \ldots L$$

Then of course L being a field is noetherian as is $\overline{\mathbb{Q}}$, but $L \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not! Indeed, $\mathbb{Q}(\sqrt[2^r]{2}) \otimes \overline{\mathbb{Q}}$ is isomorphic to the direct product of 2^r-copies of $\overline{\mathbb{Q}}$ so $L \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is the union of a sequence of subrings each being a direct product of a steadily increasing number of copies of $\overline{\mathbb{Q}}$.

PROBLEM 0.7. Show in detail that $L \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}$ is not noetherian. ✷

Geometric points

If X is a scheme a *geoemrtric point* (et geometrisk punkt) consists of an algebraically closed field k and a morphism $\operatorname{Spec} k \to X$. Giving such a geowmeyric point is equivalent to give a point $x \in X$ and a field extension $k(x) \subseteq k$.

For example

EXAMPLE 0.1. One has $\operatorname{Spec} \mathbb{Q} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Q} = \operatorname{Spec} \mathbb{Q}$. Indeed, there is only one ring homomorphism from $\mathbb{Q}$ to any ring, so $h_{\operatorname{Spec} \mathbb{Q}}(T)$ is either a singleton or empty. It follows that the fibre product $h_{\operatorname{Spec} \mathbb{Q}}(T) \times_{\mathbb{Z}} h_{\operatorname{Spec} \mathbb{Q}}(T)$ either is a singleton or empty, hence OK. ❄

PROBLEM 0.8. Show that if A is a B-algebra with property that there is at most one B-algebra homomorphism $A \to C$ for any A-algebra C then $\operatorname{Spec} A \times_{\operatorname{Spec} B} \operatorname{Spec} A = \operatorname{Spec} A$. HINT: This is a straight forward verification of the universal property of the tensor product. ✷

PROBLEM 0.9. Let p and q be two different primes. Show that $\operatorname{Spec} \mathbb{F}_p \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{F}_q = \emptyset$. ✷

PROBLEM 0.10. Show that if $\operatorname{Spec} A$ and $\operatorname{Spec} B$ are affine schemes of finite type over a field k, then $\operatorname{Spec} A \times_k \operatorname{Spec} B$ is non-empty. Is the same true if one of them is not of finite type? HINT: Yes, *e.g.,* show OK if A and B are field extensions of k. ✷

PROBLEM 0.11. Recall that $\mathbb{Z}_{(p)}$ denotes the localization of $\mathbb{Z}$ in the prime ideal (p) generated by p. Show that $\operatorname{Spec} \mathbb{Z}_{(p)} \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_{(p)} = \operatorname{Spec} \mathbb{Z}_{(p)}$. ✷

PROBLEM 0.12. Assume that p and q are two different primes. Show that $\operatorname{Spec} \mathbb{Z}_p \times_{\operatorname{Spec} \mathbb{Z}} \operatorname{Spec} \mathbb{Z}_q = \operatorname{Spec} \mathbb{Q}$. ✷

PROBLEM 0.13. Let X be the scheme obtained by glueing $X_2 = \operatorname{Spec} \mathbb{Z}_{(p)}$ to it self along the generic point. Show that $X \times_{\mathbb{Z}} X$ is obtained by glueing four copies of X_2 together along the generic points. Show that the diagonal $\Delta \subseteq X \times_{\operatorname{Spec} \mathbb{Z}} X$ is the glueing of two of them and therefore is not closed. ✷